A new recursion for three-column combinatorial Macdonald polynomials

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Abstract

The Hilbert series $\widetilde{F}_\mu$ of the Garsia-Haiman module M_μ can be described combinatorially as the generating function of certain fillings of the Ferrers diagram of μ where μ is an integer partition of n. Since there are $n!$ fillings that generate $\widetilde{F}_\mu$, it is desirable to find recursions to reduce the number of fillings that need to be considered when computing $\widetilde{F}_\mu$ combinatorially. In this paper, we present a combinatorial recursion for the case where μ is an n by 3 rectangle. This allows us to reduce the number of fillings under consideration from $(3n)!$ to $(3n)!/(3!^n n!)$.

Keywords: symmetric functions, tableaux, Macdonald polynomials

1. Introduction

The Macdonald polynomials $\{P_\mu(x_1, x_2, \ldots, x_n; q, t) : \mu \text{ is a partition}\}$ were first introduced in [9] as a basis for the vector space of symmetric functions in n variables. The P_μ can be specialized to several of the common bases for symmetric functions. For example, $P_\mu|_{q=t} = s_\mu$, the Schur functions, and $P_\mu|_{t=1} = m_\mu$, the monomial symmetric functions. A number of algebraic transformations can be used to obtain several other versions of Macdonald polynomials, including the modified Macdonald polynomials $\widetilde{H}_\mu$.

The $\widetilde{H}_\mu$ can also be thought of from a representation-theoretical standpoint as the Frobenius series of a particular doubly-graded S_n-module M_μ introduced by Garsia and Haiman [3]. The coefficient of the monomial $x_1 x_2 \cdots x_n$ in $\widetilde{H}_\mu$ is a polynomial denoted $\widetilde{F}_\mu(q,t)$, which is the Hilbert series of the Garsia-Haiman module. In [5] Haglund conjectured, and in [6, 7] Haglund, Haiman, and Loehr proved, a combinatorial definition of $\widetilde{H}_\mu$. Restricting this definition to standard fillings (defined in §2) of the Ferrers diagram of μ leads immediately to a combinatorial definition of $\widetilde{F}_\mu(q,t)$ given by

$$\widetilde{F}_\mu(q,t) = \sum_T q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)}, \tag{1}$$

where T is a standard filling and $\mathrm{inv}_\mu(T)$ and $\mathrm{maj}_\mu(T)$ are weights on T. From the combinatorial definition it follows directly that $\widetilde{F}_\mu(q,t)$ is a polynomial.

In [4] Garsia and Haiman study a rational recursion for $\widetilde{F}_\mu(q,t)$ based on the removal of corners from the diagram of μ. Their recursion is

$$\widetilde{F}_\mu(q,t) = \sum_{\nu \to \mu} c_{\mu,\nu}(q,t)\widetilde{F}_\nu(q,t), \qquad \widetilde{F}_{(1)}(q,t) = 1 \tag{2}$$

where the sum is over all partitions ν obtained by removing an inner corner from μ and $c_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t)$ is a quotient of two polynomials determined by the diagrams of μ and ν. The recursion in (2) is challenging to prove combinatorially, and the rational coefficients make it difficult to deduce that $\widetilde{F}_\mu(q,t)$ is a polynomial. Using (2) and setting $F_\mu(q,t) = t^{n(\mu)}\widetilde{F}_\mu(q,1/t)$, Garsia and Haiman proved a q-analogue of the classical hook length formula

$$F_\mu(q,q) = f^\mu \prod_{c\in\mu}[h_c]_q \tag{3}$$

where f^μ is the number of standard Young tableaux of shape μ and h_c is the length of the hook of cell c in the Ferrers diagram of μ.

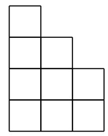
In a more recent paper [2], Garsia and Haglund present a new recursion for $\widetilde{F}_\mu$ indexed by shapes μ with two columns. Their recursion is proved using representation-theoretical techniques that do not lend themselves easily to a combinatorial understanding of the recursion. This recursion is again based on removing corners of μ. In [8], Loehr and Niese present a recursion for two-column shapes that is based on removing the bottom row of μ. They give a fully combinatorial proof of the recursion using bijections and combinatorial operations on the fillings that generate F_μ.

In this paper, we introduce a recursion for $\mu = (3^n)$ obtained by removing the bottom row of the fillings. This new recursion allows us to write F_μ as a sum of $(3n)!/(3!^n n!)$ q,t-analogues of $3!^n n!$ rather than enumerating $(3n)!$ standard fillings. Iteration of the recursion leads to a bijective proof of the fact that $\widetilde{F}_{(3^n)}(q,t)$ and $F_{(3^n)}(q,t)$ are divisible by $[n]!_t$. Note also that the product of the hook lengths of the cells in the last column of the diagram of (3^n) is $n!$, and thus, this recursion for $F_{(3^n)}(q,t)$ may be a first step toward a bijective understanding of (3).

In §2, we review the combinatorial definition of $F_\mu(q,t)$ and then, in §3 we define and analyze the combinatorial operations required to give a bijective proof of the recursion. Finally, in §4 we define a new polynomial for fillings with a fixed bottom row and state a recursion for this polynomial in Theorem 4.3.

2. Combinatorial Definition of F_μ

We first review the combinatorial definition of F_μ, which was conjectured in [5] and proved in [6, 7]. A *partition* of an integer n is a sequence $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ such that $\mu_1 + \mu_2 + \cdots + \mu_k = n$. Let $\mathrm{Par}(n)$ denote the set of all partitions of n. The *diagram* of μ is a collection of left justified boxes such that in row i from the bottom there are μ_i boxes. For example, the diagram of $\mu = (3, 3, 2, 1)$ is



A *standard filling* of $\mu \in \mathrm{Par}(n)$ is a placement of the integers $1, 2, \ldots, n$, each used exactly once, in the diagram of μ. Let $\mathcal{F}_\mu = \{T : T \text{ is a standard filling of } \mu\}$. A standard filling T of $\mu = (3, 3, 2, 1)$ is

$$T = \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 5 & \\ \hline 4 & 9 & 6 \\ \hline 8 & 1 & 7 \\ \hline \end{array} \, . \tag{4}$$

The *ascent set* of a word $w = w_1 w_2 \cdots w_k$ is $\mathrm{Asc}(w) = \{i : w_i < w_{i+1}\}$. The *comajor index* of w is $\mathrm{comaj}(w) = \sum_{i \in \mathrm{Asc}(w)} i$. A *column word* of $T \in \mathcal{F}_\mu$ is a word obtained by reading down a column from top to bottom. The μ-*comajor index* of $T \in \mathcal{F}_\mu$, denoted $\mathrm{comaj}_\mu(T)$, is the sum of the comajor indices of the column words of T. Thus, for T as shown in (4),

$$\mathrm{comaj}_\mu(T) = \mathrm{comaj}(3248) + \mathrm{comaj}(591) + \mathrm{comaj}(67) = 5 + 1 + 1 = 7.$$

Given a triple of cells in a filling $T \in \mathcal{F}_\mu$ as shown:

$$\begin{array}{|c|} \hline a \\ \hline c \\ \hline \end{array} \qquad\qquad \begin{array}{|c|} \hline b \\ \hline \end{array}$$

where the entry a is directly above the entry c and entry b is somewhere to the right of a in the same row as a, the cells form an *inversion triple* if and only if $a < c < b, c < b < a$, or $b < a < c$. This is equivalent to saying that visiting the entries a, b, and c in increasing order induces a counterclockwise traversal of the cells. If the cells containing a and b are in the first row of T, we set $c = \infty$. Then $\mathrm{inv}_\mu(T)$ is the number of inversion triples in T. For T in (4), the inversion triples are $1 < 8 < \infty, 7 < 8 < \infty, 4 < 8 < 9, 1 < 6 < 9$, and $2 < 4 < 5$. Thus $\mathrm{inv}_\mu(T) = 5$.

The combinatorial formula for F_μ is

$$F_\mu(q, t) = \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{comaj}_\mu(T)}. \tag{5}$$

The filling T in (4) contributes the term $q^5 t^7$ to $F_{(3,3,2,1)}$.

In the proof of the recursion, we will fix the bottom row of a filling of shape (3^n), and then use certain combinatorial operations to transform the filling into a "key filling" with certain properties.

3. Combinatorial Operations

In this section we define several combinatorial operations on standard fillings which will be used in the proof of the recursion in §4. For a more complete explanation of the inversion flip and cyclic shift operations, see [1, 8, 10].

Let $\mu \in \mathrm{Par}(n)$ with $\mu'_i = \mu'_{i+1}$ for some i (so columns i and $i+1$ in μ have equal height). We define the *inversion flip* move $s_i : \mathcal{F}_\mu \to \mathcal{F}_\mu$ as follows:

- Given $T \in \mathcal{F}_\mu$, let a (resp. b) be the entry of T at the bottom of column i (resp. $i+1$) and c (resp. d) be the entry of T directly above a (resp. b).
- Switch entries a and b in the bottom row of T as shown here (note that only the bottom two rows of columns i and $i+1$ are shown):

$$\begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline \end{array} \to \begin{array}{|c|c|}\hline c & d \\ \hline b & a \\ \hline \end{array}.$$

- If a, c, d and b, c, d are either both inversion triples or both not inversion triples, the move is complete. Otherwise, apply s_i recursively to the filling obtained by ignoring the bottom row of T.

Note that s_i is an involution and $s_i \circ s_j = s_j \circ s_i$ when $|i - j| \geq 2$. The effect on the statistics of using the inversion flip on a filling is indicated in the following lemma.

Lemma 3.1. *Given $\mu \in \mathrm{Par}(n)$ with $\mu'_i = \mu'_{i+1}$ for some i, and given $T \in \mathcal{F}_\mu$ with entries a and b in the first row of columns i and $i+1$, respectively, then*

(a) $\mathrm{comaj}_\mu(s_i(T)) = \mathrm{comaj}_\mu(T)$, *and*

(b) $\mathrm{inv}_\mu(s_i(T)) = \mathrm{inv}_\mu(T) + \begin{cases} 1 & \text{if } a < b; \\ -1 & \text{if } b < a. \end{cases}$

Proof. See [8, Proposition 3.4]. □

Fig. 1 shows the result of successive inversion flips.

$$\begin{array}{|c|c|c|}\hline 7 & 8 & 2 \\ \hline 4 & 1 & 5 \\ \hline 9 & 3 & 6 \\ \hline \end{array} \xrightarrow{s_2} \begin{array}{|c|c|c|}\hline 7 & 2 & 8 \\ \hline 4 & 5 & 1 \\ \hline 9 & 6 & 3 \\ \hline \end{array} \xrightarrow{s_1} \begin{array}{|c|c|c|}\hline 7 & 2 & 8 \\ \hline 4 & 5 & 1 \\ \hline 6 & 9 & 3 \\ \hline \end{array}$$

$$q^5t^7 \qquad\qquad q^6t^7 \qquad\qquad q^5t^7$$

Figure 1: Example of $s_1 \circ s_2$.

The second combinatorial operation used to prove the recursion is *cyclic shifting*. Given $T \in \mathcal{F}_\mu$, define the *cyclic shift* of T, denoted $\mathrm{cyc}(T)$, to be the filling obtained by replacing each entry c in T with $c + 1 \pmod{n}$ where we use the convention that $a \pmod{n} \in \{1, 2, \ldots, n\}$. Cyclic shifting preserves the inversion statistic and increases the comajor index by 1 when n is not in the bottom row of T. Moreover, the cyclic shift and inversion flip operations commute with each other.

Example 3.2. Let $\mu = (4, 3, 3, 1)$. Then, the result of cyclic shifting can be seen in Fig. 2.

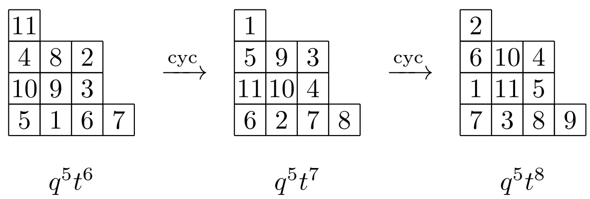


Figure 2: Successive cyclic shifts.

We summarize the above discussion in the following theorems.

Theorem 3.3. *Let $\mu \in \mathrm{Par}(n)$. For $T \in \mathcal{F}_\mu$, where n is not in the bottom row,*

(a) $\mathrm{comaj}_\mu(\mathrm{cyc}(T)) = \mathrm{comaj}_\mu(T) + 1$*;*

(b) $\mathrm{inv}_\mu(\mathrm{cyc}(T)) = \mathrm{inv}_\mu(T)$*.*

Proof. See [8, Proposition 5.3]. □

Theorem 3.4. *Let $\mu \in \mathrm{Par}(n)$ with $\mu'_i = \mu'_{i+1}$ for some i. Then $s_i \circ \mathrm{cyc} = \mathrm{cyc} \circ s_i$.*

Proof. See [8, Lemma 5.5]. □

When $\mu = (3^n)$, consisting of 3 columns of equal height n, and $3n$ is in the first row of T, then $\mathrm{comaj}_\mu(\mathrm{cyc}(T)) = \mathrm{comaj}_\mu(T) - (n-1)$ as seen in the cyclic shift operation in Fig. 3. We can use the inversion flip move to compensate for the change in the inversion statistics, as illustrated in Fig. 3 and made explicit in the following definition.

Definition 3.5. Let $T \in \mathcal{F}_{(3^n)}$ for some $n \geq 2$. The first row in T can be written as $\sigma(\mathbf{a}) = a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}$ where $\sigma \in S_3$ and $\mathbf{a} = (a_1, a_2, a_3)$ with $1 \leq a_1 < a_2 < a_3 \leq 3n$. Then $\mathrm{C} : \mathcal{F}_{(3^n)} \to \mathcal{F}_{(3^n)}$ is defined by

$$\mathrm{C}(T) = \begin{cases} \mathrm{cyc}(T) & \text{if } 3n \text{ is not in row 1 of } T; \\ s_1 \circ s_2 \circ \mathrm{cyc}(T) & \text{if } 3n \text{ is in row 1 and } \sigma = 123, 231, \text{ or } 312; \\ s_2 \circ s_1 \circ \mathrm{cyc}(T) & \text{if } 3n \text{ is in row 1 and } \sigma = 132, 213, \text{ or } 321. \end{cases}$$

Given a statement A, $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ if A is false.

$$
\begin{array}{|c|c|c|}\hline 8&2&4\\\hline 1&5&3\\\hline 9&6&7\\\hline\end{array}
\xrightarrow{\text{cyc}}
\begin{array}{|c|c|c|}\hline 9&3&5\\\hline 2&6&4\\\hline 1&7&8\\\hline\end{array}
\xrightarrow{s_2}
\begin{array}{|c|c|c|}\hline 9&3&5\\\hline 2&6&4\\\hline 1&8&7\\\hline\end{array}
\xrightarrow{s_1}
\begin{array}{|c|c|c|}\hline 9&3&5\\\hline 2&6&4\\\hline 8&1&7\\\hline\end{array}
$$

$$q^5t^7 \qquad q^3t^5 \qquad q^4t^5 \qquad q^5t^5$$

Figure 3: An example of $\mathrm{C}(T)$.

Theorem 3.6. *Let $T \in \mathcal{F}_{(3^n)}$. Let the first row of T be $\sigma(\mathbf{a})$ where $\mathbf{a} = (a_1, a_2, a_3)$ and $\sigma \in S_3$. Then for $0 \le i < 3n$,*

(a) $\mathrm{comaj}_{(3^n)}(\mathrm{C}^i(T)) = \mathrm{comaj}_{(3^n)}(T) + i - n(\chi(i > 3n - a_1) + \chi(i > 3n - a_2) + \chi(i > 3n - a_3))$*;*

(b) $\mathrm{inv}_{(3^n)}(\mathrm{C}^i(T)) = \mathrm{inv}_{(3^n)}(T)$.

Proof. Let $T \in \mathcal{F}_{(3^n)}$ with first row $\sigma(\mathbf{a})$ as above. Since $\mathrm{cyc} \circ s_k = s_k \circ \mathrm{cyc}$ for any k, $\mathrm{C}^i(T) = s \circ \mathrm{cyc}^i(T)$ where s is some sequence of s_1's and s_2's. For $0 < j \le i$, when $3n$ is not in the first row of $\mathrm{cyc}^{j-1}(T)$, $\mathrm{comaj}_{(3^n)}(\mathrm{cyc}^j(T)) = \mathrm{comaj}_{(3^n)}(\mathrm{cyc}^{j-1}(T)) + 1$. If $3n$ is in the first row of $\mathrm{cyc}^{j-1}(T)$ for some j, then $a_1 + (j-1) = 3n$, $a_2 + (j-1) = 3n$, or $a_3 + (j-1) = 3n$ and $\mathrm{comaj}_{(3^n)}(\mathrm{cyc}^j(T)) = \mathrm{comaj}_{(3^n)}(\mathrm{cyc}^{j-1}(T)) - (n-1)$. The sequence s of inversion flips does not affect the $\mathrm{comaj}_{(3^n)}$ statistic, so

$$
\begin{aligned}
\mathrm{comaj}_{(3^n)}(\mathrm{C}^i)(T) = {} & \mathrm{comaj}_{(3^n)}(T) + i \\
& - n(\chi(i > 3n - a_1) + \chi(i > 3n - a_2) + \chi(i > 3n - a_3)).
\end{aligned}
$$

To show that $\mathrm{inv}_{(3^n)}(\mathrm{C}^i(T)) = \mathrm{inv}_{(3^n)}(T)$ by Theorem 3.3(b) it is sufficient to show

(i) $\mathrm{inv}_{(3^n)}(s_1 \circ s_2 \circ \mathrm{cyc})(T) = \mathrm{inv}_{(3^n)}(T)$ when $\sigma = 123, 231$, or 312 and $a_3 = 3n$, and

(ii) $\mathrm{inv}_{(3^n)}(s_2 \circ s_1 \circ \mathrm{cyc})(T) = \mathrm{inv}_{(3^n)}(T)$ when $\sigma = 132, 213$, or 321 and $a_3 = 3n$.

Since $a_3 = 3n$, the first row of $\mathrm{cyc}(T)$ is $\tau(\widetilde{a})$ where $\tau = \tau_1\tau_2\tau_3$ with $\tau_i = \sigma_i + 1 \pmod 3$ and $\widetilde{a} = (1, a_1+1, a_2+1)$. All inversion triples above the first row remain unchanged after cyc, so it is sufficient to show that the inversions in the first rows of the fillings T and $(s \circ \mathrm{cyc})(T)$ are the same, where s is either $s_1 \circ s_2$ or $s_2 \circ s_1$. First consider $s_1 \circ s_2 \circ \mathrm{cyc}(T)$. The first row of $\mathrm{cyc}(T)$ is $\boxed{a'_1}\boxed{a'_2}\boxed{a'_3}$ where $a'_j = (a_{\sigma(j)} + 1) \mod 3n$. Then the first row of $s_1 \circ s_2 \circ \mathrm{cyc}(T)$ is $\boxed{a'_3}\boxed{a'_1}\boxed{a'_2}$. When $\sigma = 123$, the first row of T is $\boxed{a_1}\boxed{a_2}\boxed{a_3}$, which has no inversions, and $\boxed{a'_3}\boxed{a'_1}\boxed{a'_2} = \boxed{1}\boxed{\widetilde{a}_1}\boxed{\widetilde{a}_2}$ where $\widetilde{a}_j = a_j + 1$, so the first row of $s_1 \circ s_2 \circ \mathrm{cyc}(T)$ also has no inversions. When $\sigma = 231$, the first row of T is $\boxed{a_2}\boxed{a_3}\boxed{a_1}$, which has two inversions, and $\boxed{a'_3}\boxed{a'_1}\boxed{a'_2} = \boxed{\widetilde{a}_1}\boxed{\widetilde{a}_2}\boxed{1}$, which also has two inversions.

Finally, when $\sigma = 312$, the first row of T is $\begin{array}{|c|c|c|}\hline a_3 & a_1 & a_2 \\ \hline \end{array}$, which has two inversions, and $\begin{array}{|c|c|c|}\hline a_3' & a_1' & a_2' \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \widetilde{a}_2 & 1 & \widetilde{a}_1 \\ \hline \end{array}$, which also has two inversions. Thus $\mathrm{inv}_{(3^n)}(s_1 \circ s_2 \circ \mathrm{cyc}(T)) = \mathrm{inv}_{(3^n)}(T)$. A similar case analysis of $s_2 \circ s_1 \circ \mathrm{cyc}(T)$ leads to $\mathrm{inv}_{(3^n)}(s_2 \circ s_1 \circ \mathrm{cyc}(T)) = \mathrm{inv}_{(3^n)}(T)$. □

The last operation we need is called **a**-*augmentation*. Given a filling $T \in \mathcal{F}_{(3^{n-1})}$, and a vector $\mathbf{a} = (a_1, a_2, a_3)$ with $1 = a_1 < a_2 < a_3 \leq 3n$, form a new filling $A_{\mathbf{a}}(T) \in \mathcal{F}_{(3^n)}$ by replacing each entry c in T by $c+1$ if $c < a_2 - 1$, $c+2$ if $a_2 - 1 \leq c < a_3 - 2$, and $c+3$ if $c \geq a_3 - 2$, and then placing the result over a new first row $\begin{array}{|c|c|c|}\hline 1 & a_2 & a_3 \\ \hline \end{array}$. Denote the set of fillings with first row $\mathbf{a}$ by $\mathcal{F}_{\mu,\mathbf{a}}$. For example, given

$$T = \begin{array}{|c|c|c|}\hline 4 & 2 & 8 \\ \hline 5 & 1 & 7 \\ \hline 9 & 6 & 3 \\ \hline \end{array}$$

and $\mathbf{a} = (1, 4, 6)$,

$$A_{\mathbf{a}}(T) = \begin{array}{|c|c|c|}\hline 7 & 3 & 11 \\ \hline 8 & 2 & 10 \\ \hline 12 & 9 & 5 \\ \hline 1 & 4 & 6 \\ \hline \end{array}.$$

Note that $\mathrm{inv}_{(3^3)}(T) = 7$, $\mathrm{comaj}_{(3^3)}(T) = 5$, $\mathrm{inv}_{(3^4)}(A_{\mathbf{a}}(T)) = 7$, and $\mathrm{comaj}_{(3^4)}(A_{\mathbf{a}}(T)) = 8$.

Theorem 3.7. *Let $n \in \mathbb{N}$, $T \in \mathcal{F}_{(3^{n-1})}$, and $\mathbf{a} = (a_1, a_2, a_3)$ where $1 = a_1 < a_2 < a_3 \leq 3n$ for $a_2, a_3 \in \mathbb{N}$. Let the first row of T be denoted $\begin{array}{|c|c|c|}\hline b_1 & b_2 & b_3 \\ \hline \end{array}$ and the second row of $A_{\mathbf{a}}(T)$ be denoted $\begin{array}{|c|c|c|}\hline b_1' & b_2' & b_3' \\ \hline \end{array}$. Then*

(a) $b_j' < a_i$ *if and only if* $b_j + (i-1) < a_i$ *for all* $i, j \in \{1, 2, 3\}$,

(b) $\mathrm{inv}_{(3^n)}(A_{\mathbf{a}}(T)) = \mathrm{inv}_{(3^{n-1})}(T) + \chi(b_2 < a_2 - 1 \leq b_3) - \chi(b_3 < a_2 - 1 \leq b_2)$, *and*

(c) $\mathrm{comaj}_{(3^n)}(A_{\mathbf{a}}(T)) = \mathrm{comaj}_{(3^{n-1})}(T) + (n-1)(\chi(b_2 < a_2 - 1) + \chi(b_3 < a_3 - 2))$.

Proof. To prove part (a), first suppose $b_j' > a_i$. Then, if $a_2 < b_j' < a_3$, $b_j' = b_j + 2$ by definition, and if $b_j' > a_3$, $b_j' = b_j + 3$, so $b_j + i > a_i$ if $i = 2$ or $i = 3$. Next, suppose that $b_j + i - 1 \geq a_i$. If $a_2 \leq b_j + i - 1 < a_3$, then $b_j' = b_j + 2 > a_2$. If $a_3 \leq b_j + i - 1$, then $b_j' = b_j + 3 > a_3$.

For parts (b) and (c), we analyze what happens to inversions and ascents when a new first row is added. First, notice that all ascents above the first two rows of $A_{\mathbf{a}}(T)$ are in the same positions as the ascents of T, and that all inversion triples in $A_{\mathbf{a}}(T)$ that don't involve the first two rows are the same as the inversion triples in T above the first row. There is an ascent in $A_{\mathbf{a}}(T)$ between row one and two in column i if and only if $b_i' < a_i$. By part (a), this holds if and only if $b_i + i - 1 < a_i$. Thus $\mathrm{comaj}_{(3^n)}(A_{\mathbf{a}}(T)) = \mathrm{comaj}_{(3^{n-1})}(T) + (n-1)(\chi(b_2 < a_2 - 1) + \chi(b_3 < a_3 - 2))$.

To analyze the number of inversion triples in $A_{\mathbf{a}}(T)$, we need only to consider the triples

$$\begin{array}{|c|c|}\hline b_1' & b_2' \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\cline{1-1}\cline{3-3} b_1' & \multicolumn{1}{c|}{} & b_3' \\ \cline{1-1}\cline{3-3} 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \text{ and } \begin{array}{|c|c|}\hline b_2' & b_3' \\ \hline a_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Note that the first two triples have the same inversion status as $\boxed{b_1}\boxed{b_2}$ and $\boxed{b_1}\ \ \boxed{b_3}$, so the only triple of concern is the third. If $b_2 > b_3$ in T, then $b_2' > b_3'$ in $A_{\mathbf{a}}(T)$. If $a_2 < b_3' < b_2'$ or $b_3' < b_2' < a_2$, then the triple is an inversion triple in $A_{\mathbf{a}}(T)$ and $\operatorname{inv}_{(3^n)}(A_{\mathbf{a}}(T)) = \operatorname{inv}_{(3^{n-1})}(T)$. On the other hand, if $b_3' < a_2 < b_2'$, the triple is not an inversion triple. This occurs if and only if $b_3 < a_2 - 1 \leq b_2$. In this case, $\operatorname{inv}_{(3^n)}(A_{\mathbf{a}}(T)) = \operatorname{inv}_{(3^{n-1})}(T) - 1$. Similarly, if $b_2 < b_3$ in T, $b_2' < b_3'$ in $A_{\mathbf{a}}(T)$. If $a_2 < b_2' < b_3'$ or $b_2' < b_3' < a_2$, the triple is not an inversion triple and $\operatorname{inv}_{(3^n)}(A_{\mathbf{a}}(T)) = \operatorname{inv}_{(3^{n-1})}(T)$. If $b_2' < a_2 < b_3'$, the triple is an inversion triple, $b_2 < a_2 - 1 \leq b_3$, and $\operatorname{inv}_{(3^n)}(A_{\mathbf{a}}(T)) = \operatorname{inv}_{(3^{n-1})}(T) + 1$. □

We combine **a**-augmentation with cyclic shifting by defining a new operation.

Definition 3.8. For $n \in \mathbb{N}$, define $\overline{\text{cyc}} : \mathcal{F}_{(3^n),\mathbf{a}} \to \mathcal{F}_{(3^n),\mathbf{a}}$ by $\overline{\text{cyc}}(T) = A_{\mathbf{a}} \circ \text{cyc} \circ A_{\mathbf{a}}^{-1}(T)$. Similarly, define $\overline{\text{s}}_i = A_{\mathbf{a}} \circ s_i \circ A_{\mathbf{a}}^{-1}$ for $i = 1, 2$.

Theorem 3.9. *Let $n \in \mathbb{N}$ with $n \geq 2$, $T \in \mathcal{F}_{(3^{n-1})}$, and $\mathbf{a} = (a_1, a_2, a_3)$ where $1 = a_1 < a_2 < a_3 \leq 3n$. The first row of T can be denoted* $\boxed{b_{\sigma(1)}}\boxed{b_{\sigma(2)}}\boxed{b_{\sigma(3)}}$ *where $1 \leq b_1 < b_2 < b_3 \leq 3(n-1)$ and $\sigma \in S_3$. Then for $0 \leq j \leq 3(n-1) - b_3$,*

(a) $$\operatorname{comaj}_{(3^n)}(\overline{\text{cyc}}^j(A_{\mathbf{a}}(T))) = \operatorname{comaj}_{(3^{n-1})}(T) + j + (n-1)(\chi(j < a_2 - b_{\sigma(2)} - 1) + \chi(j < a_3 - b_{\sigma(3)} - 2));$$

(b) $$\operatorname{inv}_{(3^n)}(\overline{\text{cyc}}^j(A_{\mathbf{a}}(T))) = \operatorname{inv}_{(3^{n-1})}(T) + \chi(b_{\sigma(2)} + j < a_2 - 1 \leq b_{\sigma(3)} + j) - \chi(b_{\sigma(3)} + j < a_2 - 1 \leq b_{\sigma(2)} + j).$$

Proof. Let $n \in \mathbb{N}$ with $n \geq 2$ and $T \in \mathcal{F}_{(3^{n-1})}$ with first row $\boxed{b_{\sigma(1)}}\boxed{b_{\sigma(2)}}\boxed{b_{\sigma(3)}}$ as above. Let $\mathbf{a} = (a_1, a_2, a_3)$ with $1 = a_1 < a_2 < a_3 \leq 3n$. Then, for $0 \leq j \leq 3(n-1) - b_3$,

$$\begin{aligned}
\operatorname{comaj}_{(3^n)}(\overline{\text{cyc}}^j(A_{\mathbf{a}}(T))) &= \operatorname{comaj}_{(3^n)}(A_{\mathbf{a}}(\text{cyc}^j(T))) \\
&= \operatorname{comaj}_{(3^{n-1})}(\text{cyc}^j(T)) + (n-1)(\chi(b_{\sigma(2)} + j < a_2 - 1) \\
&\qquad + \chi(b_{\sigma(3)} + j < a_3 - 2)) \\
&\qquad\qquad \text{by Theorem 3.7} \\
&= \operatorname{comaj}_{(3^{n-1})}(T) + j + (n-1)(\chi(j < a_2 - b_{\sigma(2)} - 1) \\
&\qquad + \chi(j < a_3 - b_{\sigma(3)} - 2)) \\
&\qquad\qquad \text{by Theorem 3.3.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{inv}_{(3^n)}(\overline{\mathrm{cyc}}^j(A_{\mathbf{a}}(T))) &= \mathrm{inv}_{(3^n)}(A_{\mathbf{a}}(\mathrm{cyc}^j(T))) \\
&= \mathrm{inv}_{(3^{n-1})}(\mathrm{cyc}^j(T)) + \chi(b_{\sigma(2)} + j < a_2 - 1 \leq b_{\sigma(3)} + j) \\
&\qquad - \chi(b_{\sigma(3)} + j < a_2 - 1 \leq b_{\sigma(2)} + j) \\
&\qquad\qquad \text{by Theorem 3.7} \\
&= \mathrm{inv}_{(3^{n-1})}(T) + \chi(b_{\sigma(2)} + j < a_2 - 1 \leq b_{\sigma(3)} + j) \\
&\qquad - \chi(b_{\sigma(3)} + j < a_2 - 1 \leq b_{\sigma(2)} + j) \\
&\qquad\qquad \text{by Theorems 3.1 and 3.3.} \quad \square
\end{aligned}$$

We would also like an augmented version of the operation C in Definition 3.5 that increases the comajor index without changing the inversion statistic. It is not enough to simply use the composition $A_{\mathbf{a}} \circ C \circ A_{\mathbf{a}}^{-1}$, because the triple

$b_{\sigma(2)}$	$b_{\sigma(3)}$
a_2	

creates problems if $b_{\sigma(2)}$ or $b_{\sigma(3)}$ are equal to $a_2 - 1$.

Definition 3.10. For $n \geq 2$ and $\mathbf{a} = (1, a_2, a_3)$ with $1 < a_2 < a_3 \leq 3n$, we define $\overline{\mathrm{C}} : \mathcal{F}_{(3^n),\mathbf{a}} \to \mathcal{F}_{(3^n),\mathbf{a}}$ as follows. Let $U \in \mathcal{F}_{(3^n),\mathbf{a}}$. Denote the bottom row of $A_{\mathbf{a}}^{-1}(U)$ by $\sigma(\mathbf{b}) = (b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)})$ where $\mathbf{b} = (b_1, b_2, b_3)$ with $b_1 < b_2 < b_3$ and $\sigma \in S_3$. Then, when $a_2 \neq 2$ and $a_2 \neq 3n - 1$,

$$\overline{\mathrm{C}}(U) = \begin{cases}
\overline{\mathrm{cyc}}(U) & \text{if } b_3 \neq 3n-3 \text{ and not C1} \\
\overline{\mathrm{s}}_2 \circ \overline{\mathrm{cyc}}(U) & \text{if } b_3 \neq 3n-3 \text{ and C1} \\
\overline{\mathrm{s}}_1 \circ \overline{\mathrm{s}}_2 \circ \overline{\mathrm{cyc}}(U) & \text{if } b_3 = 3n-3, \sigma = 123, 231, 312, \text{ and not C2} \\
\overline{\mathrm{s}}_2 \circ \overline{\mathrm{s}}_1 \circ \overline{\mathrm{s}}_2 \circ \overline{\mathrm{cyc}}(U) & \text{if } b_3 = 3n-3, \sigma = 123, 231, 312, \text{ and C2} \\
\overline{\mathrm{s}}_2 \circ \overline{\mathrm{s}}_1 \circ \overline{\mathrm{cyc}}(U) & \text{if } b_3 = 3n-3, \sigma = 132, 213, 321, \text{ and not C3} \\
\overline{\mathrm{s}}_1 \circ \overline{\mathrm{cyc}}(U) & \text{if } b_3 = 3n-3, \sigma = 132, 213, 321, \text{ and C3}
\end{cases}$$

where

(C1) $\quad b_{\sigma(2)} = a_2 - 2$ or $b_{\sigma(3)} = a_2 - 2$;

(C2)
- (1) $\sigma = 123$, $\quad b_1 < a_2 - 2$ and $b_2 \neq a_2 - 2$;
- (2) $\sigma = 231$, $\quad b_2 < a_2 - 2$ or $b_1 \geq a_2 - 1$;
- (3) $\sigma = 312$, $\quad b_1 \neq a_2 - 2$ and $b_2 \geq a_2 - 1$;

(C3)
- (1) $\sigma = 132$, $\quad b_2 < a_2 - 2$ or $b_1 < a_2 - 2 < b_2$;
- (2) $\sigma = 213$, $\quad b_1 \geq a_2 - 1$ or $b_2 < a_2 - 2$;
- (3) $\sigma = 321$, $\quad b_1 \neq a_2 - 2$ and $b_2 \geq a_2 - 1$.

If $a_2 = 2$ or $a_2 = 3n - 1$, then $\overline{\mathrm{C}}(U) = A_{\mathbf{a}} \circ \mathrm{C} \circ A_{\mathbf{a}}^{-1}(U)$.

To show that $\overline{\mathrm{C}}$ is a bijection, it is sufficient to define an inverse function $\overline{\mathrm{C}}^{-1}$.

Definition 3.11. For $n \geq 2$ and $\mathbf{a} = (1, a_2, a_3)$ with $1 < a_2 < a_3 \leq 3n$, define $\overline{\mathrm{C}}^{-1} : \mathcal{F}_{(3^n),\mathbf{a}} \to \mathcal{F}_{(3^n),\mathbf{a}}$. Let $U \in \mathcal{F}_{(3^n),\mathbf{a}}$. Denote the bottom row of $A_{\mathbf{a}}^{-1}(U)$ by $\sigma(\mathbf{b}) = (b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)})$ where $\mathbf{b} = (b_1, b_2, b_3)$ with $b_1 < b_2 < b_3$ and $\sigma \in S_3$. Then, when $a_2 \neq 2$ and $a_2 \neq 3n-1$,

$$\overline{\mathrm{C}}^{-1}(U) = \begin{cases} \overline{\mathrm{cyc}}^{-1}(U) & \text{if } b_1 \neq 1 \text{ and not I1;} \\ \overline{\mathrm{cyc}}^{-1} \circ \overline{\mathrm{s}}_2(U) & \text{if } b_1 \neq 1 \text{ and I1;} \\ \overline{\mathrm{cyc}}^{-1} \circ \overline{\mathrm{s}}_2 \circ \overline{\mathrm{s}}_1(U) & \text{if } \sigma = 123, 231, \text{ or } 312, \text{ and not I2;} \\ \overline{\mathrm{cyc}}^{-1} \circ \overline{\mathrm{s}}_2 \circ \overline{\mathrm{s}}_1 \circ \overline{\mathrm{s}}_2(U) & \text{if } \sigma = 132, 213, \text{ or } 321, \text{ and I2;} \\ \overline{\mathrm{cyc}}^{-1} \circ \overline{\mathrm{s}}_1 \circ \overline{\mathrm{s}}_2(U) & \text{if } \sigma = 132, 213, \text{ or } 321, \text{ and not I3;} \\ \overline{\mathrm{cyc}}^{-1} \circ \overline{\mathrm{s}}_1(U) & \text{if } \sigma = 123, 231, \text{ or } 312, \text{ and I3;} \end{cases}$$

where

(I1) $b_{\sigma(2)} = a_1 - 1$ or $b_{\sigma(3)} = a_2 - 1$;

(I2)
- (1) $\sigma_1 = 1$, $b_2 < a_2 - 1$ and $b_3 \neq a_2 - 1$;
- (2) $\sigma_1 = 2$, $b_3 < a_2 - 1$ or $b_2 \geq a_2$;
- (3) $\sigma_1 = 3$, $b_2 \neq a_2 - 1$ and $b_3 \geq a_2$;

(I3)
- (1) $\sigma_1 = 1$, $b_3 < a_2 - 1$ or $b_2 < a_2 - 1 < b_3$;
- (2) $\sigma_1 = 2$, $b_2 \geq a_2$ or $b_3 < a_2 - 1$;
- (3) $\sigma_1 = 3$, $b_2 \neq a_2 - 1$ and $b_3 \geq a_2$.

If $a_2 = 2$ or $a_2 = 3n-1$, then $\overline{\mathrm{C}}^{-1}(U) = A_{\mathbf{a}} \circ \mathrm{C}^{-1} \circ A_{\mathbf{a}}^{-1}(U)$.

One can check each case to see that $\overline{\mathrm{C}}^{-1} \circ \overline{\mathrm{C}} = \mathrm{id}_{\mathcal{F}_{(3^n),\mathbf{a}}}$ and $\overline{\mathrm{C}} \circ \overline{\mathrm{C}}^{-1} = \mathrm{id}_{\mathcal{F}_{(3^n),\mathbf{a}}}$.

Theorem 3.12. *For $\overline{\mathrm{C}}$ as in Definition 3.10 and $T \in \mathcal{F}_{(3^n),\mathbf{a}}$,*

(a) $\mathrm{comaj}_{(3^n)}(\overline{\mathrm{C}}(T)) = \mathrm{comaj}_{(3^n)}(T) + \begin{cases} 1 \\ -(n-2) \end{cases}$ *and*

(b) $\mathrm{inv}_{(3^n)}(\overline{\mathrm{C}}(T)) = \mathrm{inv}_{(3^n)}(T)$.

Proof. Note that (b) follows directly from the construction of $\overline{\mathrm{C}}$. To prove (a), let $T \in \mathcal{F}_{(3^n),\mathbf{a}}$ and consider two cases: $b_3 \neq 3(n-1)$ and $b_3 = 3(n-1)$, where b_3 is the largest entry in the bottom row of $A^{-1}(T)$, as in Definition 3.10. We use the labels shown in Fig. 4 for the entries in the second rows of T and $\overline{\mathrm{C}}(T)$.

We first suppose $b_3 \neq 3(n-1)$. Note that $\overline{\mathrm{C}}$ sends x to x' in this case and $\mathrm{comaj}_{(3^{n-1})}(A^{-1}(\overline{\mathrm{C}}(T))) = \mathrm{comaj}_{(3^{(n-1)})}(A^{-1}(T)) + 1$ by the definition of $\overline{\mathrm{C}}$. It remains to show that $\overline{\mathrm{C}}(T)$ cannot have more ascents between the bottom two rows than T, and that T has at most one more ascent between the bottom two rows than $\overline{\mathrm{C}}(T)$.

Suppose first that T has no ascents between the bottom two rows. Then $y > a_2$ and $z > a_3$ in T and $y \mapsto y'$ and $z \mapsto z'$ in $\overline{\mathrm{C}}(T)$, so $\overline{\mathrm{C}}(T)$ also has no ascents between the bottom two rows.

Now consider T with exactly one ascent between the bottom two rows. Suppose that $y > a_2$ but $z < a_3$. If $z > y$ or $a_2 < z < y$, then $y \mapsto y'$ and $z \mapsto z'$ since $\mathrm{inv}_{(3^n)}(T) = \mathrm{inv}_{(3^n)}(\overline{\mathrm{C}}(T))$. The only way to lose an ascent is if $z = a_3 - 1$. There is no way for $y' < a_2$ since $y' > y > a_2$. If $y > a_2$, but $z < a_2$, then $y \mapsto y'$ and $z \mapsto z'$ provided $z \neq a_2 - 1$, and the number of ascents between the bottom two rows of T is the same as in $\overline{\mathrm{C}}(T)$. However, if $z = a_2 - 1$, then $z < a_2 < y$ is not an inversion triple in T, and to preserve the number of inversion triples between T and $\overline{\mathrm{C}}(T)$, $z \mapsto y'$ and $y \mapsto z'$ in $\overline{\mathrm{C}}(T)$. Note that if $y \geq a_3 - 1$, then there is one more ascent between the bottom rows of T than in $\overline{\mathrm{C}}(T)$, but if $a_2 < y < a_3 - 1$, T and $\overline{\mathrm{C}}(T)$ have the same number of ascents between the bottom two rows. One can do a similar analysis of the case where $y < a_2$ and $z > a_3$.

If T has two ascents between the bottom two rows, we have $y < a_2$ and $z < a_3$. Note that if $y < a_2 - 1$ and $z \neq a_2 - 1$, then $y \mapsto y'$ and $z \mapsto z'$ in $\overline{\mathrm{C}}(T)$. Then, if $z = a_3 - 1$, T has one more ascent between the bottom two rows than $\overline{\mathrm{C}}(T)$. Otherwise, they have the same number of ascents. On the other hand, if either $y = a_2 - 1$ or $z = a_2 - 1$, then $y \mapsto z'$ and $z \mapsto y'$ in $\overline{\mathrm{C}}(T)$ in order to preserve the number of inversions. This immediately leads to T with either the same number of ascents or exactly one more ascent between the bottom two rows than $\overline{\mathrm{C}}(T)$.

Now consider what happens if $b_3 = 3(n-1)$. Then it follows, by the definition of $\overline{\mathrm{C}}$ and properties of the cyclic shift move, that $\mathrm{comaj}_{(3^{(n-1)})}(A^{-1}(\overline{\mathrm{C}}(T))) = \mathrm{comaj}_{(3^{(n-1)})}(A^{-1}(T)) - (n-2)$. To show that (a) holds, we show that either the number of ascents between the bottom two rows is preserved by applying $\overline{\mathrm{C}}$, leading to $\mathrm{comaj}_{(3^n)}(\overline{\mathrm{C}}(T)) = \mathrm{comaj}_{(3^n)}(T) - (n-2)$, or that exactly one ascent is introduced between the bottom two rows, leading to $\mathrm{comaj}_{(3^n)}(\overline{\mathrm{C}}(T)) = \mathrm{comaj}_{(3^n)}(T) - (n-2) + (n-1) = \mathrm{comaj}_{(3^n)}(T) + 1$.

Suppose that T has no ascents between the bottom two rows. Then $y > a_2$ and $z > a_3$ and we must show that the bottom two rows of $\overline{\mathrm{C}}(T)$ do not have two ascents. If $a_2 = 2$, there is no possible way for $\overline{\mathrm{C}}(T)$ to have more than one ascent between the bottom two rows, so we assume $a_2 \neq 2$. If $x = 3n$ and $y < z$ in T, then $z \mapsto x'$ in $\overline{\mathrm{C}}(T)$. Since $a_2 < y < z$ is not an inversion triple in T, we must have $z' = 2$ and $y \mapsto y'$ in $\overline{\mathrm{C}}(T)$. Thus, $y' > y > a_2$, so $\overline{\mathrm{C}}(T)$ has only one more ascent between the bottom two rows than T has. If instead, $y > z$ in T, then $z \mapsto y'$ or $z \mapsto z'$ in $\overline{\mathrm{C}}(T)$, so $\overline{\mathrm{C}}(T)$ has at most one more ascent between the bottom two rows than T. Similarly, if $y = 3n$ and $z > x$ in T, then $z \mapsto y'$ or $z \mapsto z'$, so again, $\overline{\mathrm{C}}(T)$ has at most one more ascent between the bottom two rows than T has. Likewise, if $x > z$, then $x \mapsto y'$ or $x \mapsto z'$ and since $x > z > a_3$, there is at most one more ascent in $\overline{\mathrm{C}}(T)$ than in T. The case where $z = 3n$ is similar.

$$T = \begin{array}{|c|c|c|}\hline x & y & z \\ \hline 1 & a_2 & a_3 \\ \hline \end{array} \qquad\qquad \overline{\mathrm{C}}(T) = \begin{array}{|c|c|c|}\hline x' & y' & z' \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$$

Figure 4: The bottom two rows of T and $\overline{\mathrm{C}}(T)$

Suppose that T has exactly one ascent between the bottom two rows. Then either $y > a_2$ and $z < a_3$ or $y < a_2$ and $z > a_3$. We first consider $y > a_2$ and $z < a_3$ in T. Note first that if $a_3 = 3n$, $z' < a_3$ in $\overline{\mathrm{C}}(T)$, so no ascents can be lost as a result of applying $\overline{\mathrm{C}}$ to T. Thus, we assume that $a_3 \neq 3n$. Then either $x = 3n$ or $y = 3n$. Suppose $x = 3n$. If $a_2 = 2$, then either $y' = 3$ or $z' = 3$. If $z' = 3$, the number of ascents between the bottom two rows of T and $\overline{\mathrm{C}}(T)$ are the same. If $y' = 3$, then $z > y$ in T, so $z \mapsto x'$ in $\overline{\mathrm{C}}(T)$ and thus $y \mapsto z'$ in $\overline{\mathrm{C}}(T)$, so $z' < a_3$. Once again, the number of ascents is preserved. If $a_2 \neq 2$, then either $y' = 2$ or $z' = 2$, and again, at most one ascent is gained, and none are lost between the first two rows as a result of applying $\overline{\mathrm{C}}$ to T. Suppose then that $y = 3n$ and $x < z$. Then $x' = 2$ (or 3 if $a_2 = 2$) and $x \mapsto z'$ or $x \mapsto y'$ in $\overline{\mathrm{C}}(T)$. If $x \mapsto z'$, then $z' < a_3$ since $x < z < a_3$. If $x \mapsto y'$, then $z \mapsto z'$ and, since the number of inversions are preserved, $z > a_2$ and thus $y' < a_2$. A similar argument holds for the case where $y < a_2$ and $z > a_3$.

Finally, suppose $y < a_2$ and $z < a_3$. We must show then that $y' < a_2$ and $z' < a_3$ in $\overline{\mathrm{C}}(T)$. Note that $a_2 \neq 2$ in this case. If $a_3 \neq 3n$, then $x = 3n$ in T, so one of y' or z' must be 2 in $\overline{\mathrm{C}}(T)$. If $y' = 2$ and $y < z$, then $y \mapsto z'$, so $z' < a_3$ also. On the other hand, if $y' = 2$ and $y > z$, then $z < a_2$ and $z \mapsto z'$, so $z' < a_3$. Similarly, if $z' = 2$ and $y < z$, then $y \mapsto y'$. If $y < a_2 - 1$, then $y' < a_2$. Note that if $y = a_2 - 1$, then $z > a_2 > y$ forms an inversion triple in T, so y must map to z'. If instead, $z' = 2$ and $z < y$, then $z \mapsto y'$, so $y' < a_2$. Thus, T and $\overline{\mathrm{C}}(T)$ have the same number of ascents between the first two rows. If $a_3 = 3n$, then $z' < a_3$ and it remains to show that $y' < a_2$. If $a_2 = a_3 - 1$, the result follows immediately. If $x = 3n - 1$ in T, the argument is similar to the previous case. We consider the case where $z = 3n - 1$. If $x < y$ in T, then $x' = 2$ in $\overline{\mathrm{C}}(T)$. Then, if $y < a_2 - 1$, $x \mapsto z'$ in $\overline{\mathrm{C}}(T)$ in order to preserve inversions. Thus $y \mapsto y'$ and $y' < a_2$. If $y = a_2 - 1$, then $x \mapsto y'$ and $y' < a_2$. Thus, if $x < y$, T and $\overline{\mathrm{C}}(T)$ have the same number of ascents between the first two rows. We can analyze the case where $x > y$ similarly. Thus, ascents present between the first two rows of T are present between the first two rows of $\overline{\mathrm{C}}(T)$. This leads immediately to property (a). $\square$

From the case analysis in the proof of Theorem 3.12 and the definition of $\overline{\mathrm{C}}$, we can identify fillings T with the property that $\mathrm{comaj}_{(3^n)}(\overline{\mathrm{C}}^{-1}(T)) = \mathrm{comaj}_{(3^n)}(T) + (n-2)$. We call fillings with this property *key fillings.* There are six basic types of key fillings that are displayed in Table 1. In the table, X, Y, Z are the entries in the first row of the filling $A^{-1}(Ak)$ with X', Y', Z' the corresponding entries in Ak for each k.

Table 1: Key fillings.

Filling	Conditions
$A1 = \begin{array}{\|c\|c\|c\|}\hline \vdots & \vdots & \vdots \\ \hline X' & Y' & a_3+1 \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$	1. $X' \neq 2, Y \neq a_2 - 1$, $a_2 \neq 3n - 1$ 2. $X' \neq 2, Y > a_3 - 2, a_2 = a_3 - 1$
$A2 = \begin{array}{\|c\|c\|c\|}\hline \vdots & \vdots & \vdots \\ \hline X' & a_2+1 & Z' \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$	1. $X' \neq 2, a_2 \neq 2, Z < a_2 - 1$ 2. $X' \neq 2, a_2 \neq 2, Z \geq a_3 - 2$
$A3 = \begin{array}{\|c\|c\|c\|}\hline \vdots & \vdots & \vdots \\ \hline X' & Y' & a_2+1 \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$	$X' \neq 2, a_2 \neq 2, a_2 - 1 < Y < a_3 - 1$
$A4 = \begin{array}{\|c\|c\|c\|}\hline \vdots & \vdots & \vdots \\ \hline \boldsymbol{X'} & Y' & Z' \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$	1. $a_2 = 3n - 1, X = 1$ 2. $1 = X < Y < Z$, $Y \geq a_2 - 1$ and $Z \geq a_3 - 2$, or $Y \geq a_2 - 1$ and $a_3 = 3n$, or $Z = a_2 - 1 = a_3 - 2$, or $Z = a_2 - 1$ and $a_3 = 3n$ 3. $1 = X < Z < Y$, $a_3 = 3n$ and $Z < a_2 - 1$ and $Y \neq a_2 - 1$, or $Y = a_2 - 1$, or $Z \geq a_2 - 1$ and $Y < a_3 - 1$, or $Z \geq a_3 - 2$
$A5 = \begin{array}{\|c\|c\|c\|}\hline \vdots & \vdots & \vdots \\ \hline X' & Y' & \boldsymbol{Z'} \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$	1. $a_2 = 3n - 1, Z = 1$ 2. $1 = Z < X < Y$, $Y \geq a_2 - 1$ and $X < a_2$, or $X \geq a_2$ and $a_3 = 3n$, or $Y < a_2 - 1$ and $a_3 = 3n$, or $a_2 = 2$ and $X < a_3 - 1$, or $a_3 = 3$. 3. $1 = Z < Y < X$, $Y \neq a_2 - 1$ and $X < a_3 - 1$, or $X \leq a_2 - 1$, or $Y = a_2 - 1$
$A6 = \begin{array}{\|c\|c\|c\|}\hline \vdots & \vdots & \vdots \\ \hline X' & \boldsymbol{Y'} & Z' \\ \hline 1 & a_2 & a_3 \\ \hline \end{array}$	1. $a_2 = 3n - 1, Y = 1$ 2. $1 = Y < Z < X$, $Z = a_2 - 1 = a_3 - 2$, or $Z = a_2 - 1$ or $X \leq a_3 - 2$, or $X < a_2$, or $a_2 \neq 2$ and $X \geq a_2$ and $Z \neq a_2 - 1$ and $Z = a_3 - 2$, or $a_2 = 2$ and $X \leq a_3 - 2$ 3. $1 = Y < X < Z$, $a_2 \neq 2$ and $X \leq a_3 - 2 \leq Z$, or $X < a_2$, $Z \geq a_2 - 1$ and $a_3 = 3n$, or $a_2 = 2$ and $Z \geq a_3 - 2$, or $a_2 = 2$ and $a_3 = 3n$

4. Three-column Recursion

We first define a new polynomial for $m, n \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ with $1 = a_1 < a_2 < \cdots < a_m \leq mn$ by setting

$$R_{(m^n),\mathbf{a}}(q,t) = \sum_{T \in \mathcal{F}_{(m^n),\mathbf{a}}} q^{\mathrm{inv}_{(m^n)}(T)} t^{\mathrm{comaj}_{(m^n)}(T)}. \tag{6}$$

Note that $R_{(m^{n+1}),(1,2,\ldots,m)}(q,t) = F_{(m^n)}(q,t)$. We now show that, when $m = 3$, a factor of $[n-1]_t$ can be extracted from (6).

Lemma 4.1. *For $n \geq 2$ and $\mathbf{a} = (a_1, a_2, a_3)$ with $1 = a_1 < a_2 < a_3 \leq 3n$, let $\mathcal{T}_{(3^n),\mathbf{a}}$ be the set of key fillings. Then*

$$R_{(3^n),\mathbf{a}} = [n-1]_t \sum_{T \in \mathcal{T}_{(3^n),\mathbf{a}}} f_T(q,t) \tag{7}$$

where $f_T(q,t)$ is some polynomial determined by T.

Proof. We will prove Lemma 4.1 by using $\overline{\mathrm{C}}$ to decompose the set of fillings, $\mathcal{F}_{(3^n),\mathbf{a}}$, into cycles with a constant number of inversions. After these cycles are identified, we group fillings in a given cycle into sets of $n-1$ fillings identified by the key fillings from Table 1.

Let $A \in \mathcal{F}_{(3^n),\mathbf{a}}$. Since $\overline{\mathrm{cyc}}^{3(n-1)}(A) = A$, there exists some positive integer k such that $\overline{\mathrm{C}}^{3(n-1)k}(A) = A$. Note that k might not be equal to 1 since $\bar{\mathrm{s}}_i \circ \bar{\mathrm{s}}_j \circ \bar{\mathrm{s}}_i \neq \bar{\mathrm{s}}_j \circ \bar{\mathrm{s}}_i \circ \bar{\mathrm{s}}_j$ in general. In the set $\{B_i \in \mathcal{F}_{(3^n),\mathbf{a}} : B_i = \overline{\mathrm{C}}^i(A)$ for $0 \leq i \leq 3(n-1)k - 1\}$, there must be $3k$ values of i for which $\mathrm{comaj}_{(3^n)}(B_i) - \mathrm{comaj}_{(3^n)}(B_{i-1}) = -(n-2))$, and $3k(n-2)$ values of i for which $\mathrm{comaj}_{(3^n)}(B_i) - \mathrm{comaj}_{(3^n)}(B_{i-1}) = 1$.

For each i with $1 \leq i \leq 3(n-1)k - 1$, represent B_i with a left parenthesis if $\mathrm{comaj}_{(3^n)}(B_i) = \mathrm{comaj}_{(3^n)}(B_{i-1}) + 1$ and with $n-2$ right parentheses if $\mathrm{comaj}_{(3^n)}(B_i) = \mathrm{comaj}_{(3^n)}(B_{i-1}) - (n-2)$. For $i = 0$, represent B_0 with a left parenthesis if $\mathrm{comaj}_{(3^n)}(B_0) = \mathrm{comaj}_{(3^n)}(B_{3(n-1)k-1}) + 1$, and by $n-2$ right parentheses if $\mathrm{comaj}_{(3^n)}(B_0) = \mathrm{comaj}_{(3^n)}(B_{3(n-1)k-1}) - (n-2)$. Note that this means fillings represented with $n-2$ right parentheses are key fillings. Create a word by writing these parentheses in sequence starting with the parenthesis representing B_0. Since there are the same number of left and right parentheses, this word can be cyclically shifted to form a *Dyck word* with balanced parentheses[11, §5.3]. Given a pair of matched parentheses, let m be the number of right parentheses between the matched parentheses. Then the difference in comajor index between the fillings represented by the matched parentheses is given by

$$\Delta\,\mathrm{comaj}_{(3^n)} = -[(n-2) - m \pmod{n-2}].$$

This allows us to decompose the set $\{B_0, B_1, \ldots, B_{3(n-1)k-1}\}$ into $3k$ disjoint sets $\{S_{j_1}, S_{j_2}, \ldots, S_{j_{(n-1)}}\}$, $1 \leq j \leq 3k$ with the property that

$$\mathrm{comaj}_{(3^n)}(S_{j_{i+1}}) = \mathrm{comaj}_{(3^n)}(S_{j_i}) + 1$$

by letting each S_{j_1} be a filling denoted by right parentheses, and the remaining S_{j_i} be the fillings denoted by left parentheses that are matched with S_{j_1}, labeled from left to right. Then, for each j,

$$\sum_{i=1}^{n-1} q^{\mathrm{inv}_{(3^n)}(S_{j_i})} t^{\mathrm{comaj}_{(3^n)}(S_{j_i})} = [n-1]_t q^{\mathrm{inv}_{(3^n)}(S_{j_1})} t^{\mathrm{comaj}_{(3^n)}(S_{j_1})}. \qquad \square$$

We illustrate the method of proof of Lemma 4.1 with the fillings shown in Fig. 5. This cycle gives the Dyck word

$$\begin{matrix} ((&) &) & ((&) & ((&) &) &) \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{matrix}.$$

Matching balanced parentheses gives three sets of fillings $\{B_0, B_1, B_2\}$, $\{B_3, B_4, B_8\}$, and $\{B_5, B_6, B_7\}$. The factor of $[3]_t$ can be seen in the sum of the monomials $q^{\mathrm{inv}_{(3^4)}(T)} t^{\mathrm{comaj}_{(3^4)}(T)}$ for the fillings in each of these three sets.

We rewrite the recursion in terms of the first row of the unaugmented fillings by determining how many times the operation $\overline{\mathrm{C}}$ must be applied to obtain a key filling from Table 1. First, notice that there is not a unique way to express this recursion, as seen in Theorem 4.2.

Theorem 4.2. *For all $n, k, \mathbf{a}$ where $n \in \mathbb{N}$, $k \in \{1, 2\}$ and $\mathbf{a} = (a_1, a_2, a_3)$ with $1 = a_1 < a_2 < a_3 \le 3n$, let $\mathbf{a}^{(k)} = \mathrm{cyc}^{3n-a_{4-k}+1}(\mathbf{a})$ with the entries sorted in increasing order. Then*

$$R_{(3^n),\mathbf{a}^{(k)}}(q,t) = t^{(3-k)n-a_{4-k}+1} R_{(3^n),\mathbf{a}}(q,t).$$

Proof. Let $T \in \mathcal{F}_{(3^n),\mathbf{a}}$. Note that $\mathrm{C}^{3n-a_{4-k}+1}$ is a bijection from $\mathcal{F}_{(3^n),\mathbf{a}}$ to $\mathcal{F}_{(3^n),\mathbf{a}^{(k)}}$. Then, by Theorem 3.6,

$$\begin{aligned} \mathrm{inv}_{(3^n)}(\mathrm{C}^{3n-a_{4-k}+1}(T)) &= \mathrm{inv}_{(3^n)}(T) \text{ and} \\ \mathrm{comaj}_{(3^n)}(\mathrm{C}^{3n-a_{4-k}+1}(T)) &= \mathrm{comaj}_{(3^n)}(T) + 3n - a_{4-k} + 1 \\ &\qquad - n\sum_{j=1}^{3} \chi(a_j + 3n - a_{4-k} + 1 > 3n) \\ &= \mathrm{comaj}_{(3^n)}(T) + 3n - a_{4-k} + 1 - n(k) \\ &= \mathrm{comaj}_{(3^n)}(T) + (3-k)n - a_{4-k} + 1. \qquad \square \end{aligned}$$

Let $x^+ = \max(0, x)$.

Theorem 4.3. *For $\mu = (3^n)$ and $\mathbf{a} = (a_1, a_2, a_3)$ with $1 = a_1 < a_2 < a_3 \le 3n$,*

$$R_{(3^n),\mathbf{a}} = [n-1]_t \sum_{\substack{\mathbf{b}=(b_1,b_2,b_3) \text{ with} \\ 1=b_1<b_2<b_3\le 3(n-1)}} R_{(3^{n-1}),\mathbf{b}} (t^{E_1} + qt^{E_2} + qt^{E_3} + q^2t^{E_4} + q^2t^{E_5} + q^3t^{E_6})$$

((

B_0

8	3	2
6	7	11
12	10	5
1	4	9

q^5t^9

$\xrightarrow{\overline{\mathrm{C}}}$

)

B_1

10	5	3
7	8	12
11	2	6
1	4	9

q^5t^{10}

$\xrightarrow{\overline{\mathrm{C}}}$

)

B_2

11	6	5
8	10	2
12	3	7
1	4	9

q^5t^{11}

$\xrightarrow{\overline{\mathrm{C}}}$

((

B_3

11	7	6
10	12	3
8	2	5
1	4	9

q^5t^9

$\xrightarrow{\overline{\mathrm{C}}}$

)

B_4

12	8	7
11	2	5
10	3	6
1	4	9

q^5t^{10}

$\xrightarrow{\overline{\mathrm{C}}}$

((

B_5

2	10	8
12	6	3
11	7	5
1	4	9

q^5t^8

$\xrightarrow{\overline{\mathrm{C}}}$

)

B_6

3	11	10
2	7	5
12	8	6
1	4	9

q^5t^9

$\xrightarrow{\overline{\mathrm{C}}}$

)

B_7

5	12	11
3	8	6
10	2	7
1	4	9

q^5t^{10}

$\xrightarrow{\overline{\mathrm{C}}}$

)

B_8

6	2	12
5	10	7
11	3	8
1	4	9

q^5t^{11}

$\xrightarrow{\overline{\mathrm{C}}}$

Figure 5: One cycle of fillings of shape (3^4) with $\mathbf{a} = (1, 4, 9)$.

where

$$\begin{aligned}
E_1 =&(a_2-b_2-1)^+ + (a_3-b_3-2-(a_2-b_2-1)^+)^+,\\
E_2 =&(a_3-b_2-2)^+ - (a_3-b_2+b_3-3n) - (a_2+b_3-3n)^+)^+,\\
E_3 =&a_2-2+(a_3-b_3-a_2)^+ - (a_2+b_3-3n)^+\\
&+(a_2-b_2+b_3+1-3n)^+,\\
E_4 =&a_3-3+(a_3-b_2+b_3-3n)^+ - (a_3+b_3-1-3n)^+\\
&-(a_3-a_2-b_2+b_3+2-3n))^+,\\
E_5 =&a_2-2+(a_3-b_2-a_2)^+ - (a_3-b_2+b_3-3n)^+ - (a_2+b_2-3n)^+\\
&+(a_2+b_3-3n)^+ - (a_2+b_3-3n-(a_3-b_2+b_3-3n)^+)^+,\\
E_6 =&a_3-3-(a_3+b_3-a_2+1-3n)^+ - (a_2-b_3+b_2-2)^+\\
&+(a_2-b_3+b_2-2-(a_3+b_2-1-3n)^+)^+.
\end{aligned}$$

The initial condition is $R_{(3),(1,2,3)} = 1$*.*

Proof. Let $\mathbf{b} = (1, b_2, b_3)$, $U \in \mathcal{F}_{(3^{n-1}),\mathbf{b}}$, and let $T \in \mathcal{F}_{(3^n),\mathbf{a}}$ be defined as $T = A_{\mathbf{a}}(\sigma(U))$ where σ is some composition of s_1 and s_2. We define $\Delta q = \mathrm{inv}_{(3^n)}(\overline{\mathrm{C}}^i(T)) - \mathrm{inv}_{(3^{n-1})}(U)$ for $0 \le i \le 3n-3$ and $\Delta t = \mathrm{comaj}_{(3^n)}(\overline{\mathrm{C}}^i(T)) - \mathrm{comaj}_{(3^{n-1})}(U)$.
Case 1: Let $\sigma = \mathrm{id}_{\mathcal{F}_{(3^{n-1}),\mathbf{b}}}$.

Then $\overline{\mathrm{C}}^i(T)$ for $0 \le i \le 3n-3$ will have $\Delta q = 0$ if and only if $a_2 - 1 \le b_2$ or $b_3 < a_2 - 1$. If $b_2 \ge a_2 - 1$ and $b_3 \ge a_3 - 2$, set $i = 0$ to obtain $A4$ with $\Delta t = 0$. If, on the other hand $b_2 \ge a_2 - 1$ but $b_3 < a_3 - 2$, set $i = a_3 - b_3 - 2$ to obtain $\overline{\mathrm{C}}^i(T) = A1$ with $\Delta t = a_3 - b_3 - 2$. When $b_3 < a_2 - 1$, set $i = \max\{a_2 - b_2 - 1, a_3 - b_3 - 2\}$. When $i = a_2 - b_2 - 1$, $\overline{\mathrm{C}}^i(T) = A2$ with $\Delta t = a_2 - b_2 - 1$. Note that this holds even if $b_3 + a_2 - b_2 - 1 > 3n$, since the resulting filling will have the form

$\vdots$	$\vdots$	$\vdots$
$*$	a_2+1	$a_2{-}b_2{+}1$
1	a_2	a_3

and hence $\Delta t = a_2 - b_2 - 1 - (n-1) + (n-1) = a_2 - b_2 - 1$. When $i = a_3 - b_3 - 2$, $\overline{\mathrm{C}}^i(T) = A1$ with $\Delta t = a_3 - b_3 - 2$. So, when $\sigma = \mathrm{id}_{\mathcal{F}_{(3^{n-1}),\mathbf{b}}}$ and $\Delta q = 0$, set $i = \max\{(a_2 - b_2 - 1)^+, (a_3 - b_3 - 2)^+\}$. This can also be written as $i = (a_2 - b_2 - 1)^+ + (a_3 - b_3 - 2 - (a_2 - b_2 - 1)^+)^+$.

Similarly, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 1$ if and only if $b_2 < a_2 - 1 \le b_3$. Note that $b_2 < a_3 - 2$. If $b_3 + a_3 - b_2 - 2 \le 3n - 3$, set $i = a_3 - b_2 - 2$ to obtain $A1$ with $\Delta t = a_3 - b_2 - 2$. If $b_3 + a_3 - b_2 - 2 > 3n - 3$ and $3n - 1 - b_3 < a_2 - 1$, then set $i = \min\{a_2 - 2, a_3 - b_2 - 2\}$. If $i = a_2 - 2$, $\overline{\mathrm{C}}^i(T) = A3$ with $\Delta t = a_2 - 2 - (n-1) + (n-1) = a_2 - 2$. If $i = a_3 - b_2 - 2$, then

$\overline{\mathrm{C}}^i(T) = A1$ with $\Delta t = a_3 - b_2 - 2$. If $b_3 + a_3 - b_2 - 2 > 3n - 3$ and $a_2 - 1 \leq 3n - 1 - b_3$, set $i = 3n - 2 - b_3$ to obtain $A4$ with $\Delta t = 3n - 2 - b_3$. Then, $i = (a_3 - b_2 - 2)^+ - (a_3 - b_2 - 2 - (3n - 2 - b_3) - (a_2 - 2 - (3n - 2 - b_3))^+)^+$.

Case 2: Let $\sigma = s_2$.

Then $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 0$ if and only if $b_2 < a_2 - 1 \leq b_3$. Set $i = \max\{a_2 - b_2 - 1, (a_3 - b_3 - 2)^+\}$ to obtain either $A1$ with $\Delta t = a_3 - b_3 - 2$ or $A2$ with $\Delta t = a_2 - b_2 - 1$. Thus, $i = (a_2 - b_2 - 1)^+ + (a_3 - b_3 - 2 - (a_2 - b_2 - 1)^+)^+$.

On the other hand, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 1$ if and only if $b_3 < a_2 - 1$ or $a_2 - 1 \leq b_2$. If $a_2 - 1 \leq b_2 < a_3 - 2$ and $b_3 + a_3 - b_2 - 2 \leq 3n - 3$, set $i = a_3 - b_2 - 2$ to obtain $A1$ with $\Delta t = a_3 - b_2 - 2$. If $a_2 - 1 \leq b_2 < a_3 - 2$, $b_3 + a_2 - 2 \leq 3n - 3$, and $b_3 + a_3 - b_2 - 2 > 3n - 3$, then set $i = 3n - 2 - b_3$ to obtain $A4$ with $\Delta t = 3n - 2 - b_3$. If $b_2 \geq a_3 - 2$, do nothing to obtain $A4$ with $\Delta t = 0$. If $b_3 < a_2 - 1$ and $b_3 + a_3 - b_2 - 2 \leq 3n - 3$, set $i = a_3 - b_2 - 2$ to obtain $A1$ with $\Delta t = a_3 - b_2 - 2$. If $b_3 < a_2 - 1$, $b_3 + a_2 - 2 > 3n - 3$, and $b_3 + a_3 - b_2 - 2 > 3n - 3$, set $i = \min\{a_2 - 2, a_3 - b_2 - 2\}$ to obtain either $A3$ with $\Delta t = a_2 - 2$ or $A1$ with $\Delta t = a_3 - b_2 - 2$. Finally, if $b_3 < a_2 - 1$, $b_3 + a_3 - b_2 - 2 > 3n - 3$ and $b_3 + a_2 - 2 \leq 3n - 3$, set $i = a_3 - b_2 - 2$ to obtain $A1$ with $a_3 - b_2 - 2$. We can combine these cases to write $i = (a_3 - b_2 - 2)^+ - (a_3 - b_2 - 2 - (3n - 2 - b_3) - (a_2 - 2 - (3n - 2 - b_3))^+)^+$.

Case 3: Let $\sigma = s_1$.

Then $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 1$ if and only if $b_3 < a_2 - 1$. If $b_3 + a_2 - 2 \leq 3n - 3$, set $i = \max\{a_2 - 2, a_3 - b_3 - 2\}$ to obtain either $A2$ with $\Delta t = a_2 - 2$ or $A1$ with $\Delta t = a_3 - b_3 - 2$. If $b_3 + a_2 - 2 > 3n - 3$, and $b_3 + a_2 - b_2 - 1 \leq 3n - 3$, set $i = 3n - 2 - b_3$ to obtain $A5$ with $\Delta t = 3n - 2 - b_3$. If $b_3 + a_2 - 2 > 3n - 3$ and $b_3 + a_2 - b_2 - 1 > 3n - 3$, set $i = a_2 - b_2 - 1$ to obtain $A2$ with $\Delta t = a_2 - b_2 - 1$. Then, in this case, $i = a_2 - 2 + (a_3 - b_3 - 2 - (a_2 - 2))^+ - (a_2 - 2 - (3n - 2 - b_3))^+ + (a_2 - b_2 - 1 - (3n - 2 - b_3))^+$.

Similarly, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 2$ if and only if $b_3 \geq a_2 - 1$. If $b_3 + a_3 - 3 \leq 3n - 3$, set $i = a_3 - 3$ to obtain $A1$ with $\Delta t = a_3 - 3$. If $b_3 + a_3 - 3 > 3n - 3$ and $b_2 + 3n - 2 - b_3 \geq a_3 - 2$, set $i = 3n - 2 - b_3$ to obtain $A6$ with $\Delta t = 3n - 2 - b_3 - (n - 2) + (n - 2) = 3n - 2 - b_3$. If $b_3 + a_3 - 3 > 3n - 3$, $b_2 + 3n - 2 - b_3 < a_3 - 2$, and $b_3 + a_3 - b_2 - 2 - (3n - 3) < a_2 - 1$, set $i = a_3 - b_2 - 2$ to obtain $A1$ with $\Delta t = a_3 - b_2 - 2$. If $b_3 + a_3 - 3 > 3n - 3$, $b_2 + 3n - 2 - b_3 < a_3 - 2$ and $b_3 + a_3 - b_2 - 2 - (3n - 3) \geq a_2 - 1$, set $i = a_2 - 2 + 3n - 2 - b_3$ to obtain $A3$ with $\Delta t = 3n - 2 - b_3 + a_2 - 2$. Then $i = a_3 - 3 - (a_3 - 3 - (3n - 2 - b_3))^+ + (a_3 - b_2 - 2 - (3n - 2 - b_3))^+ - (a_3 - b_2 - 2 - (a_2 - 2 + 3n - 2 - b_3))^+$.

Case 4: Let $\sigma = s_2 \circ s_1$.

In this case, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 1$ if and only if $b_3 \geq a_2 - 1$. If $b_3 + a_2 - 2 \leq 3n - 3$, set $i = \max\{a_2 - 2, (a_3 - b_3 - 2)^+\}$ to obtain either $A2$ with $\Delta t = a_2 - 2$ or $A1$ with $\Delta t = a_3 - b_3 - 2$. If $b_3 + a_2 - 2 > 3n - 3$ and $3n - 2 - b_3 + b_2 \geq a_2 - 1$, set $i = 3n - 2 - b_3$ to obtain $A4$ with $\Delta t = 3n - 2 - b_3$. If $b_3 + a_2 - 2 > 3n - 3$ and $3n - 2 - b_3 + b_2 < a_2 - 1$, set $i = a_2 - b_2 - 1$ to obtain $A2$ with $\Delta t = a_2 - b_2 - 1$. Then $i = a_2 - 2 + (a_3 - b_3 - 2 - (a_2 - 2))^+ - (a_2 - 2 - (3n - 2 - b_3))^+ + (a_2 - b_2 - 1 - (3n - 2 - b_3))^+$.

Similarly, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q = 2$ if and only if $b_3 < a_2 - 1$. If $b_3 + a_3 - 3 \leq$

$3n-3$, set $i=a_3-3$ to obtain $A1$ with $\Delta t=a_3-3$. If $b_3+a_3-3>3n-3$ and $b_2+3n-2-b_3\geq a_3-2$, set $i=3n-2-b_3$ to obtain $A6$ with $\Delta t=3n-2-b_3$. If $b_3+a_3-3>3n-3$ and $b_2+3n-2-b_3+a_2-2<a_3-2$, set $i=3n-2-b_3+a_2-2$ to obtain $A3$ with $\Delta t=3n-2-b_3-(n-1)+a_2-2+(n-1)=3n-2-b_3+a_2-2$. On the other hand, if $b_3+a_3-3>3n-3$, $b_2+3n-2-b_3<a_3-2$, and $b_2+3n-2-b_3+a_2-2\geq a_3-2$, set $i=a_3-b_2-2$ to obtain $A1$ with $\Delta t=a_3-b_2-2$. Then $i=a_3-3-(a_3-3-(3n-2-b_3))^+ +(a_3-b_2-2-(3n-2-b_3))^+ -(a_3-b_2-2-(a_2-2+3n-2-b_3))^+$.

Case 5: Let $\sigma=s_1\circ s_2$.

In this case, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q=2$ if and only if $b_2<a_2-1$. If $b_3+\max\{(a_3-b_2-2)^+,a_2-2\}\leq 3n-3$, set $i=\max\{(a_3-b_2-2)^+,a_2-2\}$ to obtain either $A1$ with $\Delta t=a_3-b_2-2$ or $A2$ with $\Delta t=a_2-2$. If $b_3+\max\{(a_3-b_2-2)^+,a_2-2\}>3n-3$, and $b_2+a_2-2\leq 3n-3$, set $i=a_2-2$ to obtain $A2$ with $\Delta t=a_2-2$. If $a_2-2+b_2>3n-3$, set $i=3n-2-b_2$ to obtain $A6$ with $\Delta t=3n-2-b_2-2(n-2)+2(n-2)=3n-2-b_2$. Then $i=a_2-2+(a_3-b_2-2-(a_2-2))^+ -(a_3-b_2-2-(3n-2-b_3))^+ +(a_2-2-(3n-2-b_3))^+ -(a_2-2-(3n-2-b_3)-(a_3-b_2-2-(3n-2-b_3))^+)^+ -(a_2-2-(3n-2-b_2))^+$.

On the other hand, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q=3$ if and only if $b_2\geq a_2-1$. If $b_3+a_3-3\leq 3n-3$, set $i=a_3-3$ to obtain $A1$ with $\Delta t=a_3-3$. If $b_3+a_3-3>3n-3$, $a_2-2+3n-2-b_3\leq a_3-3$, and $b_2+a_2-2+3n-2-b_3\leq 3n-3$, set $i=a_2-2+3n-2-b_3$ to obtain $A3$ with $\Delta t=a_2-2+3n-2-b_3$. If $b_3+a_3-3>3n-3$, $a_2-2+3n-2-b_3>a_3-3$, and $b_2+a_3-3\leq 3n-3$, set $i=a_3-3$ to obtain $A1$ with $\Delta t=a_3-3$. If $b_3+a_3-3>3n-3$ and $a_2-2+3n-2-b_3+b_2>3n-3$, set $i=3n-2-b_2$ to obtain $A6$ if $b_3-b_2+1=a_2-1$ or $A5$ if $b_3-b_2+1\neq a_2-1$ with $\Delta t=3n-2-b_2$. Then $i=a_3-3-(a_3-3-(3n-2-b_3)-(a_2-2))^+ -(a_2-(b_3-b_2+1)-1)^+ +(a_2-(b_3-b_2+1)-1-(a_3-3-(3n-2-b_2))^+)^+$.

Case 6: Let $\sigma=s_2\circ s_1\circ s_2$.

When $\sigma=s_2\circ s_1\circ s_2$, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q=2$ if and only if $b_2\geq a_2-1$. If $b_3+\max\{a_2-2,(a_3-b_2-2)^+\}\leq 3n-3$ set $i=\max\{a_2-2,(a_3-b_2-2)^+\}$ to obtain either $A1$ with $\Delta t=a_3-b_2-2$ or $A2$ with $\Delta t=a_2-2$. If $b_3+(a_3-b_2-2)^+>3n-3$ and $b_3+a_2-2\leq 3n-3$, set $i=3n-2-b_3$ to obtain $A5$ with $\Delta t=3n-2-b_3$. If $b_3+\min\{a_2-2,(a_3-b_2-2)^+\}>3n-3$ and $b_2+a_2-2\leq 3n-3$, set $i=a_2-2$ to obtain $A2$ with $\Delta t=a_2-2$. If $b_3+\min\{a_2-2,(a_3-b_2-2)^+\}>3n-3$ and $b_2+a_2-2>3n-3$, set $i=3n-2-b_2$ to obtain $A6$ with $\Delta t=3n-2-b_2$ since $3n-1-b_2<a_2$. Then $i=a_2-2+(a_3-b_2-2-(a_2-2))^+ -(a_3-b_2-2-(3n-2-b_3))^+ -(a_2-2-(3n-2-b_2))^+ +(a_2-2-(3n-2-b_3))^+ -(a_2-2-(3n-2-b_3)-(a_3-b_2-2-(3n-2-b_3))^+)^+$.

Similarly, $\overline{\mathrm{C}}^i(T)$ will have $\Delta q=3$ if and only if $b_2<a_2-1$. In this case, if $b_3+a_3-3\leq 3n-3$, set $i=a_3-3$ to obtain $A1$ with $\Delta t=a_3-3$. If $b_3+a_3-3>3n-3$, $b_2+(3n-2-b_3+a_2-2)>3n-3$, and $b_2+a_3-3\leq 3n-3$, set $i=a_3-3$ to obtain $A1$ with $\Delta t=a_3-3-(n-1)+(n-1)=a_3-3$, since $b_3+a_3-3-(3n-3)<a_2-1$. If $b_3+a_3-3>3n-3$ and $b_2+(3n-2-b_3+a_2-2)\leq 3n-3$, then $a_3-3>3n-2-b_3+a_2-2$, and thus, set $i=3n-2-b_3+a_2-2$ to obtain $A3$ with $\Delta t=3n-2-b_3+a_2-2$. When $b_3+a_3-3>3n-3$,

$b_2+(3n-2-b_3+a_2-2) > 3n-3$, and $a_3-3 \geq 3n-2-b_2$, set $i = 3n-2-b_2$ to obtain $A5$ since $A = 3n-1-b_2 < a_3-1$ with $\Delta t = 3n-2-b_2-2(n-1)+2(n-1) = 3n-2-b_2$. Then $i = a_3-3-(a_3-3-(3n-2-b_3)-(a_2-2))^+ - (a_2-(b_3-b_2+1)-1)^+ + (a_2-(b_3-b_2+1)-1-(a_3-3-(3n-2-b_2))^+)^+$. □

Setting $\mathbf{a} = (1,2,3)$ in Theorem 4.3 yields

$$F_{(3^n)}(q,t) = R_{(3^{n+1}),(1,2,3)}(q,t) = [n]_t[3]!_q \sum_{\substack{\mathbf{b}=(b_1,b_2,b_3)\\ 1=b_1<b_2<b_3\leq 3n}} R_{(3^n),\mathbf{b}}(q,t).$$

Iteration of the recursion shows that $F_{(3^n)}(q,t)$ is divisible by $[n]!_t$. This recursion allows us to reduce the computation of $F_{(3^n)}(q,t)$ from requiring the enumeration of $(3n)!$ standard fillings to a summation involving $(3n)!/(3!^n n!)$ q,t-analogues of $n!3!^n$.

5. Future Work

Extending the recursion in [8] from two-column shapes to three-column shapes involved exceptionally intricate combinatorics. In the two-column case, there are no true triples between the bottom two rows since the bottom left entry is a 1. The addition of a third column introduces a triple between the bottom two rows, which proves difficult to accommodate. The general idea of decomposing the set of standard fillings into sets based on the number of inversions and their relation to some *key fillings* is promising, but the presence of $\binom{m-1}{2}$ triples between the first two rows of an n-column shape pose significant difficulties. The final recursion for (m^n) should have the basic form

$$R_{(m^n),\mathbf{a}}(q,t) = [n-1]_t \sum_{\substack{\mathbf{b}=(b_1,\ldots,b_m)\\ 1=b_1<b_2<\cdots<b_m\leq m(n-1)}} R_{(m^{n-1}),\mathbf{b}}(q,t) \sum_{\sigma\in S_m} t^{\mathrm{tstat}(\sigma,\mathbf{a},\mathbf{b})} q^{\mathrm{inv}(\sigma)} \tag{8}$$

where tstat is some statistic dependent on σ, $\mathbf{a}$, and $\mathbf{b}$, that tracks the change in the comajor index when decomposing the filling. The exponents $E_1,\ldots,E_6$ in Theorem 4.3 are special cases of tstat. It is probable that the general definition of tstat will be algorithmic, rather than an algebraic formula, and that much of the complexity of $E_1,\ldots,E_6$ may disappear once this alternative definition is discovered.

Progress in this direction is the extension of C to fillings of shape (m^n) with increasing first row.

Definition 5.1. For $m,n \in \mathbb{N}$ with $m,n \geq 2$ and $\mathbf{a} = (a_1,a_2,\ldots,a_m)$ with $1 \leq a_1 < a_2 < \cdots < a_m \leq mn$, define $C : \mathcal{F}_{(m^n),\mathbf{a}} \to \mathcal{F}_{(m^n)}$ by

$$C(T) = \begin{cases} \mathrm{cyc}(T) & \text{if } a_m \neq mn; \\ s_1 \circ s_2 \circ \cdots \circ s_{m-1} \circ \mathrm{cyc}(T) & \text{if } a_m = mn. \end{cases}$$

It was shown in [10] that, for $0 \le i < mn$, and $T \in \mathcal{F}_{(m^n),\mathbf{a}}$ where $\mathbf{a} = (a_1, \ldots, a_m)$ with $1 \le a_1 < a_2 < \cdots < a_m \le mn$,

(a) $\mathrm{inv}_{(m^n)}(C^i(T)) = \mathrm{inv}_{(m^n)}(T)$, and

(b) $\mathrm{comaj}_{(m^n)}(C^i(T)) = \mathrm{comaj}_{(m^n)}(T) + i - n \sum_{j=1}^{m} \chi(i > mn - a_j)$.

Using this extended definition of C, we can use a parentheses matching argument similar to the proof of Lemma 4.1 to establish a formula for $F_{(m^n)}(q,t)$ that exhibits a factor of $[n]_t$.

Theorem 5.2. *Let $m, n \in \mathbb{N}$, with $m, n \ge 2$. Then*

$$F_{(m^n)}(q,t) = [m]!_q [n]_t \sum_{\substack{\mathbf{a}=(a_1,\ldots,a_m) \\ 1=a_1<a_2<\cdots<a_m\le mn}} \sum_{T \in \mathcal{F}_{(m^n),\mathbf{a}}} q^{\mathrm{inv}_{(m^n)}(T)} t^{\mathrm{comaj}_{(m^n)}(T)}. \tag{9}$$

Proof. It is shown in [8] that

$$F_{(m^n)}(q,t) = [m]!_q \sum_{\substack{T \in \mathcal{F}_{(m^n)} \text{ with} \\ \text{increasing bottom row}}} q^{\mathrm{inv}_{(m^n)}(T)} t^{\mathrm{comaj}_{(m^n)}(T)}.$$

It remains to show that the factor of $[n]_t$ can be extracted combinatorially from the set of fillings of shape (m^n) with increasing bottom row. Let $T \in \mathcal{F}_{(m^n),\mathbf{a}}$ where $\mathbf{a} = (a_1, \ldots, a_m)$ with $1 = a_1 < a_2 < \cdots < a_m \le mn$. As before, for some k, $\mathrm{C}^{mnk}(T) = T$, so we generate a sequence of fillings $A_0, A_1, \ldots, A_{mnk-1}$ where $A_i = \mathrm{C}^i(T)$ for each i, $0 \le i \le kmn - 1$. We construct a word using the following rules: for $i = 0$, represent A_0 with a left parenthesis if $\mathrm{comaj}_{(m^n)}(A_0) = \mathrm{comaj}_{(m^n)}(A_{mnk-1}) + 1$ and with $n-1$ right parentheses if $\mathrm{comaj}_{(m^n)}(A_0) = \mathrm{comaj}_{(m^n)}(A_{mnk-1}) - (n-1)$. For $1 \le i \le kmn - 1$, represent A_i with a left parenthesis if $\mathrm{comaj}_{(m^n)}(A_i) = \mathrm{comaj}_{(m^n)}(A_{i-1}) + 1$ and with $n-1$ right parentheses if $\mathrm{comaj}_{(m^n)}(A_i) = \mathrm{comaj}_{(m^n)}(A_{i-1}) - (n-1)$. Note that any A_i represented by $n-1$ right parentheses must have $a_1 = 1$. Since there are an equal number of left and right parentheses, the resulting word can be shifted to obtain a Dyck word.

Given a pair of matched parentheses, let t be the number of right parentheses between the matched parentheses. The difference in comajor index between the fillings represented by these matched parentheses will be

$$\Delta\, \mathrm{comaj}_{(m^n)} = -[(n-1) - t \pmod{n-1}].$$

We can now decompose the set $\{A_0, A_1, \ldots, A_{mnk-1}\}$ into mk disjoint sets $\{S_{j_1}, S_{j_2}, \ldots, S_{j_n}\}$, $1 \le j \le mk$, where each S_{j_1} is a filling represented by right parentheses (and thus has increasing first row $\mathbf{a} = (a_1, \ldots, a_m)$ with $a_1 = 1$)

and the remaining S_{j_i} are the fillings represented by left parentheses that are matched with S_{j_1}, labeled from left to right. Then, for each j,

$$\sum_{i=1}^{n} q^{\mathrm{inv}_{(m^n)}(S_{j_i})} t^{\mathrm{comaj}_{(m^n)}(S_{j_i})} = [n]_t q^{\mathrm{inv}_{(m^n)}(S_{j_1})} t^{\mathrm{comaj}_{(m^n)}(S_{j_1})}. \qquad \square$$

It should be noted that, when $\mathbf{a} = (1, 2, \ldots, m)$, Theorem 5.2 also immediately leads to the formula

$$R_{(m^n),\mathbf{a}}(q,t) = [m]!_q [n-1]_t \sum_{\substack{\mathbf{b}=(b_1,\ldots,b_m) \\ 1=b_1<\cdots b_m \leq m(n-1)}} R_{(m^{n-1}),\mathbf{b}}(q,t).$$

In order to find a recursion of the form in (8), $\overline{\mathrm{C}}$ must also be extended to fillings of shape (m^n) with fixed bottom row, and an algorithm to compute $\mathrm{tstat}(\sigma, \mathbf{a}, \mathbf{b})$ must be determined. This will involve figuring out how to handle the $\binom{m-1}{2}$ triples between the first two rows of such a shape.

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