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Theory of "Art" : Qualitative in Nature and Quaintintative in Practice

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THE THEORY OF "ART": QUALITATIVE IN NATURE AND QUAINTINTATIVE IN PRACTICE

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ABSTRACT

The primary focus of this work is to discuss the mechanical action of the Marshall Differential Analyzer ("Art") in a mathematical context. Differential Analyzers are primarily used to calculate the solutions to Differential Equations. A full account of the uses of Marshall's Differential Analyzer is given. Furthermore, a pure mathematical justification of how the machine integrates (mechanical integration) is provided via an application of the Riemann Integral. Many simple example problems are detailed ultimately leading up to the calculation of nonlinear problems, more specifically, a nonlinear oscillator.

1. INTRODUCTION

Why do people dislike mathematics? Why does math seem to be a majority of people's worst subject? Why do some people think that they can't learn mathematics? Persons who have taught or enjoyed mathematics have asked themselves these questions. It is the belief of many that the reason some people are discouraged by mathematics is that there is a disconnect between the physical world and mathematical theory. Nothing could be farther from the truth. Theory is based on a certain set of assumptions in order to provide undeniable truth, and it is the application of this theory to the physical world that provides the missing link between theory and practice. The illusion of a disconnect from theory to practice exists due to the fact that some assumptions are not always plausible in practice. For example, some mathematical models of the environment may disregard wind resistance and this is a very important factor in the real world. Perhaps the underlying assumptions involved in mathematical theory are elements of little interest to many and even less interesting to those in the various sub-disciplines of applied science. However, from an educational perspective, making natural assumptions is necessary because classroom and textbook are not always representative of the real world. It is the task of the teacher, especially in mathematics, to provide a link from theory to practice for those who are not so inclined to enjoy the theoretical aspects of a particular subject. To do this some instructors use what is called a manipulative and they incorporate it in their lectures.

A manipulative is a physical model, or analog, that is used to clarify a person's intuition through visual and tactile interpretation. For quite some time, Dr. Bonita

Lawrence, at Marshall University, has been using one such manipulative to teach certain theoretical points in applied and pure mathematics. This manipulative is called a differential analyzer and was designed to solve differential equations. The differential analyzer is an analog computer that uses mechanical interrelationships to represent dynamical systems; it is literally a physical interpretation of a differential equation. One nice property that most manipulatives possess is that they are confined within a set of constraints or assumptions. So, by using a manipulative like the differential analyzer to teach concepts in mathematics, one may observe why an assumption needs to made and alternatively what happens when the assumption is ignored. A full account of the construction and current uses of Marshall's Differential Analyzer (called "Art" for Dr. Arthur Porter) and the finer mathematical points therein, is the focus of this work. The differential analyzer is used as a tactile teaching tool, but some students in the graduate program at Marshall University are using the machine to study more sophisticated dynamical systems. Because the differential analyzer offers a perspective not available using numerical techniques implemented by a digital computer, it can be a useful tool for advanced level research.

I am one such graduate student. My interest is in the calculation of solutions to nonlinear ordinary differential equations that do not possess closed form solutions. The Marshall Differential Analyzer is a means by which one can create solutions of up to a fourth order nonlinear differential equation. As an addition to using modern standard methods, such as a fourth order Runge-Kutta method, using the differential analyzer provides alternative insight into the abstract nature of a differential equation. Moreover, some of this insight is in the form of "real-time" knowledge about the subsequent derivatives, starting from the highest order for a particular differential equation. This type of information about the derivatives and other terms arising in a specific problem becomes especially interesting if a term in this sequence is not explicitly available in the differential equation itself. Since the differential analyzer uses a strict analog process, the "real-time" calculation of solutions to nonlinear problems may be observed through systematic mechanical processes. Ultimately, this visualization of mathematics is the appeal of using such a machine in the analysis and solving of differential equations.

The Marshall Differential Analyzer was formally revealed to the public in May of 2009. The primary construction of the machine was complete and "Art" was suitable to run up to fourth-order linear ODE's. Since the Grand Opening of "Art," "he" has been modified in several ways and can now solve certain nonlinear ODE's. This thesis concerns the transition of using a differential analyzer to solve various types of differential equations beginning with simple types and leading up to more complex problems including a final showcase of "Art's" programmability with a nonlinear example of an ODE that does not possess a closed-form solution.

In this first section, a brief account outlining the history of the differential analyzer and the chain of events leading to the conception of Marshall's Differential Analyzer is presented. In Section 2, the construction of Marshall's Differential Analyzer is detailed including pictures and descriptions of possible future components that may be added to the machine to enhance its capabilities. The third section is a mathematical justification of mechanical integration theory through an application of the Riemann integral. The discussion in Section 4 concerns the particulars of using the machine to solve certain types of differential equations; this section could be used as an operations manual for Marshall's Differential Analyzer. Finally, Section 5 gives an example of using the machine to solve a nonlinear differential equation that doesn't possess an explicit closed-form solution. Additionally, a discussion of the advantages and disadvantages of using a differential analyzer for this type of nonlinear analysis can be found in this section, along with a summary of this thesis.

The goal of this work is to provide a mathematical description of the theoretical principles described by the motion of a differential analyzer. Because the differential analyzer is a physical model of mathematical concepts, a link from the theoretical world to the physical world can be made through the study of the theory of a differential analyzer and its practical uses. These mathematical concepts include simple math operations, such as addition and multiplication. Other concepts include the study of functions, equations, integration, differentiation, solutions of systems of equations, parametric representations of relations, etc. Among the more rigorous mathematics that a differential analyzer can model are certain types of dynamical systems, such as differential equations and dynamic equations on time-scales. Along with representing mathematical ideas, a differential analyzer can provide solutions to these complicated dynamical systems so that they may be studied qualitatively. Using the differential analyzer to study dynamical systems is certainly not without its limitations, however. These limitations are discussed throughout this paper and, ideally, any reader may benefit by learning from one who has worked with such a machine and whose comments and conclusions are based on first-hand experience. "There are moments in our lives, especially if we are poets or painters or scholars or engineers or scientists, when in a figurative sense we 'strike pay dirt.' I don't mean this in a pejorative sense, but rather in the sense of revelation. Such a moment occurred when the model d.a. (differential analyzer) produced for the first time a series of graphs of, for me, unsurpassed beauty..." (Arthur Porter [14]).

These words spoken by Arthur Porter express his feelings about the first set of solution curves produced by a differential analyzer of his own design. There are many who have worked on Marshall's Differential Analyzer, including myself, who have had similar feelings about viewing a solution plot for the first time. The point should be well taken that despite using a tool made popular some 70 years ago and later considered to be obsolete, there yet still exists something elegant and beautiful about the way it works. This feeling can only be fully appreciated by the most patient of people who can observe and understand the inner workings of the machine with their own eyes.

1.1. HISTORY OF THE DIFFERENTIAL ANALYZER

The rise and fall of the popularity of the differential analyzer is unknown to most. A brief account of the advances in calculating machines leading up to the first mechanical integrating machine is presented in this section. A short description of the first fully capable differential analyzer, invented and built by Vannevar Bush, and its applications is outlined. The role the differential analyzer played in the development of technology leading up to the advent of modern computers is the missing link connecting the 19th and 20th centuries, at least from a technological and computational perspective.

Many differential analyzers were built in the era of the World War II, primarily in the U.S. and throughout Great Britain. They proved to be invaluable tools in the calculation of ballistics tables for example. Unfortunately, the differential analyzer vanished soon after the war, but its technology is not forgotten.

1.2. MECHANICAL INTEGRATION

Although using a wheel to measure arc-length was a technique employed by the ancient Greeks, the first mechanical integrating devices were planimeters. Planimeters are tools used to calculate the area of two-dimensional shapes, their use justified by an application of Green's theorem. Many inventors of the early 1800s attempted to construct a workable model, but it was the inspiring model constructed by Johann Martin Hermann in 1818 that sparked a revolution of the planimeter mechanism [3]. The interesting point of fact is that Hermann's model was a physical representation of the definite integral of a function. Unfortunately, for the time, this complex design was not suitable for accurate calculation of the area of two-dimensional irregular shapes. However, this design would be later realized into the common form, the mechanical integrator. With the aim of a new planimeter design, Tito Gonnella, invented a model that operated under a concept similar to Hermann's. After his first design Gonnella invented a disk and wheel planimeter that has been redesigned several times and is similar to a mechanical integrator. These simple machines were the start of what would be the age of the continuous calculating machines [3].

Mechanical integrators have designs that vary from wheel and disk, to disk globe and cylinder, to a wheel and cone like Hermann's design. Nonetheless, all mechanical integrators have in common three very specific movable parts. First, a rotating surface that governs the rotation of another circular surface so that the former literally turns the latter like a gear. The two are orthogonal to each other. In a mathematical context these can be interpreted as the differential and the integral motions respectively. The third movable part is the most complex and may be interpreted as the integrand. This part will displace one of the other two movable parts so that it governs the ratio of turns between them, ultimately determining the angular velocity of the second part with respect to the first. Note that if this third movable part does not turn, then the integral is that of a linear function, as the rate of change of a linear function is a constant. Moreover, by the Fundamental Theorem of Calculus, if the third part represents a derivative of a function with respect to the first part, then the second part is the function itself. This concept will be discussed in great detail throughout this entire work. It is this fundamental relationship that allows a mechanical integrator to calculate the solutions of differential equations in general.

The next significant advance for the mechanical integrator was its implementation in the calculation of linear differential equations. It was known that planimeters could calculate the definite integral of a function and several scientists were experimenting with the use of a mechanism of this type to solve various problems in computational science. William Thomson (Lord Kelvin) and his brother James Thomson conceived the idea of using a mechanical integrator to solve differential equations. James Thomson published the first paper of its kind, describing his design for a new type of mechanical integrator, using a disk, globe and cylinder. Lord Kelvin published a second paper, an extension of the first, that showed how his brother's calculating machine could be used to solve linear nth order differential equations. Later William Thomson invented a "harmonic analyzer" that would predict the tides [15].

Thomson's calculating machine consisted of several disk, globe and cylinder mechanical integrators connected together by rotating shafts. The idea was to create a closed system of equality, interrelating the various rates that were available from each integrator. This closed system of equality represented a differential equation. Finding the solution on a closed domain was a matter of tabulating values taken from shaft rotations relative to certain movable parts of an integrator. For its time Thomson's machine was very sophisticated (See Figure 1.1).

The problem with using a mechanical integrator to drive the movable parts of other mechanical integrators was that the output operated at a very low torque. The range of problems that could be solved by Thomson's machine was limited. As seen in Figure 1.1, the machine was big and heavy because the globes had to be of considerable weight in order to apply the needed torque. Furthermore, the globes were aligned to allow the smooth rolling about the surface of the disk. Note that the cylinders will only pick up a strict rotational motion. Then the rotation of the cylinder could be picked up by another movable part of an integrator.

The Thomson's inventions, led to a string of continuous calculating machines. Some of the most famous were the bombsights used in air-craft during World War I, the mechanical computers of Hanabil Ford for the U.S. Navy, and Author Pollen

Figure 1.1. A picture of Thomson's integrating machine, Author: Andy Dingley; taken from his own work

for the British Navy [3]. All of these mechanisms were capable of integrating data obtained in "real time" such as velocity in order to calculate distance for artillery type trajectories. These mechanisms led to the development of the first differential analyzer fully capable of solving a vast range of nonlinear problems.

1.3. BUSH'S DIFFERENTIAL ANALYZER

Vannevar Bush coined the term "Differential Analyzer" when he used a combination of several technologies (some of which were of his own invention, for instance, the product intergraph) to develop the first analytical analog computer, the Differential Analyzer. Although Bush credits Leibnitz "the inventor of calculus itself" [1], as the inspiration for using mechanical interrelationships to solve equations, it was the use of mechanical integrators along with mechanical torque amplifiers that would offer the capability of solving nonlinear differential equations. The mechanical torque amplifier was invented by H.W. Niemann (a colleague of Bush) and was based on a capstan principle. (The apparatus will be discussed in a later section.) Basically, a torque amplifier provides the output of an integrator with enough rotational force to drive heavy loads, such as an integrator disk, without causing slippage in the system. As previously mentioned, torque amplification was a limiting factor in previous calculating machines. The ability to drive an integrator disk with something other than a primary motor drive, such as using the output of an integrator to drive the disk of another integrator, was necessary. With such a setup we are able to integrate with respect to something other than the independent variable. For instance, if the output of an integrator represents y, where y is a function of x, then if we integrate y with respect to y it will lead to a the square of y, provided the insertion of an appropriate gear train (the square of y, a nonlinear term, has been created). Mechanically, we have used the output of one integrator to drive both the rate of change drive and the disk drive of another integrator.

Bush describes the first differential analyzer in his paper "The Differential Analyzer, A New Way to Solve Differential Equations" [1]. In this paper Bush provides several diagrams of the primary components that comprise the differential analyzer. The primary difference in Bush's machine and the mechanical integrating machines prior to that was the extensive section of interconnection and the torque amplifiers. The section of interconnection is a pathway by which the movable parts of the various integrator units could be interconnected with one another. This addition makes the differential analyzer the perfect tool for solving different types of differential equations. Because the section of interconnection was so versatile, terms contained in nonlinear problems were easily interconnected on the new differential analyzer.

The Bush-type analyzer was a model that would be built at several universities in both the U.S. and Great Britian during the pre-World War II era. These machines became important tools in the calculation of ballistics tables during the war. Both U.S. and British military leaders had differential analyzers at their disposal for the purpose of supporting the war effort. Actually, the term computer literally means one who computes; so the people performing calculations on the differential analyzer and other calculating machines were the computers of World War II.

One of the most famous uses of the differential analyzer for military purposes was modeling the bouncing bomb. The bouncing bomb was used to disable the hydroelectric dams in Germany. On May 16th and 17th 1943, "Operation Chastise" was the mission carried out by the Royal Navy, where Möhne and Edersee Dams were targeted by the "dambusters" [13]. It would be interesting to determine which differential analyzer was primarily used for the conception of the bouncing bomb built by Barnes Wallis. That information, along with the extent at which a differential analyzer was actually used during the planning of the experiments, would be an interesting study. One fact is certain; if a differential analyzer was extensively used, it was a Bush-type differential analyzer.

Bush went on to build a more elaborate machine called the "Rockefeller DA." The goal in building the Rockefeller DA was to make the differential analyzer more automatic. But there was a need for a more efficient way to perform complex computations. For this reason the differential analyzer was replaced by digital computing machines, the first of which was the E.N.I.A.C. As for the Rockefeller DA it was dismantled shortly after the war due to fear of technology leakage. There is no doubt that advancement in technology provided an advantage in war times and thereafter. The differential analyzer played a very important role in this technological advancement. It is unfortunate that the analyzer was lost to time because, as Vannevar Bush himself said, a differential analyzer could be used as a teaching tool.

Although the differential analyzer was not efficient enough to sufficiently carry out calculations with required time demand, it provided a perspective that is not otherwise available from the conventional methods of today. The method a differential analyzer uses to model differential equations is a pure analog of the dynamic as it occurs in real-time, and each movable part represents a piece of the dynamical puzzle that comprises the differential equation. This perspective on dynamic equations is the attraction for the re-creation of Bush's invention, from a mathematical point of view.

1.4. DR. PORTER AND THE DA

Dr. Arthur Porter was an original contributor to computer science. In his day the latest technology of "computers" was the differential analyzer of Bush type. Porter was first exposed to the idea of a differential analyzer by Dr. Douglas Hartree when Hartree was in need for a student to do research related to the machine. Hartree had recently visited M.I.T. and saw Bush's differential analyzer. It was after his return from the U.S. that Hartree suggested a small workable model could be built out of Meccano parts. As the cost for a full scale differential analyzer would be too high, building one out of Meccano parts would be perfect to start a research project within

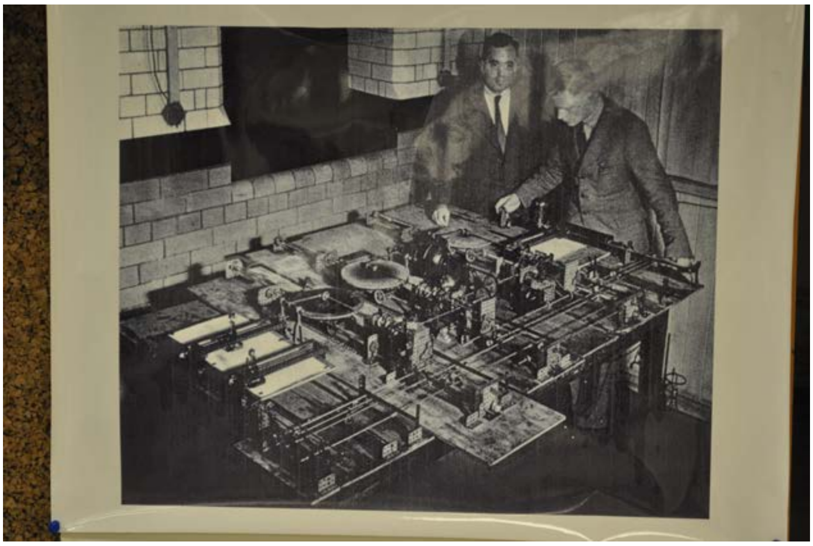

Figure 1.2. Dr. Douglas Hartree on the left and Dr.Arthur Porter on the right, working on the Manchester DA

reasonable financial means. The deal was that if Porter could qualify for the Master's program, then he could work under Professor Hartree receiving an assistantship. It turned out that Porter did qualify by building a model torque amplifier in completion of his undergraduate degree and by taking comprehensive examinations. In fact, Porter was ranked number one in his class, thus receiving the Samuel Bright Scholarship and Graduate Research Scholarship [14]. It was clear who would be Professor Hartree's assistant and who would be building the model differential analyzer. Porter soon began to design a model differential analyzer, primarily out of Meccano parts. The completion of the first Manchester Machine marked Porter's completion of his Master's Degree. Porter's thesis concerned the wave function of the hydrogen and chromium atoms. The Manchester Differential Analyzer was used to calculate the atomic structures of the hydrogen and chromium atoms in the form of a several plots. The results in Porter's thesis can still be found to this day in, as he states in his memoir, "Calculation Machine Section of the National Science Museum in South Kensington, London" [14].

Due to the success of the Meccano model differential analyzer, Hartree soon sought funding to build a full-sized differential analyzer. It was 30 feet long by 15 feet wide and was equipped with eight integrator units. Additionally, the new machine had a special input table built for it. This table could handle special functions with a "time-lag." That is, non-homogeneities which were defined on delayed time intervals. Porter's Ph.D. dissertation was concerned with problems in control theory that implemented this special input table.Hartree and Porter's work sparked a trend of universities building differential analyzers. The fact that the Manchester Machine was the first differential analyzer outside of the U.S. provided other universities, such as Cambridge, with the idea of using such calculating machines for research.

After Porter completed his Ph.D., he was accepted for post doctoral work at M.I.T, working under the direction of Vannevar Bush. The opportunity was a great honor for Dr. Porter. His primary task was to design a "function unit" which was a device used to eliminate the human error element in empirical data input. This component was used to enhance the capability of the famous Rockefeller DA. This most advanced differential analyzer must have been a marvelous sight, consisting of 2000 vacuum tubes and incorporating over 100 electric motors. As previously mentioned, the Rockefeller DA was designed to be more automatic than previous models. In addition to the electronic torque amplification and automatic control restart motors, the machine used a type of punch card incorporated with a large telephone-exchange for more variable programmability. The Rockefeller DA was the first analog/digitial hybrid computer and its conception started a computing revolution leading up to the advent modern of computers.

Dr. Porter was a member of the differential analyzer staff at M.I.T. under Bush for two years (1936-38). During his time at M.I.T. Porter was an integral part of the staff. Although Dr. Porter, being the most modest of men, would have said he served a small purpose in the completion of the Rockefeller DA, Bush wrote to Professor

Hartree addressing the great pleasure he enjoyed while working with Dr. Porter "He proved to be a very valuable member of the group....carried on one particular aspect of the work on his own responsibility, making the analyses and some of the engineering designs and supervising one part of the work" [14].

Dr. Porter went on to have a very successful career working for the Military College of Science, the Imperial College of Science and Technology, the University of Saskatchewan, and the University of Toronto. Porter and Hartree remained good friends throughout the rest of Hartree's life. Although working with differential analyzers represents a small subset of Porter's life and many accomplishments throughout his career, Porter refers to those times as "the most exciting times of my life." Dr. Arthur Porter passed on in February 2010, at the age of 99. For most, Dr. Porter is remembered as a loving father, loyal friend, and an original computer pioneer. For me, Dr. Porter is an icon who will forever stand in my thoughts as the "lighthouse" that marks my own journey on a new path in my life.

1.5. THE MARSHALL DA TEAM

The Marshall Differential Analyzer Team (DA Team) began with a classroom announcement of a research opportunity. It was in an entry-level undergraduate proof-writing course where I first met Dr. Bonita A. Lawrence. On the first day of class, she said she had seen a static display of a machine that solves differential equations. I was very intrigued that a mechanical machine could be used for the calculation of solutions to differential equations, a subject that was of interest to me at the time. I teamed up with two other students and we decided to work with Dr. Lawrence and try to get some extra course credit. At the time we would have never imagined what would become of the original trio. Our first task was to find a working model of some type of differential analyzer so that we could learn about it. It didn't take much time to search and find Tim Robinson's differential analyzer. Tim is a

Meccano enthusiast and electronics engineer who decided to build his own differential analyzer out of Meccano parts for sport. We quickly e-mailed him and asked him about his machine, more specifically we asked a question about how an integrator works. He referred us to a nice webpage that describes the various components of his machine. Upon browsing through Tim's webpage, we went to Dr. Lawrence and told her of our success. She quickly got in touch with Tim Robinson who then told her of Dr. Arthur Porter, an original computer pioneer who had worked on differential analyzers in the time their inception. Dr. Porter was 94 years old at the time and Tim told us he lived in Advance, North Carolina. Needless to say, our first trip as the DA Team was to visit Dr. Porter at his home in North Carolina.

Meeting Dr. Porter was one of the many great pleasures I have had as a DA Team member. At the time I didn't know the magnitude of the scientist I was about to meet. It is only now that I can appreciate the great honor that was given to me when I shook the hand of Dr. Arthur Porter. At his home, Dr. Porter and his wonderful wife Patricia greeted us with the warmest hospitality. He told us of his endeavors as a computer pioneer and told us how the differential analyzer was a direct precursor to the modern day digital computer. Dr. Porter also devoted an entire day to explaining how a differential analyzer worked. It was funny that when he first started to explain the concept of a Bush schematic to us via an example of simple harmonic motion, he said "Oh... I have done this so many times before." The experience was one which has proved to have a remarkable impact on my life. During our goodbye's Dr. Porter signed a copy of his memoir, which he had recently completed, and then we were on our way back with a great deal of inspiration.

The next adventure for the DA Team was a trip to California to see Tim Robinson and his machine in action. When we arrived, Tim greeted us and we went to his lovely home in the hills south of San-Francisco. Down in Tim's basement, in the billiards room, was a differential analyzer built entirely out of Meccano parts. Tim also devoted several full days to try to explain the concept of a differential analyzer to us. Later he took us on a tour through the Computer History Museum, where he works as a docent. In the Museum was an electronic differential analyzer that was build in the 1950s. Surprisingly, the machine still worked and Tim ran a phase plot that produced a Cornu Spiral. Although we still by no means fully understood how the differential analyzer worked, our confidence was renewed when we saw a workable model. Tim and his wife Lisa are a lovely couple and both are considered to be an integral part of the DA Team.

Our first task, when we returned to Marshall, was to build a scaled-down and workable model differential analyzer. We named the small machine “Lizzie” after the “Tin Lizzie” of automobiles. It took us less than a semester to build the model and, after a quick visit from Tim Robinson, it was workable within a few days. Building Lizzie was just the exercise we needed to learn the finer principles of how a differential analyzer works. Although Lizzie runs much slower than a full-scale differential analyzer, watching the machine allowed us to observe the mechanical interrelationships as they represent mathematical principles.

The DA Team, at this point, had grown to about 15 members, and Dr. Lawrence was planning several conferences where our undergraduate research could be presented. The first of these conferences was a trip to New Orleans, for the National Mathematics Meetings. We presented Lizzie along with a poster display in a session of about 170 different mathematics research topics. The problem set up to run was dampened harmonic motion and Lizzie ran gracefully. The next journey for Lizzie was a trip to Washington D.C. for the “Posters on the Hill” session. I had the distinct pleasure of explaining the workings of the machine to U.S. Senator, Jay Rockefeller. It was quite an experience to explain the machine to Senator Rockefeller because the most famous and elaborate differential analyzer ever built was funded by the Rockefeller Foundation. Additionally, the DA Team had the opportunity to meet

Congressman Nick J. Rahall, and several education advisors for U.S. senators representing West Virginia, Ohio, and Kentucky. We also displayed Lizzie in the poster session with 400 other undergraduate posters on various subjects.

The most exciting time for me was the building of "Art", a four integrator differential analyzer that has the capacity of solving nonlinear problems. We started building in the fall semester of my final year as an undergraduate and were finished with the construction by the end of the spring semester. Soon after the primary construction was complete Tim Robinson came for another visit. Tim had designed a new means of torque amplification based on a previous design of servo-mechanisms. The DA Team built the necessary components to complement the design, then Tim programmed a micro-processor to control the servo-mechanisms, that is, the torque amplification. Although I wasn't enrolled as a graduate student yet, the DA Team spent long hours in the summer working out the kinks in the new machine. One full year later, "ART" was displayed for "his" Grand Opening, a highlight for the Mathematics Department at Marshall University.

The DA Team has had a varying number of members over the years, from as few as 3 up to 15. I am proud to say that I have been an active member of the DA Team since its beginning. The completion of this paper marks the end of this era in my life. As all other members have, it is time to graduate and start the next chapter. However, my hands have not seen their last differential analyzer. The project continues, Dr. Lawrence and Tim Robinson continue, and the spirt of Dr. Arthur Porter lives on with the Marshall DA Team.

2. CONSTRUCTION DETAILS OF THE MARSHALL DIFFERENTIAL ANALYZER

This section concerns the specific details of the mechanical operation of the Marshall Differential Analyzer. It is necessary for the reader to pay close attention to this section, because, the correlation between movable parts and mathematical principle is often hidden within the details of design. The goal of this discourse is to provide a visual for the machine. Many schematic drawings are provided and actual pictures of Marshall's machine are available here along with a specific description of the picture in the caption. Although there are no precise blueprints provided, there is enough builder-specific information for enthusiastic readers to build these components themselves.

2.1. INTRODUCTION

The Marshall Differential Analyzer is equipped with all the major components required to solve up to fourth order problems. There are four integrator units, all of which have a corresponding servo-mechanism that operate individually, for torque amplification. Additionally, the machine includes a 6 by 1 by 0.5 foot section of interconnection, four adders, an Input Table and an Output Table. A description of a Multiplier is also given and a description of its construction can be found in Section 2.12. It took the DA Team less than one semester to build our first model "Lizzie" (Refer to Figure 2.1), and at that point we were on our way to understanding how to operate a differential analyzer. Now it was time to build a full-scale model that had the capacity to solve nonlinear problems.

Figure 2.1. The First Differential Analyzer Built by Marshall's DA Team.

All of the main components in a differential analyzer operate in unison if the problem being solved requires those parts. For example, some problems like simple harmonic motion require the use of two integrators so the other two integrators are static for this problem. The integrators in use all work together. Among these components, four integrators, the section of interconnection, the adders and the input and Output Tables, the integrators are the primary components. They are used to relate various rates of change that are described by a given differential equation. This relation is essentially the differential equation itself and its particular parameters. The section of interconnection, along with the adders is used to connect the integrators together. The Input Table is used to input certain special functions into the section. Last, the Output Table plots the desired result.

It is important to stress the fact that all the parts move at the same time as this is a key difference between mechanical integration and numerical integration. The motion of the machine, beyond limited mechanical properties, is a continuous analog computer. Moreover, a digital computer operates discretely using boolean logic, which restricts a numerical method to computing one calculation at a time, as opposed to an analog machine, which can compute multiple calculations at any given time. The continuous nature of the differential analyzer has to be maintained very carefully using mechanical intuition. Limited mechanical errors, such as backlash and slippage, can be made negligible by careful consideration of the tolerances of the parts. For instance, to make a minor adjustments in the vertical height for two angle-girders, the extra space in a slot can be used. To avoid slippage, the two heights must be precisely aligned. Suppose two pieces of angle need to be lined up at the exact same height so that a rod may be journaled through them and spun freely. Both pieces of angle can be adjusted about an eight of an inch in either direction on some vertical axis, but when the measurement is taken the two pieces may not line up. The difficulty arises because it is hard to make very small adjustments on something that is designed to move an eighth of an inch when it needs to be adjusted perhaps a sixty-fourth of an inch. This small tolerance of adjustment is simply not available in the design of the parts. The solution is to loosen the adjustments on the angle girder (nuts and bolts) so there is some play in the adjustment. Next journal (pass or guide) the rod through the two pieces of angle and spin the rod until it spins freely. This rod will spin freely now because the adjustments are loose. Now ever so gradually tighten the adjustments. Which must be done essentially while the rod is spinning. What this procedure does is precisely align the two pieces of angle so the rod can spin with very little play. Note that if the angle girder measurements are taken a second time then the marks will seem nearly unchanged. However, "nearly" is the key word. One might not be able to discern the difference with the naked eye and say a set of Vernier calipers. Nonetheless, the rod does spin freely now. By spinning the rod and loosening the adjustments a very precise alignment is forced, one so precise that it could not be measured feasibly. These types of situations occur often in the construction processes as well as in maintenance. In a lot of cases it is best to adopt a more mechanical approach instead of a theoretical one.

2.2. PARTS DESCRIPTION

The parts for the machine were ordered from India via an independent distributor, Ashok Banerjee. Ashok machines parts that are reproductions of the the original Meccano. Although these parts are very simple, in most cases they prove to be sturdy and reliable. The framework of the machine is mostly constructed from angle-girders. That is, a strip of metal bent 90 degrees that has half-inch spaced holes on one side and half-inch spaced slots on the other. The slots are convenient because they allow minor adjustments to be made in the alignment of the various components.

Other parts that are essential to the construction are tiny set-screws, gears, and couplings. The couplings connect the steel rods or bus shafts together and the set-screws allow the gears to be attached to the rods. The gears are bored through the center so that the rod may pass freely through them. The gear also has a hole tapped in its surface that is perpendicular to the hole for the rod. Basically, a set-screw is fitted into this tapped hole in the gear. When the set-screw is tightened down it makes contact with the rod and fixes the gear in place to the rod. Now when the rod is turned so is the gear.

Although there are many more parts that are necessary for the construction, gears, rods, sets-screws and angle-girders are the basic building blocks of Marshall's machine. Other parts will be mentioned as needed.

2.3. HOW AN INTEGRATOR WORKS.

The integrator units are the primary components on any differential analyzer. Their function is to integrate any derivative term in a differential equation, thus reducing the order of the derivative by one. An integrator consists of a rotating disk mounted on a movable carriage and a wheel that rests on the surface of the disk perpendicular to its surface. Hence, as the disk is rotates so does the wheel. Because all the wheel's weight rests on the surface of the disk, the force of friction between wheel and disk is the driving force of the wheel. When the carriage is moved back and forth, the distance from the edge of the wheel to the center of the disk is changed, thus changing the revolutions per minute of the wheel. From a mechanical perspective, an integrator is simply a variable gear train where the corresponding radius of the wheel and its position on the disk from the center represent the gear ratio. Note that the force of friction from the contact between the surface of the disk and the edge of the wheel is the only driving force from disk to wheel so that the output of the wheel operates at a very low torque. Nonetheless, with a very light load this force of friction is enough to maintain a continuous connection between disk and wheel. (Heavier loads will be discussed later.) Usually (in a variable speed motor, for example) gears are made of teeth, and the ratio of two gears is the number of teeth on one gear divided by the number of teeth on the second gear. The gear turned by direct drive determines the dividend of the quotient. In the case of the integrator, the two gears (disk and wheel) don't have teeth. So in order to find the gear ratio of the two, one must consider the radius of each. Because the disk is mounted on a carriage, it may be moved back and forth in a strict linear path with the wheel being centrally located on it. This movement changes the distance from the center of the disk to the edge of the wheel. This distance is the radius that is used for determining the ratio of disk to wheel. The radius of the wheel is fixed so we shall denote it by a. Also because the disk has the direct drive connection, the ratio of disk to wheel is the radius on the disk divided by the radius of the wheel. So in order to calculate the number of turns of the wheel with respect to the number of turns of the disk, we must fix the radius of the disk. Denote the distance from the center of the disk to the edge of the of the wheel, which is the radius on the disk, by y. Now let Δx be any portion of a turn of the disk. The total number of turns of wheel is

$$\frac{(y * \Delta x)}{a},$$

where Δx is a portion of a turn of the disk, y is the radius on the disk and a is the radius of the wheel. Hence, the ratio of disk to wheel is $y : a$, or $\frac{y}{a}$.

Keep in mind that the value of y can be changed while the disk is turning the wheel. In fact, this will be the case when solving most problems. At this point, one may note that the integrator wheel adds up consecutive differences in gear ratios in the form of tiny "arc-lengths" as prescribed by the disk. But for further interpretation consider the bird's-eye view of the integrator disk, in three discrete cases with a fixed y displacement for each case (Refer to Figure 2.2). Note: We will return to this diagram in Section 3, where a pure mathematical justification of mechanical integration is given.

Again, y denotes the displacement from the center of the disk to the edge of the wheel (in the diagram, the edge of the wheel is the starting point in each discrete "arc-length"), and now let Δx denote any portion of a turn of the disk, which is essentially some angle θ and s_i denotes the arc that is the result of turning the disk through this angle θ. Moreover, since θ is represented by Δx and y is represented by the radius on the disk, then $s_i = y_i * \Delta x_i$. The wheel rests at distance y_i from the center of the disk and the the arc-length is picked up by the wheel as the disk is turned. Note: For the three cases shown in Figure 2.2, the distances y_1, y_2, y_3 are changed in a discrete manner before the next Δx_{i+1} occurs. That is, the wheel is set in place at a fixed y_1 then the disk is rotated through some portion of a turn Δx_1, and s_1 is created. Then the distance from the edge of the wheel to the center of the disk is changed to y_2 and again the disk is rotated through Δx_2, yielding, s_2, and so

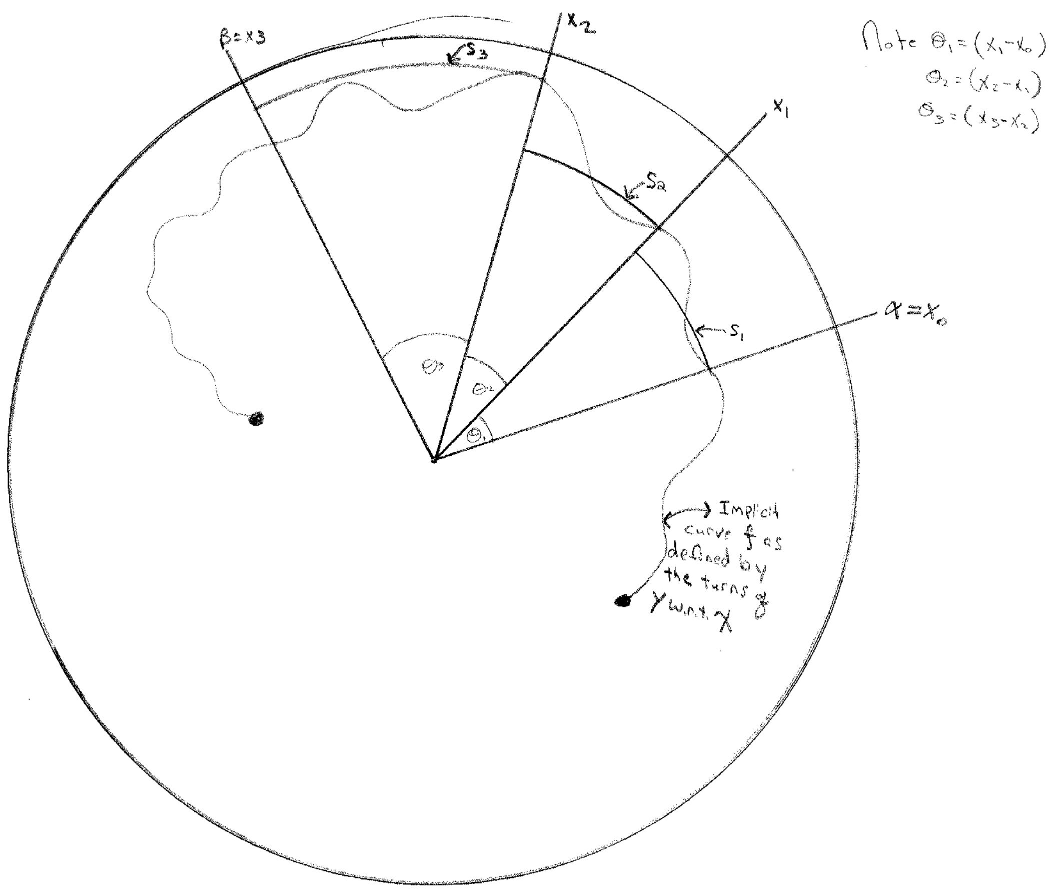

Figure 2.2. This is the "Bird's-Eye-View" of three discrete cases of arc-lengths picked-up by the wheel. Each sector depicted in the circle represents a discrete case respectively. The distance y is not labeled, however. y_i is the radius that corresponds to each s_i (arc-length) in each discrete case. Note: the implicit function "f" will be explained in Section 3.

on. The total arc-length is then $s_1 + s_2 + s_3$ or

$$\sum_{i=1}^{n} s_i,$$

or as by the definition of arc length defined above, the total arc-length is,

$$\sum_{i=1}^{n} y_i * \Delta x_i.$$

In Section 3 we verify that as Δx approaches zero this sum approaches the Riemann integral.

2.4. THE CONSTRUCTION OF AN INTEGRATOR

The original Bush design, of a wheel and disk integrator, is comparable to the integrator's at Marshall University, because the machine was, in essence, modeled after Bush's machine. Tim Robinson modeled his machine after the Bush design and the DA Team modeled Marshall's machine from Tim Robinson's design. As a matter of fact, Tim Robinson's machine on the surface looks much like Marshall's machine. Robinson's machine is made from original Meccano parts (he is quite the Meccano enthusiast), and the Marshall DA was made from reproduction parts. However, there are many instances where the two machines are very different. For example, the torque amplification systems are different.

The carriage, made from angle-girders and flat plates, has dimensions of approximately 5 by 6 inches. It is very important that the carriage be made as sturdy as possible, so gussets are placed in each of the four corners. These are in place so that the angle of the carriage at each corner is maintained at exactly 90 degrees. On both ends of the carriage a six-inch rod is journaled through the two pieces of angle girders on opposite ends of the carriage. Then two 1 inch pulleys are fixed to both ends of each rod. This makes the square box look like a cart. (See Figure 2.3).

A cart or carriage is an appropriate name for this piece, because it carries the disk in a strict linear path. This path is made of 11 inch angle girders, connected on top of a frame, so that the girders act like rails for the carriage. Next there must be some apparatus that drives this motion. This motion is achieved by a long screw called a lead screw that is fixed to the carriage by a small threaded boss. The carriage is able to be pulled or pushed via the lead screw. The threaded boss is tapped to match the pitch of the lead screw so the boss acts like a nut. Instead of a screw and

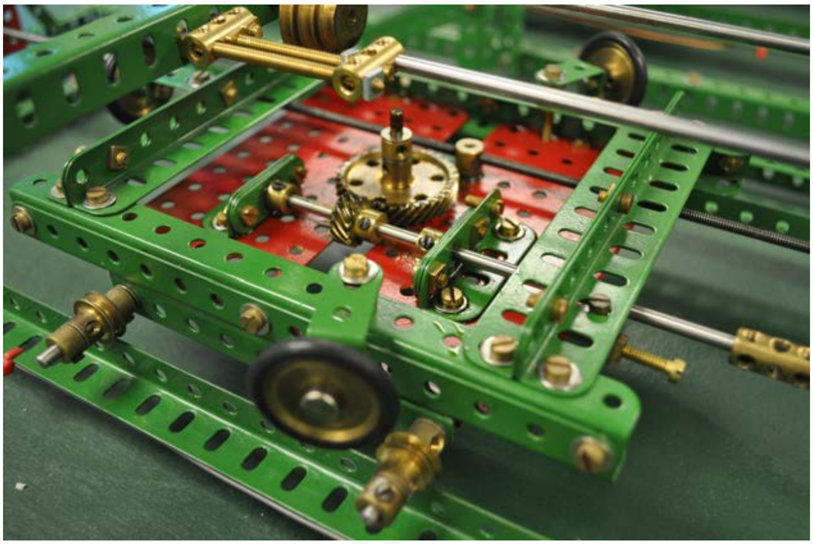

Figure 2.3. An Integrator cart riding along its rails made from angle girders.

a nut holding two pieces of frame together, the screw and nut work together so that, when the screw is turned, the nut is displaced toward one end of the screw. Because the nut or boss is fixed to the carriage, the carriage is displaced. For Marshall's machine the pitch of all threads is 32 threads per inch. When the lead screw is fixed to a rod via a coupling, the carriage is allowed to ride on the rails. As the lead screw rod is turned, the carriage moves back and forth along the rails at a rate proportional to the angular velocity of the lead screw rod. Because the pitch of the lead-screw is 32 threads per inch, when the lead screw rod is turned 32 times, the carriage will be displaced one inch.

All quantities of the machine are measured in terms of shaft rotations. To keep track of this, a revolution counter is placed in the back of each integrator and connected to the lead-screw shaft through a gear train of 3:1. And because the counter is geared to read 10 for every shaft rotation, the counter reads 10/3 for every one revolution of the lead-screw shaft (See Figure 2.4).

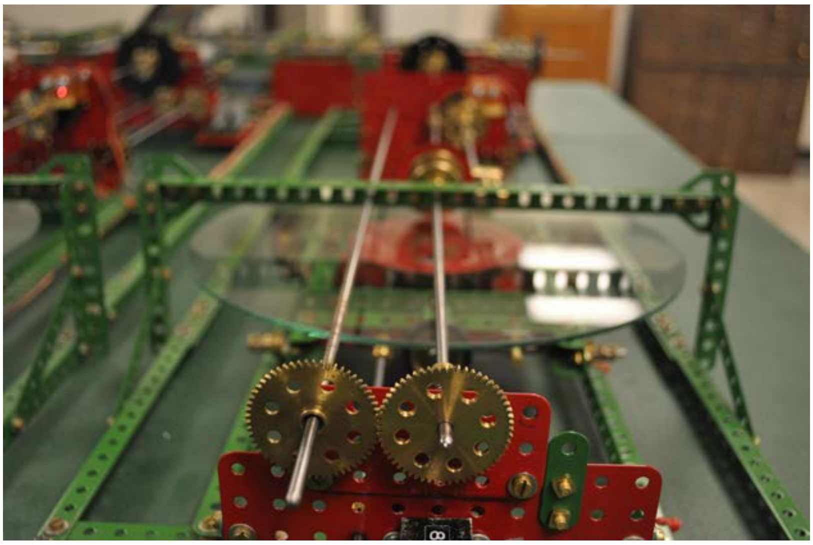

Figure 2.4. Back View of an Integrator Unit

When mounting the disk to the movable carriage, the disk must be fixed on a shaft, and that shaft must be held in some kind of bushing so that the disk cannot move up and down in the bushing but can rotate freely. This task is achieved by another simple yet necessary part, the dog collar (or collar). The name couldn't fit the part any better, because it acts like a collar for a dog. The collar is a thick spacer, like a washer, but has an added feature of set screw hole tapped into its surface. When the set screw is tightened the collar is fixed to the shaft. When a collar is attached to a shaft, the rod can't slip out of a bearing as it is being turned. Conversely, a gear, or bushing may be held in place by two collars so that it does not slide on the rod. These collars are used in 1,000 different places throughout the machine.

The disk is made out of pane glass and cut to a diameter of 11 inches. Once the center is found, a concave red plate is glued to the surface of the disk. The plate must be centered on the disk. Additionally, the plate has holes drilled at four points on a circle with a radius of one inch from the center of the plate. These are convenient because a screwed rod socket can be attached to the plate. It is important to note

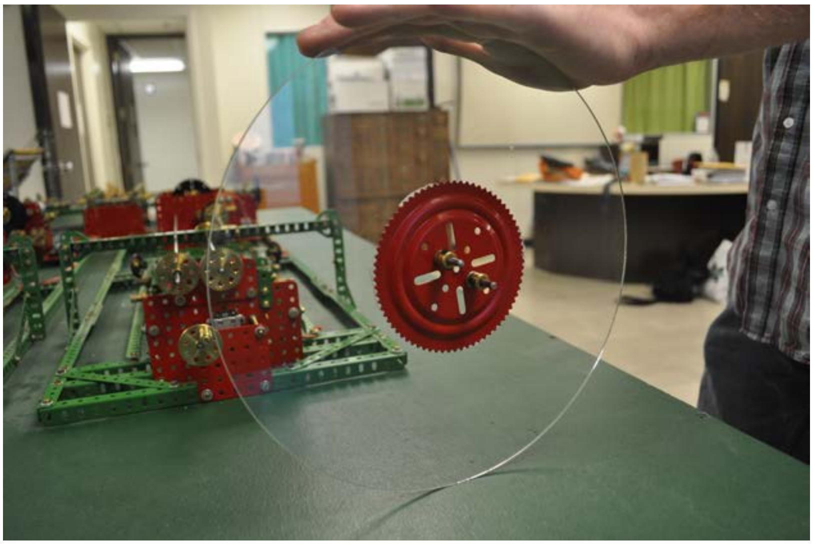

Figure 2.5. Integrator Disk made from pane glass. The rod-socket attachment and connecting plate are shown.

that the plate is conical with the concave side glued facing the disk so that there is some space between the disk and the plate. This extra space is necessary in order for the screwed rod socket to be attached the the plate (See Figure 2.5).

The purpose of the screwed rod socket is to provide a fixture for the disk, in the form of a male end piece, by inserting tiny rods in the sockets. Now a female connection must be built in the carriage so the two pieces can connect. In the very center of the carriage is a bushed wheel, and inside the bushing of this wheel is a rod approximately four inches long. Making sure the rod is permanently fixed, via a set-screw in the bushed wheel, a gear is journaled on the rod so that it can spin freely. This is a 2.5 inch helical gear, and, conveniently, there are holes drilled around four points on the circle of half inch radius from the center of the gear. These holes are used as the connecting piece for the rods in the rod sockets of the disk fixture. Tiny half-inch rods inserted into the two rod sockets are fixed to the disk-plate. The disk is lined up so that the center hole on the disk plate fits over the top of the extending rod protruding from the carriage. Note that the center hole in the plate is a guide for the disk axial rod that extends upward, orthogonal to the plane of the carriage. Additionally, the tiny rods extending from the rod sockets line up with the holes in the disk gear. When the rods are lined up, the rotation of the disk gear will turn these rods, which rotate coaxial with the disk itself. So the disk drive has a little more torque, coming from a gear that spins freely on a rod, as opposed to a rod that turns a fixed gear. Also the other two half-inch rods inside the adjacent rod sockets provide a rigid driving force so that the helical gear may turn the disk. Also collars are positioned to stop the disk gear from moving up its axial rod.

In general, helical gears mesh orthogonally, which is convenient because we want the disk drive to extend in the same direction as the lead screw drive. A small helical gear is set in place via a three-holed angle girder and then fixed to a rod that extends outside the perimeter of the carriage so that a portion of a turn of that rod results in some portion of a turn of the disk. The gear ratio between the two helical gears is 2:5, that is, for every five turns of the small gear the big gear turns twice. The rotation of the disk is geared down, so now the output of the wheel has changed. This pair of gears is called the reduction gear of the disk, and this ratio is of great importance when quantifying the integrator unit in Section 4 (See Figure 2.6, and 2.7).

The wheel must be mounted on top of the disk in such a way that, when the disk is displaced on the rails, the edge of the wheel must remain collinear with the center of the disk. This alignment is achieved by centering the wheel's edge on the center of the disk (See Figure 2.8.). The fact that the carriage rails are in a straight line keeps the wheel's edge in-line with the center of the disk. Once the wheel has been centered on the disk, a perpendicular bridge is built across the top of the disk, constructed approximately one inch above the surface of the disk. The bridge is made out of a 11 inch piece of angle girder and is placed slots down. The wheel is a conical disk with a radius of 15/16 of an inch and is fixed to a rod with a collar against the beveled side and a bushed wheel fixed in place on the concave side (See Figure 2.8). The wheel is

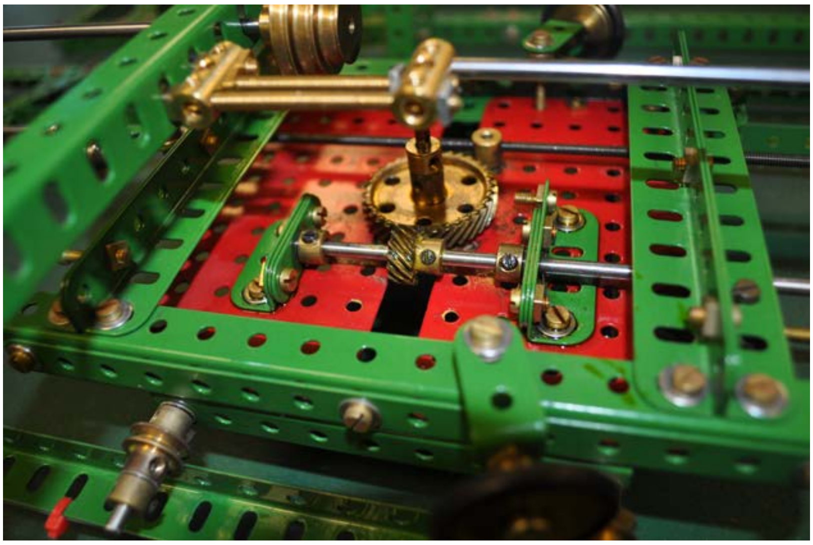

Figure 2.6. Inner components of an Integrator. The large helical gear attached to the orthogonal axis normal to the plane of the carriage is shown. Also the small helical gear the goes to the spline-shaft. This gear is driven by the differential shaft.

therefore trapped in place on the rod. The wheel rod is journaled through the middle slot on the angle-bridge constructed above the disk. The wheel is held in place so that the only motion picked up by the wheel is the rotation of the disk. The slots are used to restrict the horizontal movement but allow a small vertical adjustment so that the wheel may freely rest on the surface of the disk. Hence, the only force that holds the wheel to the disk is the force of gravity against the mass of the wheel and connecting rod. Additionally, there is a rotator bearing on the connecting wheel rod that provides extra weight for the force of friction between the disk and the wheel's edge. It is important not to make the force of friction between the disk and wheel's edge too large because, in addition to being turned, the wheel must also slide on top of the disk's surface so that the gear ratio from disk to wheel can be changed. The rotator bearing also serves another purpose that will be discussed in the next section.

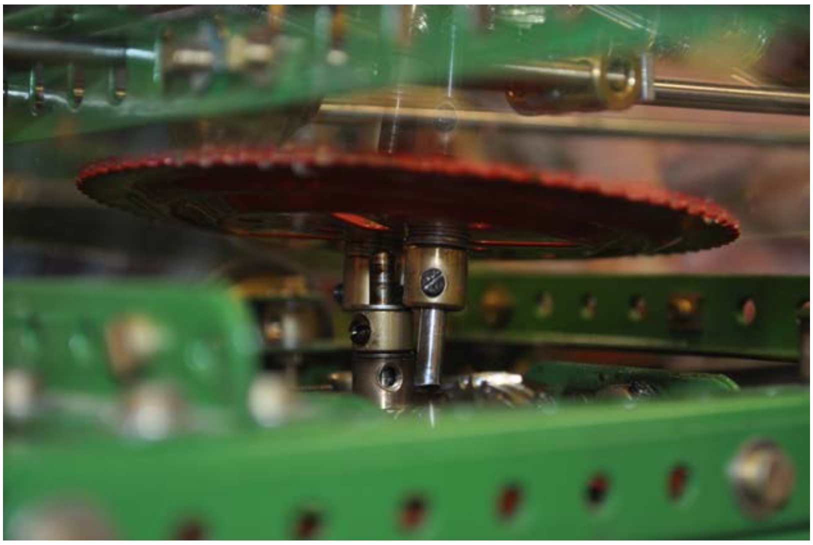

Figure 2.7. Showing the Integrator Disk being connected to the carriage by lining up the rods in the rod sockets with the holes in the helical gear in the carriage.

When the wheel is mounted properly, one should be able to turn the helical disk-gear, turning the disk, thus turning the wheel. The ratio of turns of the disk to turns of the wheel is proportional to the distance from the center of the disk to the edge of the wheel and wheel's radius. For example, if the distance from the edge of the wheel to the center of the disk is 15/16 of an inch, then one turn of the disk gives one turn of the wheel, or a 1:1 ratio. Because the radius of the wheel is 15/16 of an inch, any multiple of this radius set as the distance from the edge of the wheel to the center of the disk yields integer multiple gear ratios; 1:2 for a distance set twice the radius, etc.

There are two input shafts and one output shaft for each integrator unit; the two inputs are the disk drive and the carriage drive (lead screw), and the wheel-shaft is the output. The disk may be turned by one shaft, and the gear ratios may be changed by the carriage drive or lead-screw shaft. As a result, when the two input shafts are being turned at the same time, the disk drive moves with the carriage as the disk drive

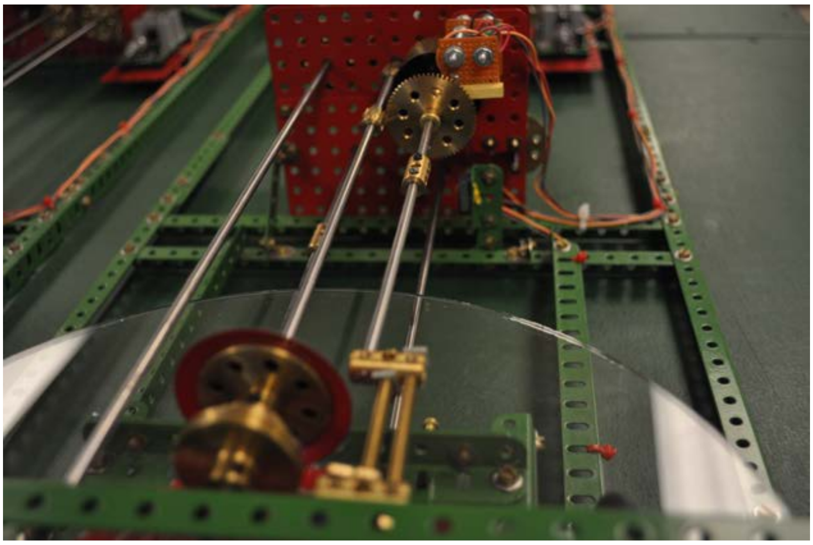

Figure 2.8. Behind View of an Integrator Wheel. On the back of red conical wheel is the bushed wheel used to trap the wheel in place on the rod, along with a collar on the reverse side.

is a direct-drive connection. This mechanical contradiction poses a problem when the disk drive is rigidly connected to a continuous motor drive. Thus, the shaft that connects the disk drive needs to move back and forth while maintaining its rotation. The freedom to rotate the disk and displace displace the carriage, through its driving mechanism, at the same time can be achieved several different ways. One way to achieve a “duel-drive” is by means of a key-way rod. A key-way rod is a rod that has a grove cut into the length of the rod at a depth which is approximately the radius of the cylindrical rod. A set-screw is specially designed so that when a gear is tightened down and fixed to the rod; the setscrew fits specifically into the grove of the key-way rod. Hence, the setscrew is the “key” and the grove is the “way.” When the disk input shaft is a key-way rod, and when the two input shafts are turned at the same time, the key-way rod will pass through the connecting gear, while maintaining its rotation as the carriage is moved by the lead-screw displacement. Marshall’s machine,

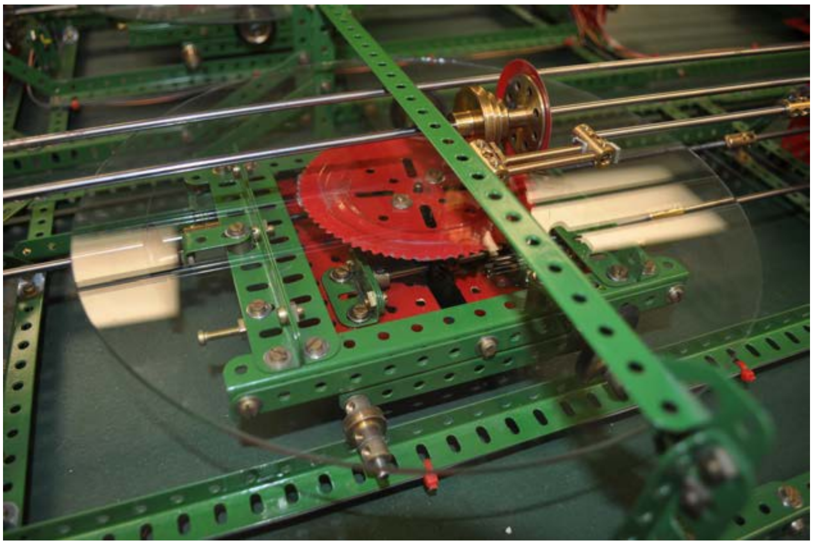

Figure 2.9. Side View of an Integrator, the weight of the disk has extra support using four rubber wheels attached to the frame of an integrator carriage via pivot bolts.

or “Art” as we like to call him, uses another technique, the spline-shaft modeled after Tim Robinson’s design.

2.5. SPLINE-SHAFT

Dr. Arthur Porter designed the original spline-shaft [9]. A spline-shaft assembly consists of at least four rods of the same length (See in Figure 2.10 the four rods at the bottom right). On Art, two of the rods are held in place by two 1 1/2 inch gears, but they are not journaled through the center of the bushing in the gear. They are held by the adjacent holes drilled into the outer circumference of the gear between the edges of the teeth and the center of the gear-bushing. Moreover, the rods are prevented from sliding through these holes by two collars pressed against both surfaces of the gear and attached to each of the four ends on both rods (eight joints total). When the two holding gears are rotated the two rods spin coaxial with the center of the gears. Additionally, in a similar fashion, the two coaxial rods carry a bushed wheel that is

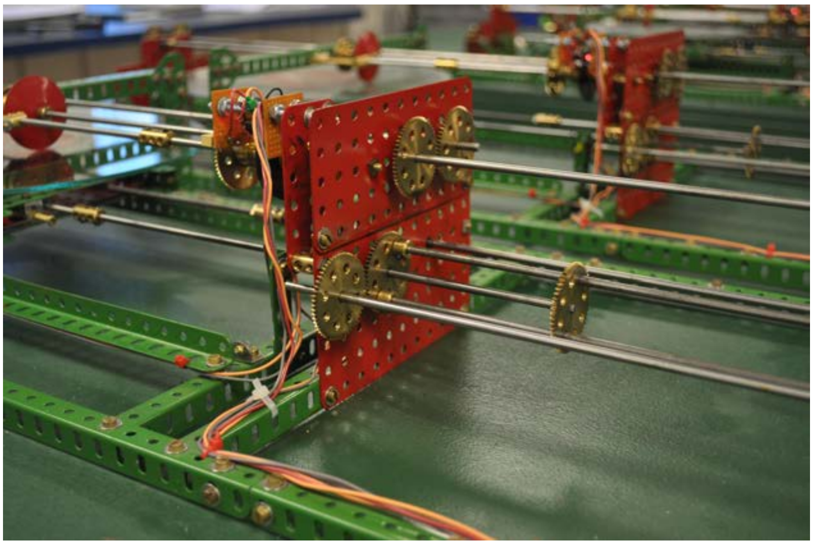

Figure 2.10. Half View of spline-shaft. The connection to the disk drive gear train and the spur gears connecting the assembly are depicted on the left.

visible in the center of Figure 2.11. A third rod is journaled through the center of one of the gears but not set in place by a setscrew; the gear bushing acts like a bearing for this center rod. The center rod passes through the center of the assembly and is fixed into the center of the bushed wheel, via a setscrew, so that the two coaxial rods act as guides for the bushed wheel and center rod. Moreover, when the center-rod is turned, so is the bushed wheel; thus, the two coaxial rods are rotated about that center. Now the center rod is able to be turned and be displaced along the guide-rods at the same time. Figure 2.10, and Figure 2.11 show the spline-shaft assembly on "Art."

The center rod is directly attached to the small helical gear in the disk drive, and the other end of the assembly is held by a flat plate attached to the frame via a small rod fixed to the gear on that same end. The idea is that the center rod attached to the bush wheel and the disk drive is able to move back and forth with the carriage because the bushed wheel can slide along the guide-rods. At the same time those guide rods rotate coaxially with the center rod essentially rotating the disk. The only issue left is how to drive the coaxial rods. Remembering that those coaxial rods are held in place by gears, a fourth rod is journaled through the frame parallel to the center rod. This fourth rod (the lowest rod seen in Figure 2.10) is equipped with two 1 1/2 inch gears that will mesh with the gears fixed to the two coaxial rods so that the fourth rod is the input rod for the disk. It is important to note that the input rod turns two gears at the same time, so the drive for the spline-shaft assembly is a direct drive at two places simultaneously. This design is necessary so that the spline-shaft assembly will not twist causing extreme backlash. Backlash occurs when there is a significant time-lag between the rotation of two rods that are connected through a gear train. (This is sometimes called “slop” or “play” in the gears.) As the input rod is turned, so are the gears fixed to it. Those gears are meshed with the gears on the spline-shaft assembly, and, therefore, the assembly is rotated coaxially with the center rod. The two coaxial rods rotate the bushed wheel, which is fixed to the center rod, which is, in turn, fixed to the helical gears that rotate the disk. All the while, the carriage may be moved back and forth with the bushed wheel moving back and forth along the two coaxial rods acting as guides for the bushed wheel. Hence, the rotation of the disk drive is maintained while the carriage is displaced. The gears on the spline-shaft have a 1:1 ratio so the reduction gear from input shaft to the actual rotations of disk itself remains a 5:2 gear ratio. And again, if the input shaft is turned five times the disk rotates twice.

The integrator unit is designed so that two inputs yield one output. If the lead-screw input remains stationary (i.e., the lead-screw does not turn) then, for any number of turns of the disk input shaft, the wheel output shaft will turn some constant multiple of that particular number of turns depending on the distance the wheel’s edge is from the center of the disk. However, if both the lead-screw input shaft and disk input shaft turn at the same time (which is most generally the case when solving DE’s) the number of rotations of the wheel output shaft, with respect to

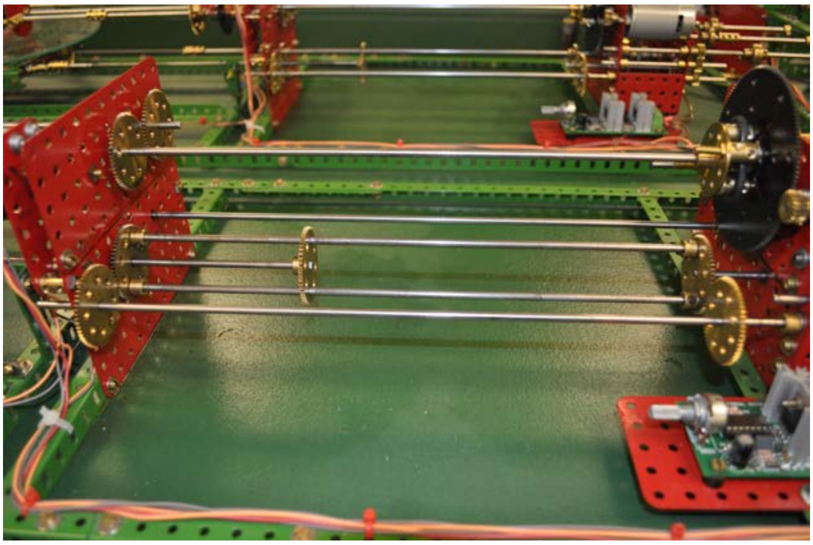

Figure 2.11. Spline-shaft Assembly; the length between the two connecting frames are consistent with the diameter of the disk.

the number of rotations of the disk, is not so easily quantified. It is in this case, when the lead-screw input is non-stationary, that the spline-shaft assembly is necessary. This allows the disk to be turned while the carriage moves back and forth.

2.6. TORQUE AMPLIFICATION

When both disk and lead-screw input shafts rotate simultaneously, the mathematics and the mechanics become much more complicated to quantify and maintain, respectively. From a mechanical point of view, when the lead-screw shaft does not turn, the wheel remains at some fixed distance from the center of the disk. But if the lead-screw turns simultaneously with the disk, then the wheel must slide on top of the surface of the disk. So, if the lead-screw is to move the carriage while the disk turns, then the force of friction between the wheel and the disk must be relatively low. This lack of significant torque poses an obvious problem if the force of friction between the disk and wheel is the only driving force for the rotations of the wheel and subsequent gear trains. Also note that, if the force of friction was increased (for example, using a rubber wheel, or adding weight to the wheel) the lead-screw would be more difficult to turn. The force of friction must not be too large because that force must be overcome for the wheel to slide. So the driving force (force of friction or torque of the wheel) must be relatively low for the integrator to operate properly. To solve nonlinear differential equations using the mechanical integration technique, the wheel-shaft (or integral shaft) must often displace a heavy load (the displacement of another integrator, for instance). So the problem with the machine has always been the amplification of torque.

Bush and Niemann were the first to provide an answer for this problem. Together they designed a torque amplifier based on the concept of a capstan. The mechanical torque amplifier essentially consisted of friction bands wrapped around oppositely rotating drums and fixed to two armatures rotating coaxial with these drums. The idea was that, one armature is rotated, one of the friction bands is tightened against the drum as the other is loosened; thus, the slack in that band is then instantaneously corrected by the rotation of the armatures, causing the latter armature to rotate with an increased torque. The latter armature, fixed to a shaft, would essentially have the same rotation as the former armature fixed to the integrand shaft. As long as the rotation of the armature did not exceed the rotation of the drums, the correction process was instantaneous. Hence, the output of the wheel would be able to turn a considerably heavier load without causing the wheel to slip.

Dr. Bush had much success with his mechanical torque amplifier, but others who used them in the operation of a differential analyzer were not so satisfied with their stability and reliability. Scientists sought a new method to amplify torque. Dr. Author Porter, one such pioneer, was one of the first scientists to design electronic servo-mechanisms to address the torque amplification problem. A servo-mechanism is a device used in control systems to adjust a particular mechanism by means of an error-sensing feedback. A differential analyzer type servo needs an error sensor, potentiometer, control-loop, and servomotor. The error sensor detects the displacement of the wheel, the potentiometer measures the error, the control-loop takes the error signal and re-sends it to an H-bridge that regulates voltage, and the servomotor turns by an amount consistent to that of the wheel.

The Marshall differential analyzer has such a device to amplify the output torque of the wheel. It was designed by the project's technical advisor, Mr. Tim Robinson, and is at least conceptually similar to Porter's first designs. The primary difference in Tim Robinson's design is a micro-processor (called a Motor-Vater) that is used to interpret the error signal as a digital number and then relay that signal to the servomotor. The error of the displacement of the wheel is proportional to the derivative of the error, the integral of the error, and the error itself. The Motor-Vater is convenient because when one tries to get the servomotor to mimic the rotation of the wheel in "real-time," these proportionality constants are parameters of a dynamical system of three equations and three unknowns. Using the Motor-Vater as the control-loop provides a programable environment so that these parameters may be easily adjusted with the use of a computer instead of soldering new wiring connections on a circuit board. Although the Motor-Vater is a micro-processor and the torque amplification is semi-digitally controlled, the solving of differential equations by the differential analyzer is strictly an analog and continuous process. It is worth noting that the purpose of the servo-system is to track the motion of wheel in "real-time." The Motor-Vater sends the signal to the servo motor, but the Motor-Vater gets all of its information from the rotation of the wheel, and it is the wheel's rotation that governs the servo-motor.

The set-up for the error detection component consists of an L.E.D. light emitter and sensor, which is wired to a potentiometer. Placed in between the emitter and the sensor are two circular plates of polarized material. The polarization of these plates

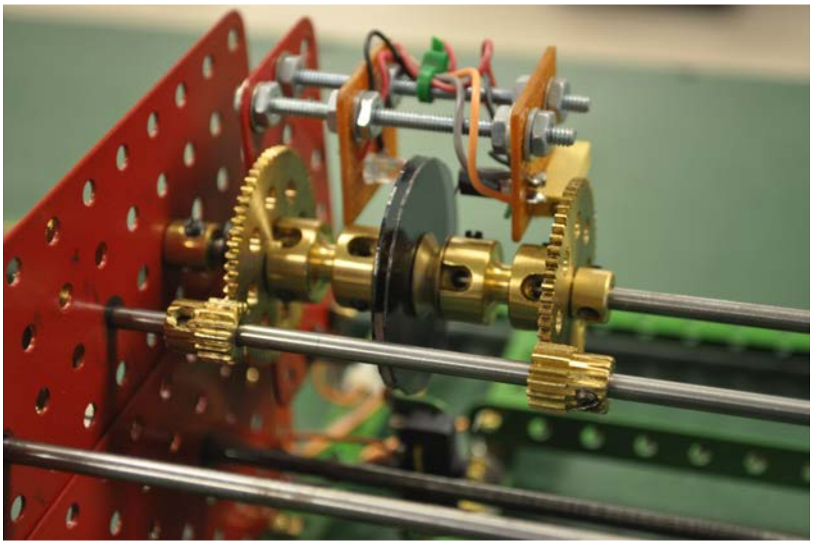

Figure 2.12. L.E.D. Error Detection assembly; The circular polarized plates and the light sensor and emitter are shown.

is such that, when the two of them are superimposed, some of the light is able to pass through the two plates. Furthermore, the amount of light that passes through them depends on the angular displacement of one plate with respect to the other plate. For example, suppose Plate 1 rests at some equilibrium position that is a fixed point in the polar plane of the plate. There is a constant amount of polarization at that point. Now let Plate 2 turn through some portion of a turn in the clockwise direction. As Plate 2 is rotated, the amount of light passing through the two Plates will increase. So the polarization of the two plates increases when the angle increases. When one plate is rotated 180 degrees with respect to the other plate no light passes through, and with another 180 degree rotation of that same plate all light passes through. The equilibrium position of the two plates occurs when the two plates are at the same degree of polarization. So when one plate is rotated clockwise more light passes through, and when that plate is rotated counter-clockwise less light passes through.

Figure 2.12 is the side view of the error-detection system; both plates are glued to two separate pulleys that are socket-coupled with their own 60 tooth gear wheel.

These two assemblies have a single rod journaled through their center with the rod fixed so that it may not be rotated; it is held in place by a conjoining rod-socket attached to a flat plate mounted on the frame of the integrator. The two plate assemblies are free to rotate independent of one another. For additional support another rod coming from the wheel bridge is connected to the assembly rod via a three hole connector. (This support structure can be seen in Figure 2.8 to right of the wheel.) This structure helps hold the assembly parallel to the wheel rod so the the 60 tooth gear on it may be properly meshed with the gears on the wheel rod.

Journaled on the wheel rod is a 15 tooth pinion gear placed so that it will mesh with the 60 tooth gear on the Plate 1 assembly. Another rod journaled through the assembly plate carries another 15 tooth pinion gear that meshes with the second 60 tooth gear on the Plate 2 assembly. These two rod are lined up end to end but rotate independently. It looks as though they are one, but the wheel rod simply rests in the lip of the second pinion, which acts as bearing for the wheel rod. Behind the Plate 2 assembly apparatus is another flat plate, providing support for the rod carrying the second pinion. Also attached to this rod is a 57 tooth gear wheel meshed to another 57 tooth gear wheel attached to a rod extending beyond the spline-shaft area to a 133 tooth gear wheel mounted on another flat plate structure. The large black 133 tooth gear wheel is driven directly by the servomotor (7:1 gear ratio) meshed with a 19 tooth pinion attached to the motor sprocket. The servomotor is also mounted on this same flat plate structure (See Figure 2.13).

The brass pulley seen behind the wheel in Figure 2.9 acts as a rotator-bearing for the wheel rotation. There is a break between the rotator bearing and the wheel rod. That is, they are essentially two separate rods. The rotator bearing rod is connected to the 133 tooth black gear wheel through a set of 1:1 gear trains. When the servomotor rotates the 133 tooth gear it also turns the rotator-bearing. The

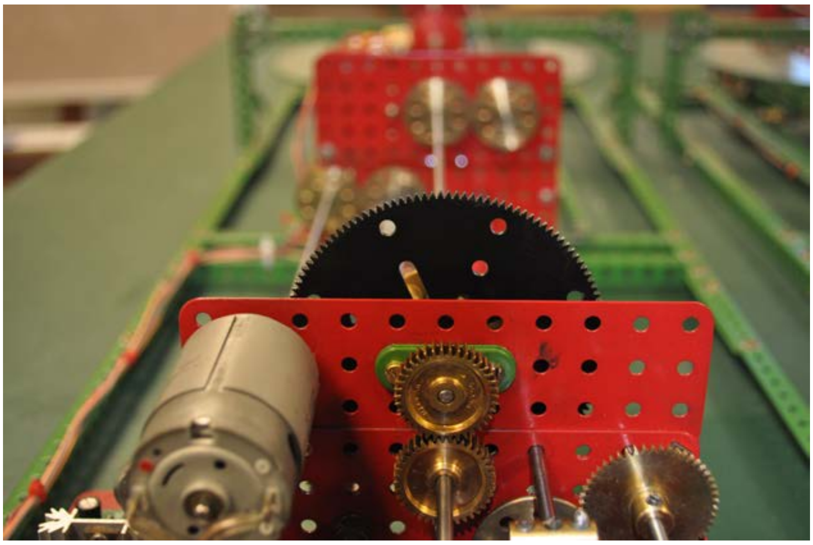

Figure 2.13. Back View of the Servo Motor.

wheel rod rests in the lip of the rotator-bearing, and as the wheel rotates the rotator-bearing rotates with the wheel, however. The rotator-bearing is being driven by the servomotor. Because the rotation of the wheel and the servomotor are designed to be the same, the rotation of wheel is essentially frictionless.

To recap, when the wheel turns, the Plate 1 assembly turns with a 4:1 ratio to that of the wheel. When Plate 1 turns there is a change in light (positive or negative). This change in light is measured by the potentiometer; the potentiometer sends that signal to the micro-processor; the micro-processor turns that signal into a digital number; that number is relayed to an H-bridge that regulates voltage; and that voltage is sent to the servomotor. When the servomotor is turned, the big 133 tooth gear is turned and its connecting rod is turned as well. The connecting rod turns a 57 tooth gear, meshed to another 57 tooth gear, which turns the pinion meshed with a 60 tooth gear attached to Plate 2 assembly. Hence, the change in light is now corrected and the polarized plates are again at their equilibrium position, now at a different reference degree of polarization. Also note that the rotator-bearing turns at the same rate as the big 133 tooth gear driven by the servomotor, and again the wheel is essentially frictionless. The reason this system works properly is that the process happens very fast; the reaction time of the servomotor is nearly instantaneous, due to the fact that we are measuring the error as a change in light. As far as reaction time is concerned, if there was a time lag from wheel to output shaft, all other integrators would presumably have that same time lag. Since the machine operates relative to itself, those differences could be quantified and subtracted out of a particular solution.

When a considerable time lag occurs, such as when a reversal of the wheel takes place, a frontlash unit can be installed in most integrators after the torque amplification. The frontlash unit adds a shaft rotation or two to the output before the reversal takes place, compensating for the time lag. The Marshall DA team has plans to build a frontlash unit because a considerable lag does occur when an integrator wheel passes through the center and reverses direction. All things considered, the error that a single integrator produces, from input to servomotor output, is small enough to be considered negligible; and, if need be, the scale of a unit variable may be expanded so as to get more precise results. This concept is discussed in further detail in Section 4.

2.7. CLUTCHES

The motion of the 133 tooth gear now tracks the motion of the wheel and hence is the output for the integrand shaft. One very convenient component on the integrand shaft is the spring-loaded clutch assembly which is geared to it. The flat plate structure to which the tracking gear is mounted has a 1:1 gear train meshed to it. This transfers the motion of the output shaft to the same vertical plane as the differential and integrand shafts. The gear ratio is 50:57 from input shaft to the actual integrand shaft that is coupled to the lead-screw (See Figure 2.14).

The addition of a clutch allows easy disconnection of the integrand shaft from the rest of the machine. This is convenient because the initial displacement of the integrator wheel from the center of the disk represents an initial condition for a particular differential equation. An initial displacement is also the lower limit for the definite integral calculated by the integrator. (If the clutch was not there the 50:57 gear train would have to be disconnected via the loosening of a setscrew each time an initial condition is changed.)

In front of the flat plate structure where the tracking gear is mounted is a rectangular box structure where the clutch assembly is mounted. The box acts as a traveling medium for all three primary integrator shafts, integrand, differential, and integral shafts. The box is constructed using four equal sized flat plates connected together by four double angle strips, and the base of the box is rigidly fixed to the base of the integrator frame. The actual clutch assembly, a Tim Robinson design, consists of one 50 and one 57 tooth gear, meshed on a diagonal (1 hole apart) with the center of the smaller 50 tooth gear in the same vertical plane as the other two primary integrator shafts.

Journaled through the box and holding the clutch gears in place, are two independently rotating rods. The lower rod carrying the 57 tooth gear also carries a bushed wheel, socket-coupling, small spring, and several collars. The socket-coupling is connected to the 57 tooth gear so that the socket-coupling and gear rotate together; neither is attached to the rod. The bushed wheel is in front of the socket coupling assembly and is attached to the rod with a setscrew. Hence, when the rod is turned so is the bushed wheel; but the gear socket-coupling assembly is allowed to spin freely on the rod. Extending from the holes drilled in the outer circumference of the 57 tooth gear are two rod adapters. Because the boss head on the rod adapter is threaded, it is easily fixed to the outer circumference of gear via a bolt. In addition to spinning freely on the rod, the gear socket-armature may be slid along it. Hence, the rod adapters are able to slide in and out of the collinear holes in the bushed wheel. A collar and spring are in place behind the gear socket-coupling armature such that the spring applies constant sliding force to the gear assembly, in the forward direction, by pressing against the collar that is fixed to the rod. So when the gear socket-coupling assembly is slid backwards, the rod adapters slide back through the holes in the bushed wheel but are not free from them. The holes in the bushed wheel act as a guide for the 57 tooth gear, keeping the gear and the bushed wheel in the same angular alignment, as well as the assembly rod. But when the assembly is slid backwards, the 50 and 57 tooth gears are now out of mesh and disconnect the rod that that turns the lead-screw (assembly rod) from the rod carrying the 50 tooth gear.

So now the lead-screw may be turned and an initial condition adjusted without affecting any other motion of the machine. When the gear assembly is released, the rod adapters snap forward through the holes in the bushed wheel so that the 50 tooth gear, in the fixed position on the adjacent rod diagonal to the assembly rod, is back in perfect mesh with with the 57 tooth gear journaled on the assembly rod. Hence, when the assembly is released the motion of the lead-screw is transferred through the gears so as its motion may be governed via the gears that feed it. Another rod is journaled through the box one hole directly above the assembly rod. Its purpose is to carry a lever that will move the gear socket-coupling assembly back and hold it in place so that the gears remain out of mesh. The lever is made out of a three-holed rod coupling with the center hole on the lever rod and two other short rods extending downward from the other two holes in the rod coupling, orthogonal to the socket-coupling assembly. These two rods snag the lip of the socket-coupling assembly, so that, when the lever-rod slides back and forth through the holes in the box, the gear socket-coupling assembly slides with the lever on the assembly rod and pulls the gears in and out of mesh. Collars are fixed to the lever rod to provide a stop, and a double

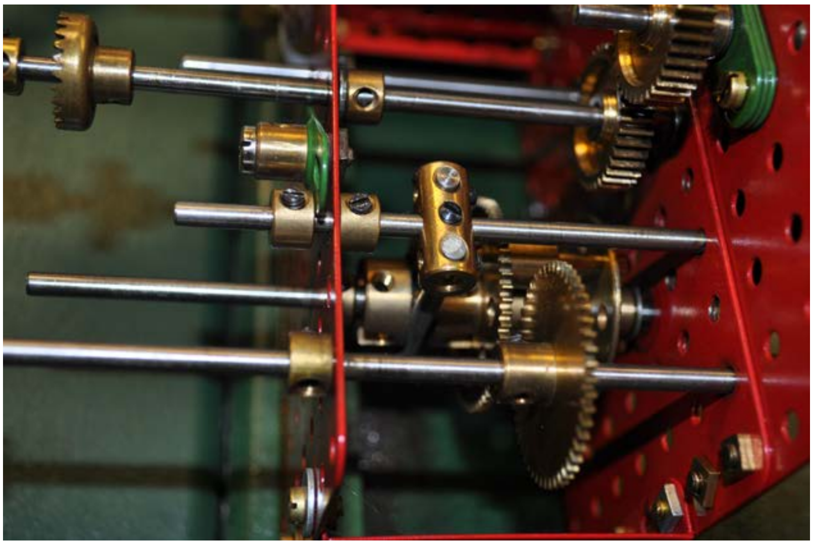

Figure 2.14. Top view of clutch assembly.

arm crank is in place next to the lever rod so that it is able to hold the lever rod in place by catching the lip of the stop collar (See Figure 2.14).

The clutch assembly, very important and necessary, was an addition that the Team added to the machine. Having a clutch makes resetting an integrator a quick and easy procedure. The use of a clutch also saves wear and tear on various parts that would result in replacement.

2.8. SECTION OF INTERCONNECTION

A section of interconnection is analogous to a grid-pathway on a circuit-board. The purpose is to transmit the motion of any output to any corresponding input among the various components in the machine. For example, when solving differential equations, it is often necessary to use the output of an integrator to displace the lead-screw input of a second integrator. Within the section of interconnection which is used to transfer motion, there are 16 helical gearboxes (See Figure 2.16). The helical gearboxes allow the motion to be transmitted in an orthogonal direction from whence it came so that it may be connected to another gearbox, essentially driving some designated input shaft on another integrator. The majority of the section is constructed from 36 holed angle-girders connected together by 5 by 5 holed flat plates at the corners. Two 36-holed angle-girders and two square flat plates create a two dimensional rectangular structure. Two of these are connected together by two 25 holed angle-girders, one top joint and one bottom joint (See Figure 2.16). This three-dimensional rectangular structure (sub-section) is 36-by-25-by-5 holes and serves as the frame for the channel bearings that guide the rods. To create a proper-sized avenue of connection, we used four of these boxed structures. Any two pieces of angle may be connected together long-ways by staggering a short third piece of angle in the break between the two pieces. This construction is done to two sub-sections to create one-half of the full section of interconnection. The two halves are connected by another means.

It is convenient to have the availability to shut down half of the section when it is not necessary for operation. This disconnection is easily achieved by simply not allowing any rod motions to pass through the guide holes, called channel bearings, at the break in a half-section. (In this case these bearings are referred to as cross-ways channel bearings.) The channel bearings for the rods are simply the holes that already exist in the pieces of angle, although it helps to cover the angle with flat strips, lining up the corresponding holes, creating a more sturdy and rigid bearing. The two input shafts and the output shaft of an integrator or any other component are in the same horizontal plane with a set of channel bearing holes. (These channel bearings are orthogonal to the cross-ways channel bearings.) When a 15-inch rod is journaled through the channel bearings in the section, the section rods and the corresponding component input/output shafts line up perfectly. The two are connected together with two contrate gears so that, when a component shaft turns, so does the section rod (or vice-versa) (See Figure 2.21).

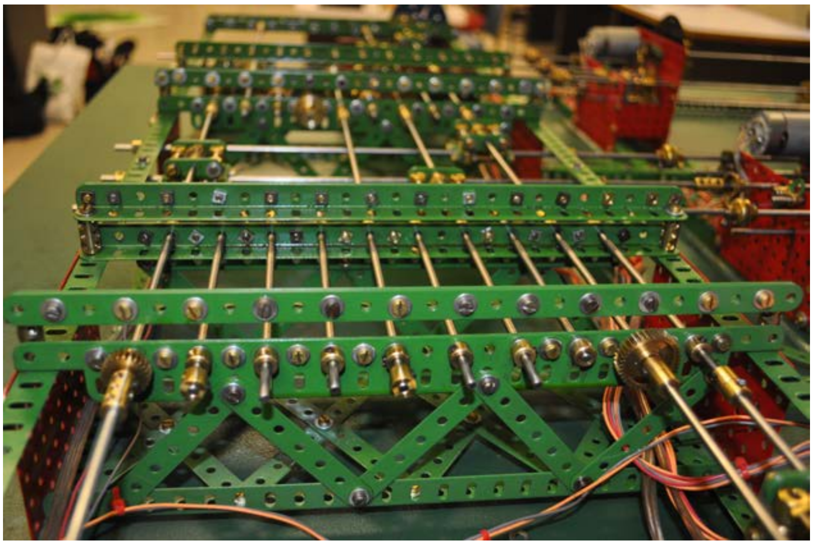

Figure 2.15. This is a view of the channel bearings that carry the cross-shafts in the section, simply made from angle girders.

The other set of channel bearings (cross-ways channel bearings) are connected on top of the first set of channel bearings. These cross-ways channel bearings are double-stacked in that they are mounted on the existing channel bearings that have already been mentioned. That is, two 25-holed angle-girders are connected together, one on top of the other by a long bolt and a three holed connector, acting as a spacer, then fastened down to the section with a nut (See Figure 2.15).

The crossways channel bearings are also supported with corresponding flat strips. The reason for the double stack is so that a gear train may be meshed on a diagonal between holes in the two girders in the stacks. The standard mesh space for a typical pair of gears is one inch. The diagonal could be thought of as a hypotenuse of a right triangle where the distance, measured from center of hole, between holes on adjacent horizontal and vertical planes are the legs. Since the spacing of holes on an angle-girder is 1/2 inch and we want the diagonal to be one inch, we know we need to make the height between stacks measured from center of hole to center of hole, approximately 0.866 inches, by the Pythagorean Theorem. It turns out that using a three-holed coupling as a spacer provides a proper mesh.

Four double stack channel bearings are mounted in the middle third portion of each half section, and four single stacks are mounted on the ends. These are arranged in such a way so that one half section is divided into three 25-holed portions, two 7-holed portions and two 1-holed portions. So starting from either end there is mounted a channel bearing structure, then 25 holes down another channel bearing, then 7 holes down another channel bearing, skip 25 holes, and again; this pattern repeats until the other end of the section is reached where the last channel bearing is placed. Each pair of channel bearings has an appropriate rod length journaled through them that is fixed into place by collars. Note that throughout, all rods journaled through crossways channel bearings will be called cross-shafts, and the other section rods will be called bus shafts. So every hole in the lower deck of the crossways channel bearing will have a cross-shaft journaled through it. However each cross-shaft will not span the entire length of the section. Instead, each one will only span through a pair of channel bearings. The cross-shafts need to be long enough to carry a washer, gear, and female dog clutch on both ends protruding from the channel bearing. So for the 7-holed pairs of available space, the cross-shaft length is 6.5 inches and for the three holed pairs it is 3.5 inches. There are 23 holes available in each cross-ways channel bearings, and one cross-shaft is journaled in every other hole in the lower deck (Note that a cross-ways channel bearing spans the width of the section so that, when any cross-shaft is journaled through them, the cross-shaft will be orthogonal to any bus-shaft). All together there are 32 cross-shafts in one half section, including nine 7-inch rods contained in two 7-holed areas, and six 11-inch rods contain in two 25-holed areas. Remember that the bus-shafts run orthogonal to the cross-shafts. There are three bus-shafts placed in the 25-holed area between two pairs of cross-ways channel bearings corresponding to the inputs and outputs of various components. The setup is similar in the other bus shafts in the subsequent 25-holed areas in the rest of the section. There is space between the height of a bus-shaft and the height of a cross-shaft. This space allows two helical gears to be meshed from bus-shaft to cross-shaft so that a bus-shaft may turn a corresponding cross-shaft and vice-versa. Due to the flexibility of the thin rods, a gear-box is in place so that the helical gears do not slip out of mesh.

The gearbox design is very simple yet extremely reliable. Tim Robinson's design consists of twelve 3-holed flat strips, four threaded bosses, four screwed rod socket bosses, four 0.5 inch threaded rods, and eight bolts, washers and nuts. One 1.5 inch threaded rod is screwed through the holes tapped in the side of two threaded bosses. The bosses are spaced about one inch apart on the threaded rod. Two nuts are in place in between the two bosses on the threaded rod as well. These are to be tightened against the boss in order to hold it in place. This process is repeated three more times, making sure that threaded bosses are in pairs together on the same threaded rod and screwed rod sockets are in pairs together as well. The threaded bosses must be screwed on all four screwed rod sockets in order to create a box. Note: This procedure is a very tricky because each threaded boss must be screwed on a screwed rod socket first and then the threaded rod must pass through the adjacent holes. Once a box is formed, the 3-holed flat-strips are joined, three at a time, to make a channel bearing for a cross-shaft and a bus shaft. They are attached to the holes in the bosses via a washer and bolt (See Figure 2.17).

The gearbox cross-shaft is 11 inches and the gear box bus-shaft is 15 inches. Bus-shafts are journaled through the local section channel bearings and each gearbox cross-shaft is journaled through a gearbox. All other cross-shafts are journaled through cross-ways channel bearings. As there are three bus shafts in each middle third section, there is a total of nine gear boxes in a one half section. The idea is that any gear box may slide along the bus-shaft and align with any desired cross-shaft.

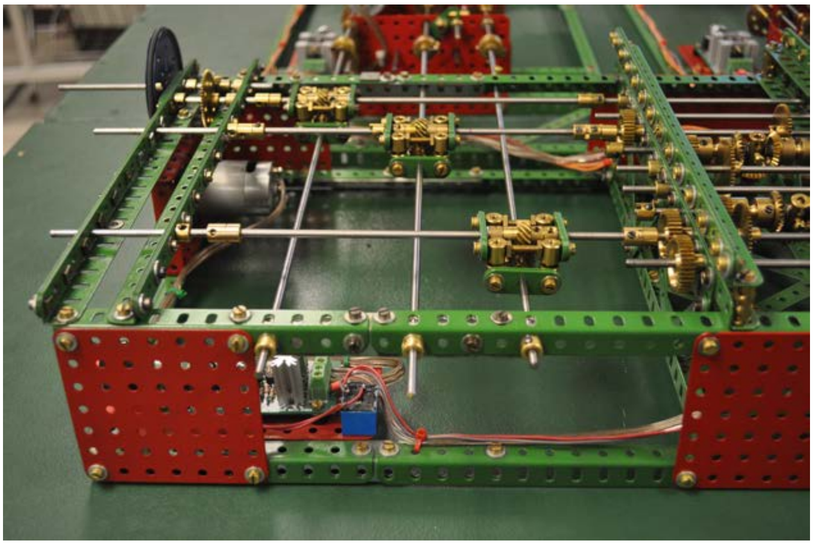

Figure 2.16. View of one subsection with Gear Boxes in place as they are connected various components.

The gears may be attached to the rod with a set screw. Journaled on the bus-shaft and cross-shaft of each gear box are helical gears, but they are within the confining perimeter of the channel bearings. Spaced properly with collars and washers, the gears mesh perfectly in the center of the gear box. Corresponding cross-shafts, from section to gear box, are connected with a female and male dog clutch. With this setup, any corresponding bus-shaft may turn any desired cross-shaft by simply moving the gear box as needed; thus creating an avenue of interconnection for the inputs/outputs of various components of the machine.

2.9. ADDER

The adder component resides in the section of interconnection where the 7-holed area of space is provided. The space is defined by two double stacked cross-ways section bearings. The adders are placed on the rods that are journaled through the lower deck of the bearings (See Figure 2.18). An intricate design of differential gears, the adder is used to combine two shaft rotations as a sum. (This includes variable

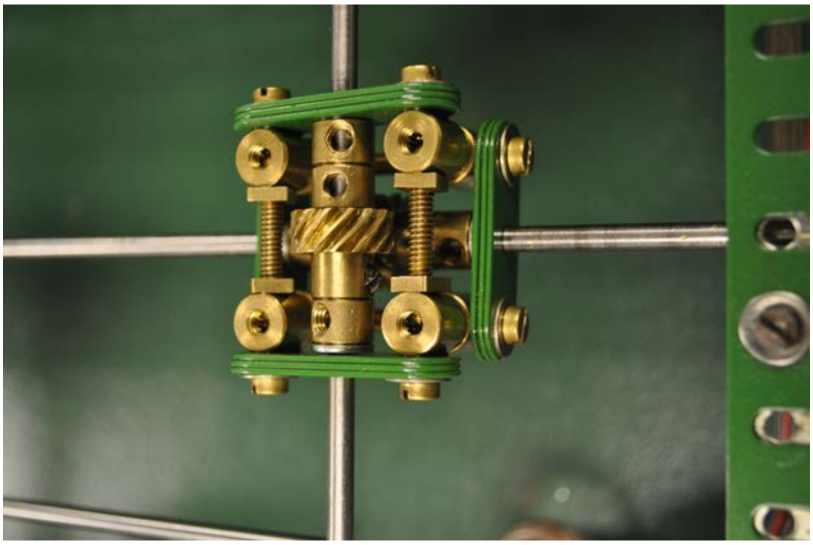

Figure 2.17. Up Close View of the Helical Gear Box; the rod journaled below is a bus-shaft that goes to the integrator, and the rod journaled above is a cross-shaft that continues along the section.

shafts.) The adder is also used to create different types of gear ratios by adding shaft rotations of two rods that are already geared up or down by particular gear trains. For example, suppose a parameter of 5/6 is needed. Among the gear trains available, 1/2, 1/3, and 1/4, an operator could create a gear train of 5/6 by adding a 1/2 gear train to a 1/3 gear train. The adder has two inputs and one output, and, from its design, when a quantity is imputed, that quantity is negated. (Two gears naturally provide a negation because the gears in a meshed-pair will turn in opposite directions.) So if a positive quantity is to be added it must be negated by a gear train before it is sent into an input.

Among the parts required to build an adder are three 25 tooth bevel gears, two socket couplings, two 25-tooth pinion gears, two 50-tooth gear wheels, one threaded coupling, one pivot bolt, and various setscrews, washers, collars, and bolts. First one 25-tooth bevel gear is socket-coupled to one 25-tooth pinion gear, this assembly is repeated so that there are two pieces. The third bevel gear is attached to the threaded

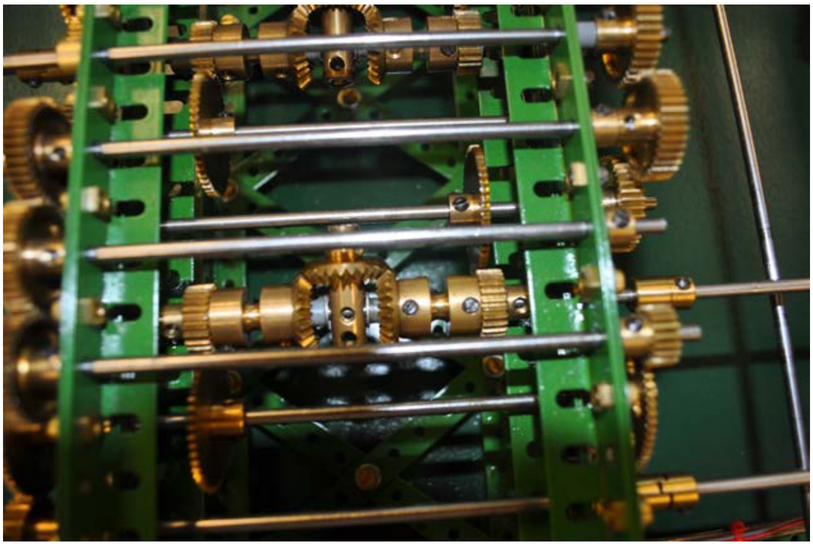

Figure 2.18. A view of a differential gear system used as an adder.

coupling with a pivot bolt so that the gear is free to spin on the smooth axle of the pivot bolt. These three bevel gears are journaled on a rod, with the pivot bolt bevel in the center to create a differential gear, properly spaced with washers so that the rotation is smooth.

A differential gear has an interesting property; the rotation of the pivot gear is determined by the rotations of two socket gears at the same time. The differential in the adder is analogous to the differential gear that drives the wheels through the axle on a car or, in the case of the adder, the wheels are the drive for the axle. Because the two bevel gears are socket-coupled to the pinion gear, the rotation of the socket-coupling assembly does not affect the rotation of the assembly rod. When a socket gear is turned so is the pivot gear, and, when both socket gears are turned at the same time, the rotation of the pivot gear is affected by both gears. Note that when the two socket gears are turned at the same time and in the same angular direction, the pivot gear locks and will rotate coaxially with the socket gears along with the assembly rod. We will use this property to add the motion of two quantities measured in terms of shaft rotations.

As a consequence of the design, when the pivot bolt rotates the threaded coupling rotates coaxial with the axle rod. If a set screw is added to the threaded coupling, when the coupling revolves about the axle rod, the axle rod itself turns as a result. So the rotation of the coupling and the axle rod have the same magnitude. Remember that the socket gears are not attached to the rod so the rotation of the those gears and the rotation of the axle rod are not in conflict. There is a 2:1 ratio between the socket gears and the rotation of the axle rod. That is, if a socket gear is turned twice, the axial rod is turned once. Note that if both socket gears are turned once at the same time, then the axle rod turns once. So one divided by two plus one divided by two is one; this is good. Now we can cancel out the division by two by simply multiplying each input by two before it reaches the pivot gear. This reduction is achieved by meshing the 25-tooth pinion gear on the other side of the socket-coupling with a 50 tooth gear wheel. (See Figure 2.19.) Two 50-tooth gear wheels are journaled and attached to rods placed adjacent to the assembly rod (also, now interpreted as the axial rod) on each side. Both serve as inputs for the adder. Now because we have accounted for the division by two of the differential gear, by simply multiplying each input by two, one rotation of both adder inputs at the same time, and in the same direction results in two rotations of the axle rod. A sum is produced in terms of shaft rotations. Note that a sum is the result of the two input shafts for any portion of a turn at any moment in time. Also if the two input shafts are rotated at the same time, at the same speed, but in opposite directions, the two inputs cancel each other out and the axial rod does not turn. This cancelation is due to the fact that the pivot gear is a differential gear. Another way of interpreting this is that the motion of one socket gear is passed through the differential (pivot gear) and received by the second socket gear. In this case, the differential gear is a

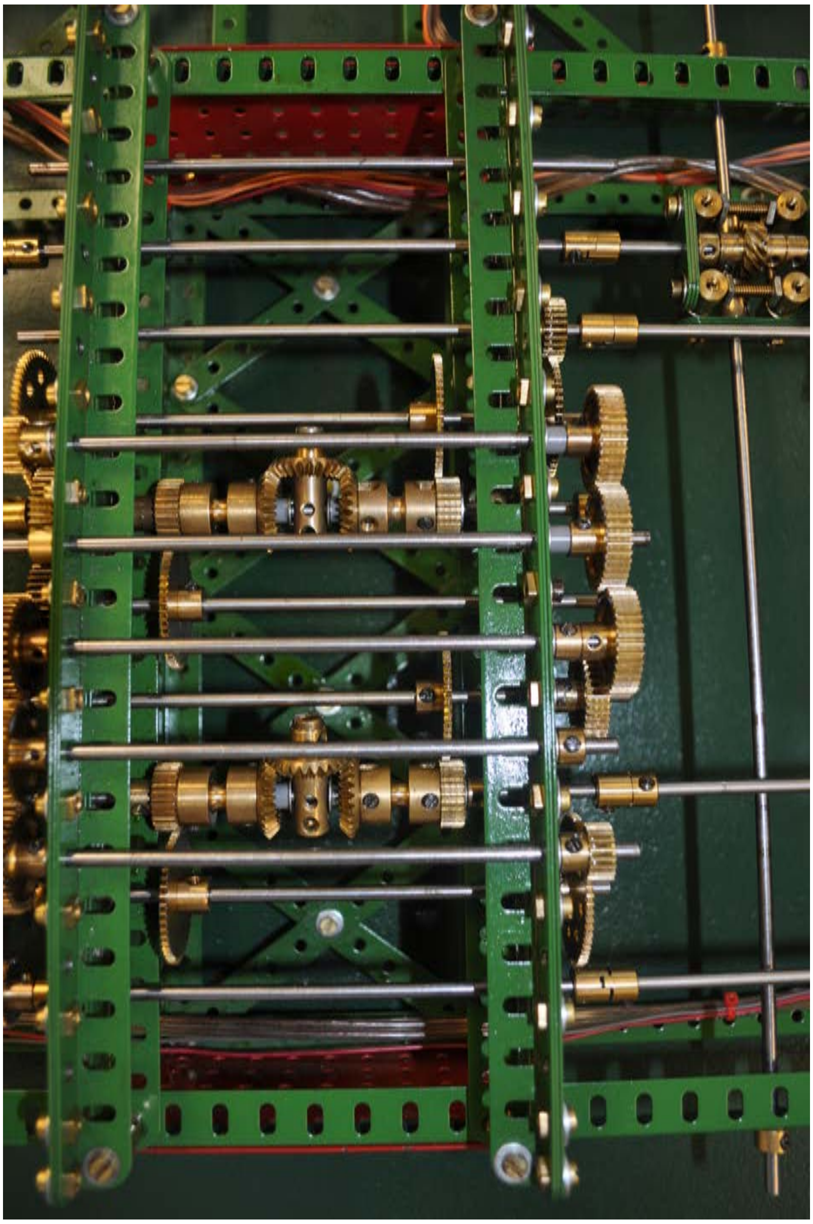

Figure 2.19. Two Adders as they are available inside the section.

medium that preserves magnitude but reverses direction for the motion of the two socket gears.

2.10. INPUT TABLE

The Input Table is designed to convert a graph, plotted in a Cartesian coordinate system, into an analogous quantity in terms of shaft rotations so that it may be sent into the interconnection of the machine and represent some non-homogeneity of a particular differential equation. The concept is similar to an Etch-a-Sketch, where rotational quantities correspond to horizontal and vertical motions. The plotting surface of the table is an 11 by 17 inch piece of 3/4 inch plywood. The plywood is attached to a primordial frame so that the plotting surface rests on the 45 degree angle formed by two surfaces perpendicular to each other. The surface is literally held in place by the hypotenuses of the two right triangles that define the ends of the primordial frame. The frames base is built out of angle-girders. At its base two 7-holed angle-girders are connected with two 36-holed angle-girders with one hole extending from each end. This makes a rectangular base. The 25-holed angle-girder is attached up-right to the base perpendicular to it. This construction is repeated on both ends creating two surfaces orthogonal to each other. A final 15-holed angle-girder connects the indices of the two surfaces to complete the primordial frame. Several 36-holed girders are attached to the back of the plywood, connected to the girders that connect the indices of the frame. This frame also acts as a carriage so there are three axle rods journaled through the base of the frame. On the axle rods are pulleys, creating a moveable carriage. The rails act like the rails for the integrator carriage, but are made out of 36-holed pieces of angle. Also on this frame is a threaded boss that will drive the table carriage, similar to that of the integrator. (See Figure 2.20.)

Because the table needs to be displaced 17 inches in both directions, some care was taken in the placement of the rods in the frame carriage. The rods need to be journaled in the end holes in the carriage so that the wheels don't have to roll over a break in rail road because two 36-holed girders are used to cover the entire 34 inches required for the displacement of the carriage. Additionally, the third axle rod is journaled through the center hole of the base of the frame carriage, so it needs just one girder to displace the full range of the carriage. Note that there are only three wheels riding on rails that support the frame carriage, two in the front and one in the

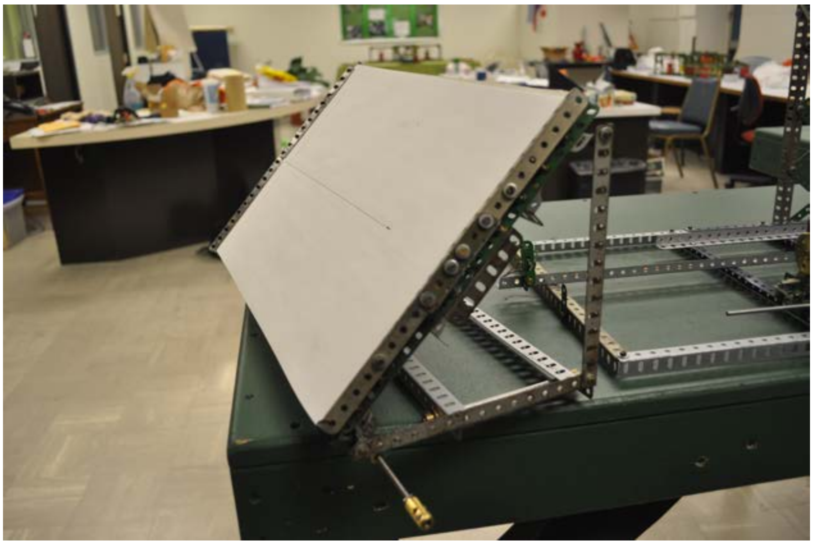

Figure 2.20. A Side View of the Input Table Cart.

back. This configuration is designed to account for any discrepancy in the alignment of the two parallel rails, because several pieces of angle had to be pieced together to make the rail road and the angle ends are beveled.

Once a movable surface in the direction of abscissa (horizontal axis) is established, an ordinate (vertical axis) direction must be established. Another movable carriage is mounted above the surface of the table, also on a 45 degree tilt, to serve as the ordinate. Located behind the frame carriage, mounted on the rail road frame, is a rectangular tower, also made from 15-holed girders and supported by flat strips along its height for rigidity. The tower extends over the height of the frame carriage; at the peak of the tower is a rod across its width, journaled through two flat truions, mounted on each end of the top of the tower. On this rod is one screwed eye bolt that pivots on the rod. At the bottom, in front of the frame carriage, is a box made from 5-holed angle girders and large bushed wheels. One rod is journaled through the bushings in the large bushed wheels and another is journaled through the girders

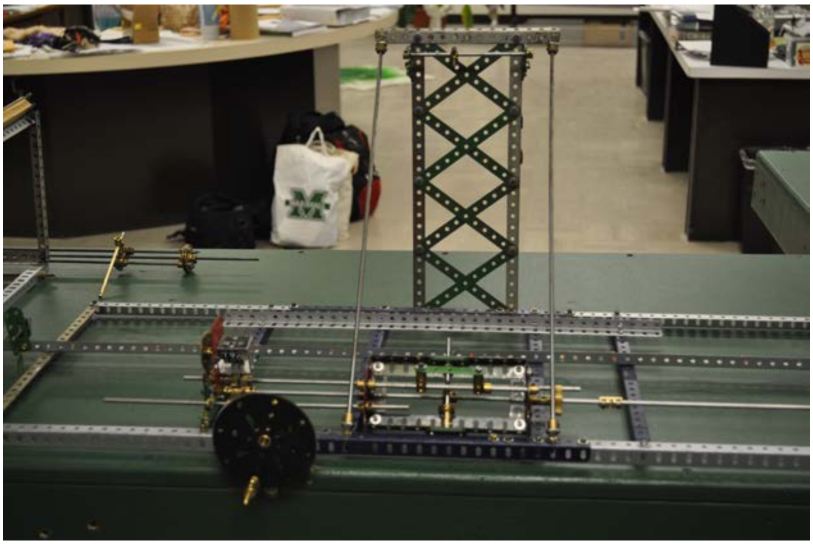

Figure 2.21. The Input Table frame is comprised of a tower located behind a hand crank (Black Gear) drive mechanism. The tower carries the ordinate rails on which the Input Table carriage rides. The helical gear train drives the ordinate carriage.

that connect the bushed wheels. The two rods journaled through the box are orthogonal to each other so that one rod is parallel to the cross-section and the other is parallel to the surface of the table. Two helical gears are on these rods and in mesh with one another as well. Additionally, on the rod parallel to the table is a threaded coupling. Attached to the coupling is a long threaded rod that is in line with the center threaded eye bolt at the top of the tower (See Figure 2.21).

The structure of the carriage is simple; it's made from two three-holed couplings connected with a three holed flat strip. In the center hole of the flat strip is a threaded boss. This simple carriage is journaled on two long rods, held in place by a bushed wheel and collars, one on each end of the two rods similar to the spline-shaft assembly. The carriage assembly serves a much different purpose than the spline-shaft. The bushed-wheels act as bearings for the long threaded rod that lines up with the eyebolt at the top of the tower, and that same threaded rod is screwed through

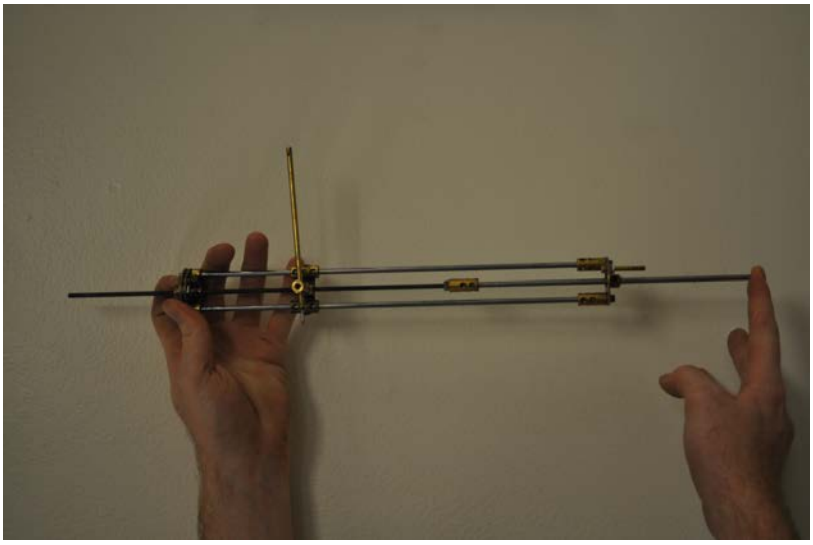

Figure 2.22. This is the Ordinate Carriage Assembly for the Input Table.

the threaded boss in the carriage. Once everything is lined up, the boss head in the upper bushed wheel of the assembly is fitted overtop of the eyebolt on the tower rod. The eyebolt is fixed to the tower rod, via a set screw, so that it no longer pivots (See Figure 2.22).

A reference pointer, or pen attachment, is also mounted to the vertical carriage so that the Input Table may also be used as an Output Table. Two rods are added from the top of the tower to the rail frame to provide extra support for the tower. There is a revolution counter mounted to both horizontal and vertical shafts with the same 10/3 ratio reduction, and a clutch box separates the horizontal and vertical shafts from the section. This design is similar to the integrator clutch boxes. The rod that essentially displaces the vertical shaft also leads to a train of gears that leads to a hand crank. This configuration allows the operator to move manually the carriage up and down the vertical axis of the table. The hand-crank shaft goes through several 1:1 gear trains, through the clutch box and into the section where it may be connected to a bus shaft on the machine to which the counter is connected (See Figure. 2.23).

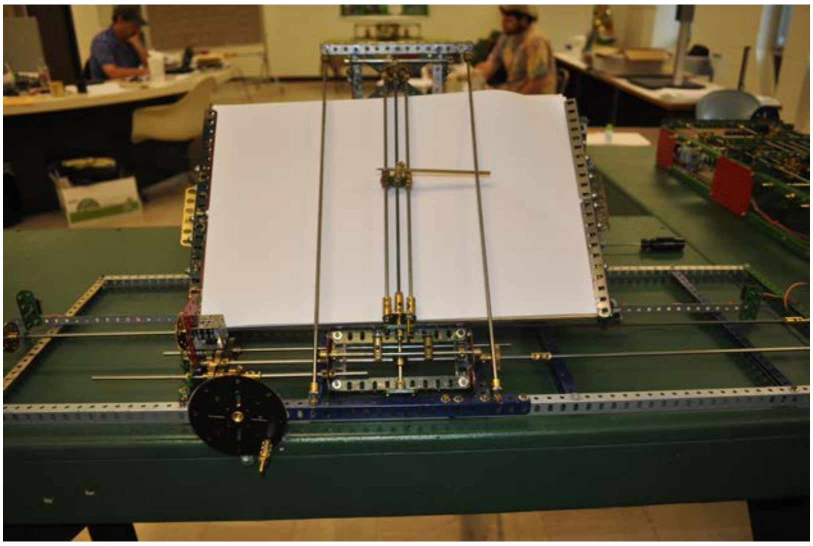

Figure 2.23. Full view of the Input Table as a whole unit; the black gear in front is the hand crank used to control the ordinate carriage and essentially transfer motion to the appropriate cross-shaft in the section.

In order to create a function on the table, it is best to let the differential analyzer do the drawing. An operator simply sets up an equation designed to provide a solution to a desired function. Once a solution is provided by the machine, an operator resets the carriage to its initial reference point, the origin, and an operator turns the black crank located in the front of the table in Figure 2.23. The independent variable for the function moves the table in the horizontal direction and as the table is displaced an operator turns the crank in order to keep the pointer on the pre-drawn curve. As long as the pointer stays on the curve, the number of turns of the crank represents the behavior of the function plotted on the table in terms of shaft rotations. The

Input Table is most convenient when a piece-wise continuous function is needed, but the concept of inputting functions will be discussed further in a later section.

2.11. OUTPUT TABLE

Conceptually, an Output Table is similar to an Input Table. The only differences are both the horizontal and vertical motions come from the machine and the reference pointer is replaced with a pen. The purpose is to plot a solution or phase plot of a differential equation. The Output Table is designed to have a horizontal surface as opposed to the upright angled Input Table, making the construction of an Output Table much simpler. The table's plotting surface is an 11 by 17 inch piece of 3/4 inch plywood. A rectangular cart or carriage is built from 36 and 25 holed girders and the plywood is screwed to the frame. Three axle rods are in place in each end of the carriage, and one is also journaled through its center. Three pulleys are fixed to the axle rods, one on the middle axle on one side and one on each end axial rod on the other side, allowing the plane of the carriage to be governed by only three points. This arrangement is designed to avoid potential minor discrepancies in the alignment of the rail road frame as it will be pieced together like the rail road frame for the Input Table described in the last section (See Figure 2.24).

The rail road frame is an elaborate joining of 36 and 25 holed angle girders, the dimensions of which are 72 by 36 inches. Several 25 holed girders are in place along the length of the frame in order to make the structure more sturdy along the width and to reinforce the 90 degree angles of the frame. The primary rails on which the carriage rides are two 36-holed girders, placed end to end. This design allows the side of the cart with two pulleys to ride along and span a full 17 inch distance without rolling over the break in the girders. For the side of the cart with one pulley, one 36-holed girder is centered on the break 21 holes from the adjacent rail (See Figure 2.25). The square perimeter of the rails are centered on the frame, so that the plotting

Figure 2.24. Side View of an Output Table Cart; its surface lies flat as opposed to the Input Table Cart.

table will also be centered on the frame. These rails are in the direction of abscissa, and there is in place a bridge that spans the center width of the frame that will carry another set of rails, 25 holes across, high above the former set. These rails are called the ordinate rails, carrying another carriage with a pen mount in the direction of the ordinate with respect to the plotting surface. For clarification, the direction along the width of the table is the ordinate (dependent variable/vertical axis) and the direction along the length of the table is the abscissa (independent variable/horizontal axis) (See Figure. 2.26).

The structure on top of the bridge above the ordinate rails was originally designed for stability for the bridge frame. Although, its dimensions are 11 by 17 inches, it serves well as a standard sized paper tray. A train held in place by two adjacent rods journaled through the frame of the tower appears on the left side of the bridge tower. These gears, starting from the bottom up, are two 1 1/2 inch spur gears and two bevel gears. The upper spur gear rod also carries one of the bevel gears as well as a three holed coupling journaled so that the length of the coupling is perpendicular

Figure 2.25. Under View of the Output Table Cart. The axial attachments for the wheels and the lead-screw driving rod are depicted.

to the rod at the couplings center and acts as a bearing for the second bevel gear. In the protruding space available in the three way coupling there is fitted a small rod, carrying the second bevel gear and a threaded coupling. The two bevel gears mesh orthogonally and the motion of the gear trains are sent in the same direction as the ordinate. The three-way coupling on the spur gear rod actually pivots on the rod that is screwed in the threaded coupling. Essentially connected to the three way coupling is a long threaded rod held in place by two lock nuts. This threaded rod will drive the ordinate carriage that carries the pen and rests in the center between the ordinate rails. (See Figure. 2.26.)

The ordinate carriage is a combination of three-way couplings, two-way couplings, and small rods pieced together with setscrews. A three way coupling is very versatile; there are two setscrew holes tapped into each end of the coupling and one available in the center. Additionally, there is a cylindrical rod hole drilled in the center of its length and three more orthogonal rod holes bored through the side of the coupling along its length. Two way couplings are similar. These various holes

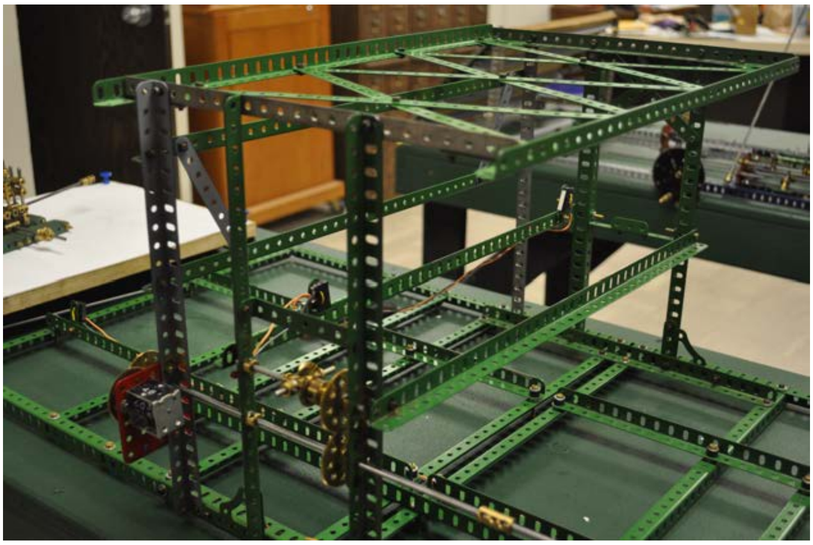

Figure 2.26. The frame of the Output Table. The tower and rail road are shown, where the above rails are for the ordinate carriage. The rails below are for the abscissa carriage.

allow for many options for connecting rods to make the ordinate carriage (See Figure 2.27).

The wheels for the ordinate carriage are actually above the center of mass. That is, the carriage hangs on the ordinate rails. In the center of mass of the carriage is a three-way coupling placed so that the center screwed hole lines up with the threaded rod centered on the ordinate rails. Hence, when the corresponding gear trains are turned the threaded rod turns and the ordinate carriage is displaced along the rails. The pen attachment is a simple 5/32 inch copper tube that fits in one of the couplings extended from the end of the carriage. The cartridge for a standard ink pen will fit snugly inside the tubing. Additionally, the pen attachment is spring loaded so as to keep the pen in contact with the table at all times. If need be, for resetting purposes, the pen may be pushed up through the coupling and held in place by small screwed-rod. The flat strips mounted on top of the carriage are to add weight, so that the force of the spring does not lift the carriage from its rails (See Figure 2.27).

Figure 2.27. Side View of the Ordinate Carriage of the Output Table. The various couplings used in its construction and the spring loaded pen are shown. Also, green flat strips are screwed together and used to stabilize the carriage on the rails.

The Output Table also has a clutch box, separating it from the section of interconnection, similar to the one on the Input Table and integrators. On the rod inside the clutch box is a threaded coupling along with a threaded rod inserted serving to drive the table carriage in the direction of abscissa. Working together, the table and ordinate carriages will provide a plot in the two dimensional plane of any two available variable motions in the section (See Figure 2.28).

2.12. MULTIPLIER

A multiplier is used to produce the product of two variable quantities. One variable shaft may always be multiplied by a constant by adding a gear train to it. However, multiplying two variables together is much different as they both are always changing. Although Marshall's machine is not equipped with a multiplier, one is being designed by the Marshall DA Team. The addition of a multiplier will expand

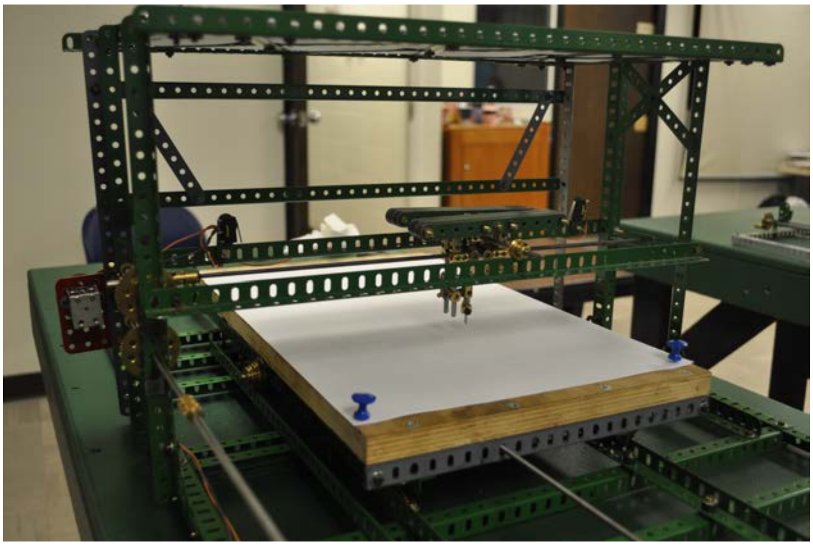

Figure 2.28. This complete view of an Output Table shows the gear train located on the left hand side leading to the the clutch box which is governed by some dependent variable from the section.

the programmability of the machine as a whole. Although the multiplier is subject to human error, much like the Input Table, it will still be a very much needed addition.

A product of two variables is always available with the use of two integrators; however, two integrators will not always be available for some higher order problems. This idea will be discussed further in a later section, but a brief mention of using a multiplier is given. The design of a multiplier unit can be thought of as a hybrid integrator/Input Table. Referring to a diagram from a paper written by Bush from the Journal of the Franklin Institute (See Figure 2.29), observe the bar in the center of the diagram. This bar is mounted to a vertical axle and rotates like an integrator disk. There is an accurately scribed line in the center of the bar, a reference line, serving a purpose very similar to the curve on an Input Table. This bar will revolve around the axis mentioned. However, the bar rotates only 90 degrees due to the drive mechanism. On the left side of the diagram there is a long threaded rod with a movable carriage. The movable carriage has a pivoting bearing mounted on it.

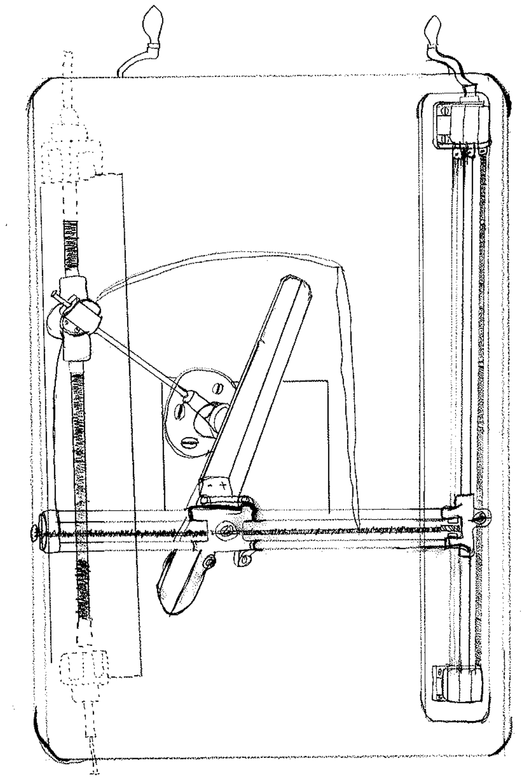

Figure 2.29. Schematic diagram of a multiplier. The Drawing is a replica of the diagram in the original Bush Paper.

Journaled in this pivot bearing is a sliding rod rigidly connected to the axle that carries the bar. So, as the threaded rod is turned, the carriage is displaced and thus the sliding rod moves up or down, essentially moving the angle on the bar. On the right side of the diagram, there are two parallel rods. The left leads to a helical gear train and then to a threaded rod that moves a pointer carriage in the direction of the ordinate. The right rod leads to a threaded rod moving the same pointer carriage

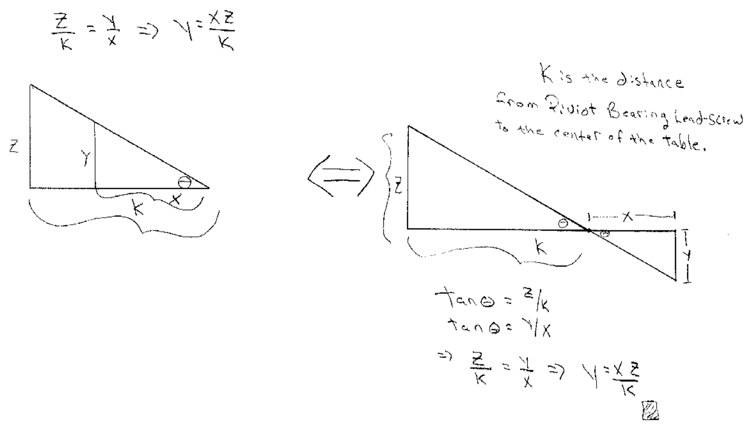

Figure 2.30. Abstraction of the action of a multiplier interpreted as similar triangles.

in the direction of abscissa. The pointer carriage is like an Input Table, but the difference in design is that the plotting surface, or table, does not move.

Figure 2.30 shows how the apparatus produces the product of shaft rotations. The common baseline for both triangles is the initial position of the rotating bar represented by the diametrical line scribed through its center. The two corresponding hypotenuses will represent the final angular position of the bar with respect to the initial position. In order for the bar to be rotated through some angle θ, the threaded rod on the left side of the Bush diagram needs to rotate through some portion of a turn. The rotations of this shaft, call it z, are proportional to the tangent of the angle θ, where the constant of proportionality depends on the fixed distance from the center of the bar to its end denoted by k in Figure 2.30. The position of the reference pointer in the direction of abscissa is x, and the position in the direction of the ordinate is y. Note that both x and y are displaced by corresponding shaft rotations. When the z-shaft is turned, a right triangle is formed. The hypotenuse that crosses through both triangles is the diametric line scribed through the center of the bar. The two corresponding legs are y, for its height and x for its base. Figure 2.30 depicts two cases: the left shows the case when the x-displacement is positive given a positive z-displacement and the right is the case where the x-displacement is negative given a positive z-displacement. Note that a negative z-displacement would flip the diagram in both cases.

The length of the hypotenuse can be easily found, given a fixed z, but it is not necessary to find it because the tangent of the angle θ corresponds to the ratio of the two legs. Also, the tangent of θ is the relationship that connects the rotations of z with that of x and y. Suppose in addition to the z-shaft turning some portion of a turn, the x-shaft also turns at the same time. The turning of the x-shaft will result in a displacement of the reference pointer along the abscissa axis. If we drop a perpendicular line from the endpoint of the x-line, to the diametric line scribed on the bar, we have two similar triangles (Refer to the right hand side of Figure 2.30). Because the angle θ is the same, the relationship, $z/k = y/x$, implies $y = (z * x)/k$, where k is the fixed distance from the center of the bar to the edge of the bar when it rests in the trivial position.

With proper gear trains, resulting in $k = 1$, the product of two variable shafts as the rotation of a third shaft will be obtained. An operator stationed at the y-shaft can turn an appropriate hand-crank and keep the reference pointer on the diametric line scribed in the bar at all times by changing y. For example, if the z-shaft turns three times and the x-shaft turns twice, then an operator would need to turn the y-shaft hand-crank six times in order to keep the pointer on the line. Note that the idea is the operator will keep the pointer on the line at all times, so using a multiplier is subject to errors depending on the operator. Therefore, if it is available, the use of two integrators is always more accurate than that of the multiplier. The degree of accuracy depends on the nature of the problem and using a multiplier may be just enough to observe the qualitative properties desired.

2.13. CONCLUSION

Building a differential analyzer is somewhat of a delicate process. The underlying principle of the machine is that all instances of precision and/or accuracy are simply relative to the machine itself. If two rods are spinning well then they are aligned straight, and for continuity gear trains are inserted in various places for smooth operation of the machine. Moreover, because all quantities are measured in terms of shaft rotations, a revolution counter is placed on every integrand shaft, for each integrator, and inputs and outputs of plotting surfaces. Additionally, limit switches are in place, hard wired to the main motor drive, to avoid any component continuously moving beyond its range. Dr. Lawrence and the DA Team, with the guidance of Tim Robinson, constructed the Marshall DA in about nine months. It has taken some time to get used to the operation of the machine; when running the machine, making sure things are mechanically operating as normal is imperative. The parts are actually very reliable with a rare occurrence of a part breaking entirely. Rods will occasionally get bent and small adjustments of collars and gears are periodically necessary. The most important aspect of maintaining the machine is keeping the moving parts oiled and running the machine often. In fact, in the rare occurrence of a break, it usually comes after the machine has been static for an extended period of time. Experience for efficient operation is necessary, but the parts are designed so that even persons without any mechanical background can follow a systematic plan, and become Meccano inventors themselves.

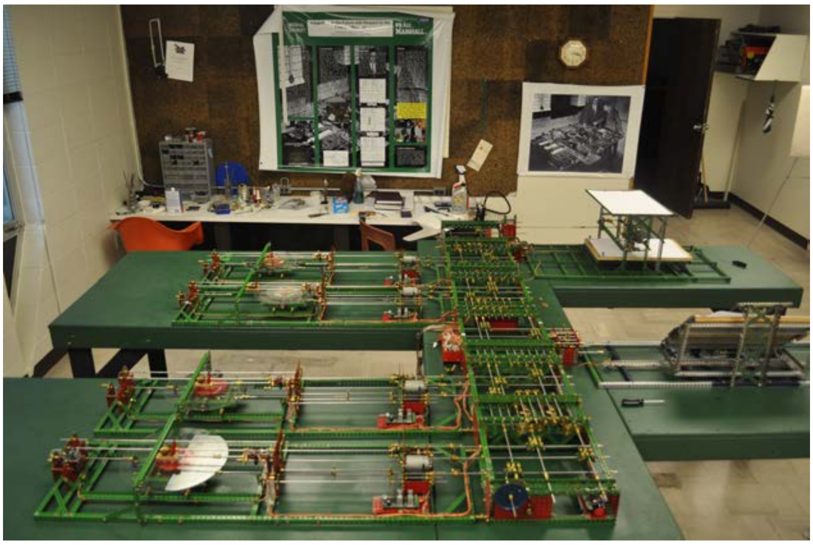

Figure 2.31. Above Frontal View of "Art" Marshall's Differential Analyzer: Shown in the center, between the two inner integrators, is the Motor-Vater.

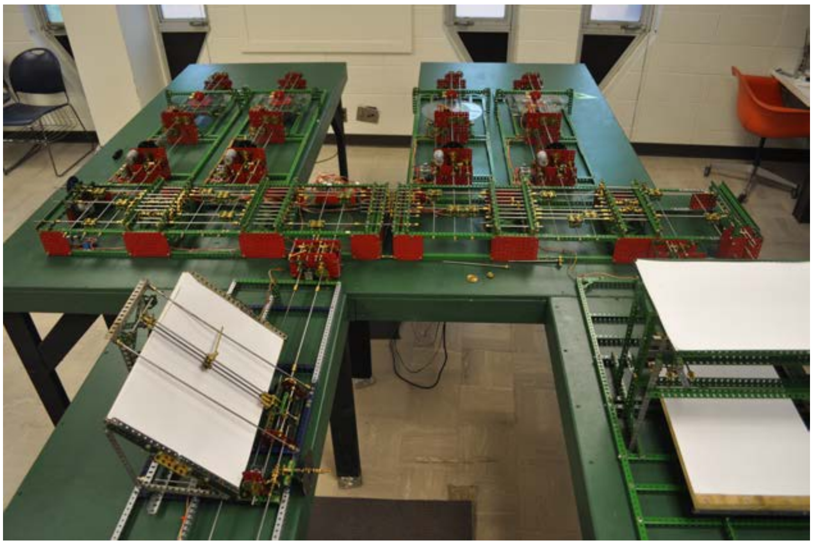

Figure 2.32. Side View of Marshall's Differential Analyzer, "Art."

3. MATHEMATICAL JUSTIFICATION OF MECHANICAL INTEGRATION

This section concerns the theoretical aspects of integration in a context consistent with the mechanical integrator. A pure mathematical justification of why an arbitrary function on the integrator satisfies the condition for Riemann integrability is given and proved in the fullest regard. The approach is an application of the construction of the Riemann integral as given in Kirkwood [12]. Also verification of the fact that the rotation of an integrator wheel produces the integral of the wheel's variance from the center of the disk with respect to the rotation of the disk is given in the classical sense. This type of application has, at least to this author's knowledge, not been given for the specific case of a mechanical integrator of this type.

3.1. NOTATION

In order to talk about the main mathematical points, we need to correlate the movable parts of the machine to the mathematical symbols that are standard for mathematicians. So first think about the rotating glass disk mounted on a movable carriage with a steel wheel placed perpendicular on top of the surface of the disk such that the disk turns the wheel by the force of friction between the glass disk and the steel wheel. The disk carriage is moved back and forth by the rotation of the lead screw. Note that the carriage is allowed to be displaced while the disk is turning the wheel, ultimately changing the turning ratio from disk to wheel and governing the revolutions per minute of the wheel's rotation (See Figure 3.1).

If the wheel is replaced by a pen and the disk turns as the carriage moves back and forth concurrently, the pen draws a somewhat deceiving polar curve on the

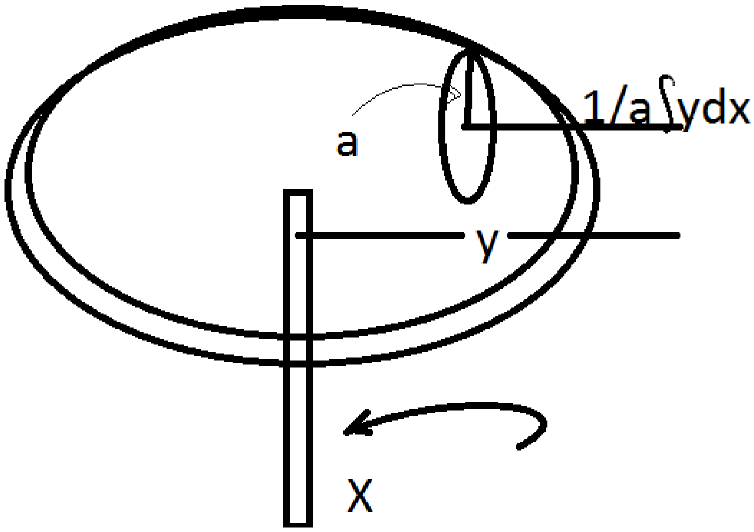

Figure 3.1. Schematic of an integrator

surface of the disk (See Figure 3.2). For example, suppose the rotation of the disk and the motion of the carriage have the same magnitude. That is, both drives are turned at the same constant rate. What will be drawn on the surface of the disk is a spiral (See Figure 3.3). However, the spiral does not represent a polar curve; the plot will be interpreted as a linear function. As another example, if the lead-screw shaft does not turn at all, the resulting curve on top of the disk is a circle (See Figure 3.2). However, this is still a function because the dependent variable is always one specific value at all times. In fact, the function is interpreted as a constant function for some fixed y value. Comparing some function plotted on the surface of the disk to that same function plotted in a Cartesian coordinate system, the independent variable is the rotation of the disk, and the dependent variable is the distance a point on the curve is from the center of the disk, within a constant. Thus, if the dependent variable and the independent variable have the same magnitude, then the function is $y = x$. Here, y is the motion of the carriage due the rotation of the lead-screw, and

x is the rotation of the disk, both of which are measured in terms of shaft rotations. An analogous interpretation for the function described by these two motions was elegantly given by Dr. Bonitia A. Lawrence. Imagine the independent variable axis of a Cartesian coordinate system being manipulated into a long coil spring, spiraling around a common center, and let the measure of some point on that axis to the common center be the dependent variable. The geometric interpretation would be a spring with different radii throughout its length, one way of visualizing the input motions of an integrator as a function. The reason for using a spring example is so that there is no confusion with the graph on the surface of the disk being a function or not. If the original interpretation is considered, the disk would turn once and at the same time the carriage would be displaced one inch from center. Stop time or the rotation of the disk and look at the distance of a corresponding point on the graph to the center of the disk, call it y_1. If the disk turns one more full rotation, and in the meantime the carriage is displaced one inch forward and subsequently one inch back, then after the disk is stopped again for another observation, the distance of the point from the center (y_2) is the same distance as the previous case (y_1). So it would seem that y is not a function. However, note that the disk is at a new x displacement; hence, the y values are still unique. Therefore, for every x-turns of the disk, there is one unique y-distance from the center, and the definition of a function is still satisfied. The y distance from the center may only be in one place, at some given portion of a turn x, such that $y = f(x)$ (Note: When interpreting the motion of the integrator in a mathematical context, $y = f(x)$. However, there aren't any mechanical interconnections that would provide a dependence of shaft rotations from the motion of the disk to the motion of the lead-screw.) When interpreting this mapping as it is in the spring example, one cannot conceive of two y positions corresponding to one x because it is impossible to have two radii at the same given point on the spring.

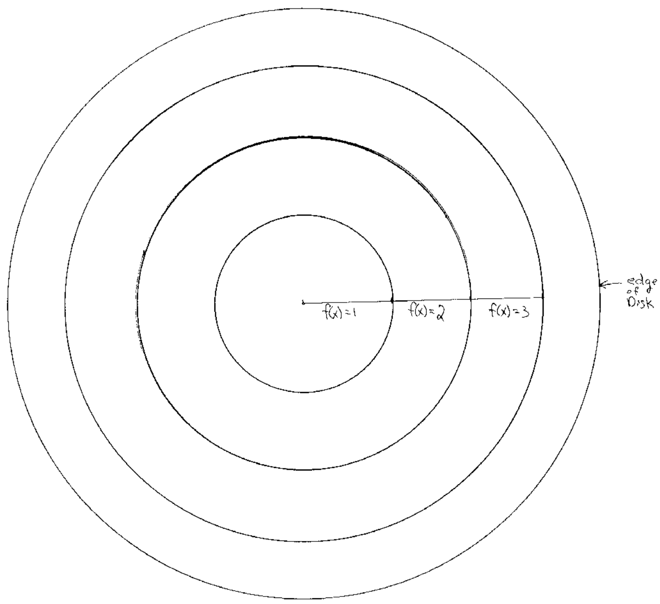

Figure 3.2. This is an example of three different constant functions that would be drawn by the Integrator on the surface of the disk.

Now that we see the relationship between the motion of the disk and the motion of the carriage via the lead-screw represents a function, we may talk about them in terms of x and y, where x is the rotation of the shaft that turns the disk and y is the displacement of the carriage due to the rotation of the shaft that turns the lead-screw, with the usual notation $y = f(x)$. At this point we are assuming that there are no gear trains inserted between either the motion of the carriage via the lead-screw (integrand shaft) and the motion of the disk via the differential shaft. Also, it is important to note that, when the wheel is mounted properly, only rotational motion may be picked up by it. That is, the wheel by design will only turn through an arc-length prescribed by the motion of the disk and determined by the wheel's distance from center of disk. At this point, we are just defining some function whose

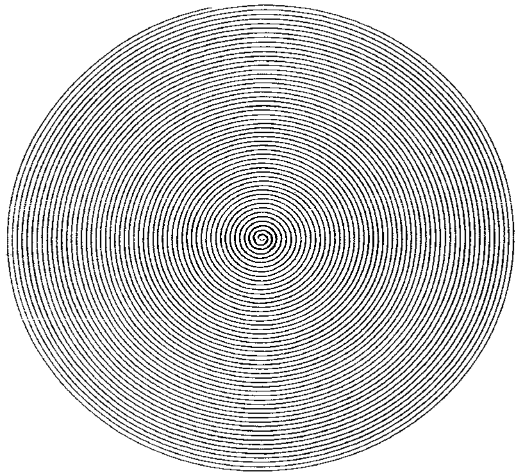

Figure 3.3. This graph was actually drawn by "Art" when the wheel was literally replaced by a pen. The integrand and differential shaft drives were turned at the same rate. The result is this spiral and it is to be interpreted as the function f(x)=x.

y motions are governed with respect to x motions. The implicit curve drawn on top is a luxury that exists for theoretical purposes, but that is not seen when using the machine to solve differential equations.

3.2. PRELIMINARY RESULTS

We will first need to show some known facts about the $Supremum$ and $Infimum$ of bounded sets of real numbers. Throughout the various definitions there are remarks that will relate the physical parts of the machine to the analogous mathematical definitions.

Definition 3.1. A set is a well-defined collection of objects.

Remark 3.2. All sets, for our purposes, may be interpreted as shaft rotations and the accumulation thereof.

Formally, the function that exists on top of the surface of the disk is

$f : X \mapsto Y$, where X and Y are subsets of real numbers and may be interpreted as numerical shaft rotations.

Definition 3.3. [12] Let A be a set of real numbers. If there is a real number b for which $x \leq b$ for every $x \in A$, then b is said to be an *upper bound for A. A set that has an upper bound is said to be bounded above. If there is a number c such that $c \leq x$ for every $x \in A$, then c is said to be a* lower bound for A. A set that has a lower bound is said to be *bounded below.* A set that is bounded above and below is said to be *bounded.* A set that is not bounded is said to be *unbounded* ([12]).

Remark 3.4. The idea of bounded for the rotations of shafts on the integrator is simply that the shaft begins turning then it stops. The upper or lower bound of any set will depend on the rate at which the shaft is turned. Suppose a shaft is turned at a monotonic rate until it is stopped; then the upper and lower bounds will be the end points of the set. However, if the shaft is turned at a non-monotonic, or non-constant rate, then the position where the shaft stops is not necessarily the largest or smallest value. The magnitude of accumulation of turns depends of the designated value of where it starts (usually 0). Because the shaft may turn in either direction, positive or negative, one would need to keep track of the largest value by numerical shaft rotation counters, if that value is desired. Unbounded sets are only available on the machine if we assume that its corresponding shaft never stops turning. Note there are still infinitely many elements between two points.

Definition 3.5. [12] Let A be a set of real numbers that is bounded above. The number b is called the *supremum* of the set A, denoted by $\sup A$, if

(i) b is an upper bound of A, and

(ii) If c is a upper bound of A, then $b \leq c$.

Definition 3.6. [12] Let A be a set of real numbers that is bounded below. The number b is called the *infimum* of the set A, denoted by $\inf A$ if

(i) b is a lower bound of A, and

(ii) If c is also a lower bound of A, then $c \leq b$.

We will need the following results about sets of real numbers to establish are main result.

Theorem 3.7. *[12] Let A and B be non-empty bounded subsets of real numbers, with $A \subseteq B$, then* $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$

Proof. As A and B are both bounded, then by definition, A and B are bounded above and below.

Let the $\sup A = \alpha_S$, $\sup B = \beta_S$ and $\inf A = \alpha_I$ and $\inf B = \beta_I$.

By the definition of *Supremum*, for all $x \in A$, $x \leq \alpha_s \leq \alpha$, where α is an upper bound for the set A, and $x \leq \beta_S \leq \beta$, for all $x \in A$, where β is an upper bound for the set B. Since, $A \subseteq B$, then β_S is an upper bound for A. Let $\alpha = \beta_S$. Then $x \leq \alpha_S \leq \alpha = \beta_S$. And so, $\alpha_S \leq \beta_S$, Hence, $\sup A \leq \sup B$.

Similarly, by the definition of $Infimum$, for all $x \in A$, $\alpha_0 \leq \alpha_I \leq x$, where α_0 is a lower bound for A.

Furthermore, since $A \subseteq B$, $\beta_0 \leq \beta_I \leq x$, for all $x \in A$, where β_0 is a lower bound for the set B. Hence, β_I is some lower bound for the set A and since α_I is the $\inf A$ we may let $\beta_I = \alpha_I$. Thus, we have that $\beta_I = \alpha_0 \leq \alpha_I \leq x$, which implies $\alpha_I \geq \beta_I$,

Therefore, $\inf A \geq \inf B$. □

The next result will be extremely useful when we prove the function on top of the glass is Riemann Integrable.

Theorem 3.8. *[12] (i) Suppose A and B are nonempty sets of real numbers such that if $x \in A$ and $y \in B$, then $x \leq y$. Then* $\sup A$ *and* $\inf B$ *are finite, and* $\sup A \leq \inf B$.

(ii) Suppose that A and B are as in part (i). Then $\sup A = \inf B$*, if and only if, for every $\epsilon > 0$, there exist an $x(\epsilon) \in A$ and $y(\epsilon) \in B$ such that $y(\epsilon) - x(\epsilon) < \epsilon$.*

Proof. Let $x \in A$ and $y \in B$, with $x \leq y$. So, for every $x \in A$, x is a lower bound for the set B, so then $x \leq \inf B$. Which implies that the $\inf B$ is an upper bound for the set A. Thus, $\sup A \leq \inf B$.

This takes care of part(i) and for part (ii) we have to show "iff". Let $\epsilon > 0$ be given, and let $x(\epsilon) \in A$ and $y(\epsilon) \in B$, such that $y(\epsilon) - x(\epsilon) < \epsilon$. We have from part (i) that $\inf B \geq \sup A$ and by the definition of infimum and supremum, we know that $y(\epsilon) \geq \inf B$, and $x(\epsilon) \leq \sup A$, for all $x \leq y$. Thus, $\inf B - \sup A \leq y(\epsilon) - x(\epsilon) < \epsilon$

Going the other direction, suppose $\inf B = \sup A$. Given $\epsilon > 0$, there exist $x(\epsilon) \in A$ and $y(\epsilon) \in B$ so that

$\sup A \geq x(\epsilon) \geq \sup A - \frac{\epsilon}{2}$ and $\inf B \leq y(\epsilon) \leq \inf B + \frac{\epsilon}{2}$, since A and B are sets of real numbers.

Denote $\alpha = \inf B = \sup A$, and we have,

$\alpha \geq x(\epsilon) \geq \alpha - \frac{\epsilon}{2}$ and $\alpha \leq y(\epsilon) \leq \alpha + \frac{\epsilon}{2}$

Which implies, $0 \leq y(\epsilon) - x(\epsilon) \leq (\alpha + \frac{\epsilon}{2}) - (\alpha - \frac{\epsilon}{2}) = \epsilon$ Since the implication has been shown in both directions we have shown, given the assumptions,

$\inf B = \sup A$ if and only if for any $\epsilon > 0$, there exist $x(\epsilon) \in A$ and $y(\epsilon) \in B$, such that $y(\epsilon) - x(\epsilon) < \epsilon$.

□

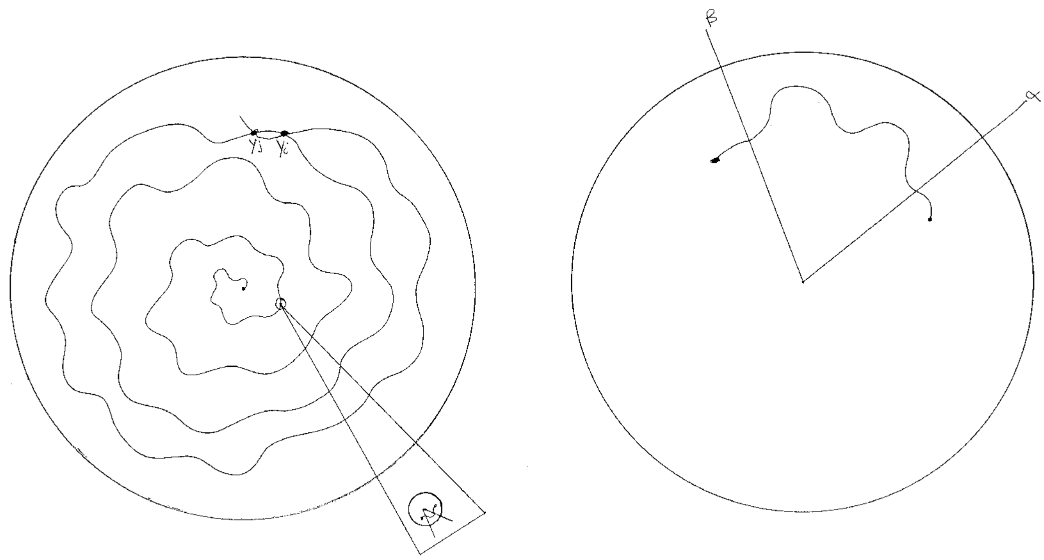

Figure 3.4. Generalized depiction of a function on top of the disk. The function can be discretely approximated into small arc-lengths prescribed by the disk.

In the next sub-section we define a Riemann sum as it pertains to a function represented on top of the integrator disk. We first want to create a generalized representation of a function on top of the integrator disk, and then use the Riemann sum to add up the differences prescribed by the function.

3.3. DEFINING A RIEMANN SUM

Referring to the Figure 3.4, recall when we described what would happen when the wheel of an integrator was replaced by a pen. The graphical representation of the function will, in practice, be spiralled in nature. However, for our purposes we want a general depiction of a function that would include all cases of what a function could be on a small closed interval. We choose a small interval so there isn't any confusion about the modular nature of the rotation of the disk. That is, if we were to let the disk turn though several full rotations, it could be difficult to discern whether or not the curve is a function, because the curve essentially draws over itself. In any case, the accumulation of turns is taken into account for each full turn of the disk, which provides a new x value for every portion of a turn therein. So to avoid this ambiguity, we consider the closed interval, $[\alpha, \beta]$, which lies within one full turn of the disk, or rather some portion of a turn.

Definition 3.9. A partition denoted by P=$\{x_0, x_1, x_2, ..., x_n\}$ of the interval $[\alpha, \beta]$ is a finite set of numbers $x_0, x_1, x_2, ..., x_n$ such that

$$\alpha = x_0 < x_1 < x_2 < .... < x_n = \beta.$$

Remark 3.10. It is very important to note that here, although some portion of a turn of the disk may be taken as an angle measurement, the relationship between polar coordinates and rectangular coordinates is not related by the rules learned in trigonometry. That is $x = rcos(\theta)$, and $y = rsin(\theta)$ do not provide a means to calculate the an angle in this partition given some numerical shaft rotation value of the differential shaft. Although, there may be a reduction gear train inserted between the driving mechanism of the differential shaft and the disk itself, let's suppose that there is no reduction gear for the disk. Then given one turn of the differential shaft x, the disk turns once as well. So for $\frac{1}{4}$ of a turn of the differential shaft, the disk turns through an angle $\frac{\pi}{2}$. If we further suppose that the carriage is at some arbitrary yet fixed position y, some ordered pair $(1, y)$ in rectangular coordinates, corresponds to a polar coordinate with and angle measure less than $\frac{\pi}{2}$. Regardless of what position the carriage takes at this point in the partition, the values of ordered pairs do not have a trigonometric relationship. The reason for using x in the partition is because even though portions of turns are angles, the surface of the disk need not be confused

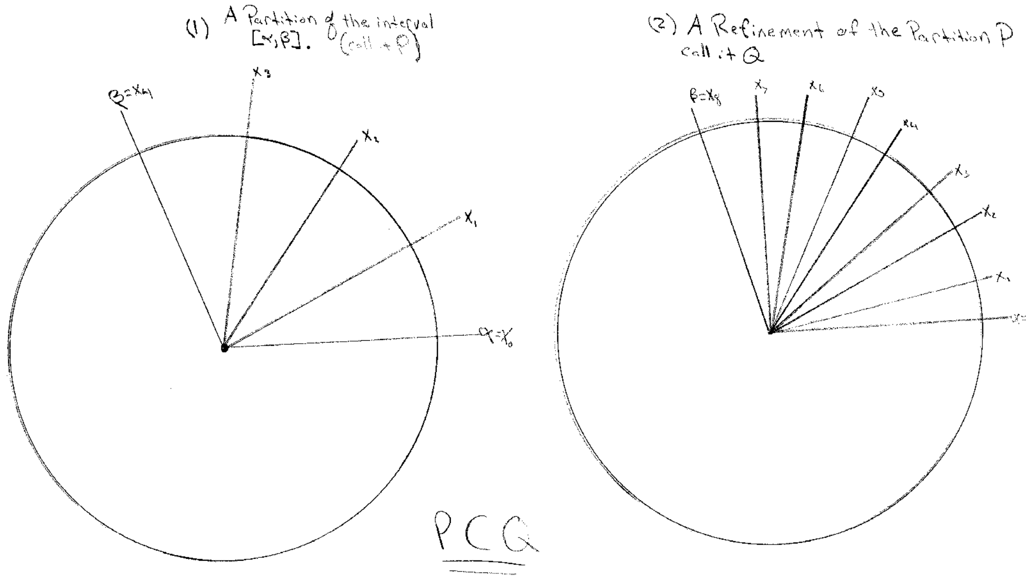

Figure 3.5. On the left is a partition of the disk, on the right is a refinement of that partition.

with a polar coordinate system. It is simply a way to keep track of the accumulations of shaft rotations that have been defined to be sets of real numbers.

Definition 3.11. Let P and Q be partitions of $[\alpha, \beta]$. Q is said to be a refinement of P if $P \subset Q$.

Partitions are used to divide an interval into smaller closed sub-intervals. Refinements are partitions, but more specifically, refinements are partitions of the sub-intervals in a partition. The union of sub-intervals in a refinement is equal to the sub-interval of the partition that has been refined.

Example 3.12. Suppose the interval $[\alpha, \beta] = [0, \frac{\pi}{2}]$, and let $P = \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$ be a partition of that interval. The partition $Q = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ is also a refinement of P, as $P \subset Q$.

Note that the interval $[0, \frac{\pi}{2}]$ of the partition P is equivalent to the union of subintervals $[0, \frac{\pi}{6}] \cup [\frac{\pi}{6}, \frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{\pi}{3}] \cup [\frac{\pi}{3}, \frac{\pi}{2}]$ in the refinement Q (Refer to Figure 3.5).

The reader should keep in mind that the function on top of the surface of the disk is two dimensional and bounded on $[\alpha, \beta]$ where the coordinate system is of its own convention, entirely. So if we want to define a Riemann sum, we must do so within the surface of the disk. We have developed partitions of the set of x values, that is the set of turns for the disk. We now denote a set that will be representative of the $y = f(x)$ values for the function as well.

Let $P = \{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[\alpha, \beta]$ with sub-intervals $[x_{i-1}, x_i]$, $i = 1, 2, .., n$. Denote y^*=$\{f(x)|x \in [x_{i-1}, x_i]\}$, as the set of all $y = f(x)$ values with respect to the partition P.

We will need the greatest and least elements of these sets, so let

$$m_i f = Inf\{f(x)|x \in [x_{i-1}, x_i]\}$$

and

$$M_i f = Sup\{f(x)|x \in [x_{i-1}, x_i]\}.$$

Note that $m_i f$ and $M_i f$ are unique. Furthermore, let $\Delta x_i = x_{i-1} - x_i$, $i = 1, 2, ..., n$ be the length of some arbitrary sub-interval in the partition. Note that Δx_i is the particular portion of a turn of the disk that corresponds to some sub-interval of the partition and is always positive.

Definition 3.13. Let $P = \{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[\alpha, \beta]$. For each subinterval$[x_{i-1}, x_i]$, choose $x^* \in [x_{i-1}, x_i]$. Then

$$\sum_{i=1}^{n} f(x^*)\Delta x_i$$

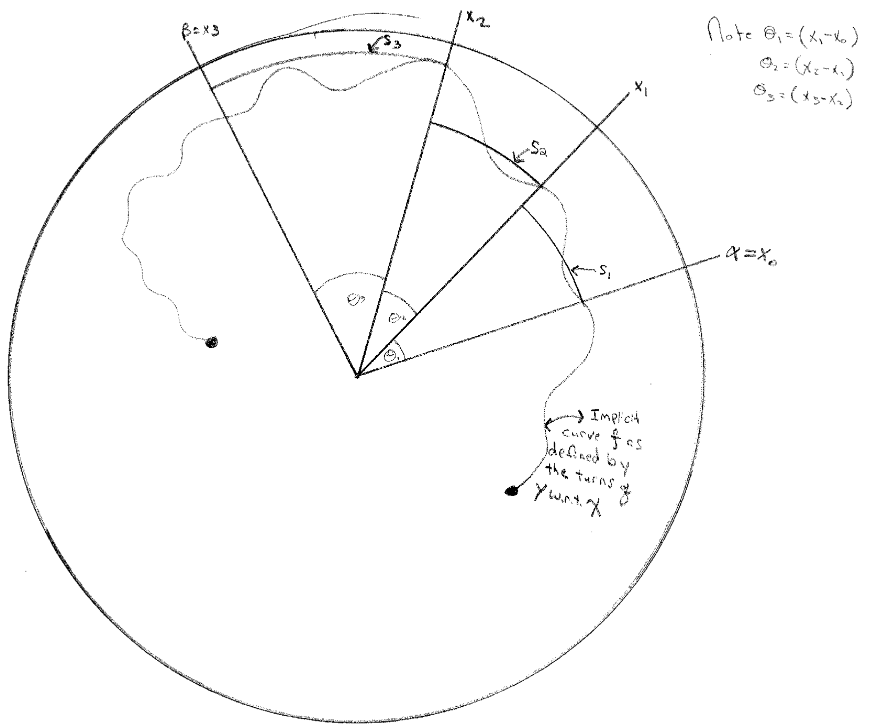

Figure 3.6. Defining a Riemann Sum over the function f. ($[s_1+s_2+s_3]$, is an example of a Riemann Sum.)

is called a *Riemann sum* of f on $[\alpha, \beta]$ with respect to the partition P and is denoted by $S(f;P)$.

At this point it is instructive to return to the three discrete cases previously mentioned in the last section on the integrator mechanics (Refer to Figure 3.6, which represents three discrete cases of the motion of the wheel when given some fixed y for some portion of a turn of the disk x.). The function is plotted on the surface of the disk, and the three cases are superimposed on top of the generalized depiction of the function f. The key point of fact is, the rotations of the wheel, as represented in the discrete cases, is by mechanical design a Riemann sum with respect to the partition. The partition P=$\{x_0, x_1, x_2\}$ corresponds to the three cases of the discrete rotations of the disk x_1^*, x_2^*, x_3^*. To see that rotations of the wheel do in fact represent a Riemann sum we need to define what the rotations of the wheel are mechanically. As discussed in the last section, the wheel is mounted in such a way that its rotation can only be affected by the motion prescribed by the disk. Hence, as the disk x turns the wheel at a fixed radius y, the wheel turns through some arc-length. To find this arc-length s we have the formula $s = r\theta$, where r is the radius or distance from the center of the disk to the edge of the arc, and θ is the angle prescribing the arc. The correlation between arc-length and a Riemann sum as defined above for our f is that x governs θ and y governs r. So to calculate the total shaft rotation for the wheel we have

$$s_1 + s_2 + s_3 = \sum_{i=1}^{3} r_i \Delta\theta_i = \sum_{i=1}^{3} f(x_i^*)\Delta x_i = S(f; P).$$

Note that our first $f(x_i^*)$, $f(x_1^*)$ in this simple case corresponds to the element x_1^* in the sub-interval $[x_0, x_1]$. What we are trying to prove in theory is that our function f is Riemann integrable, and the wheel will be the mechanism that calculates it for us.

Definition 3.14. Given the assumptions that f is bounded on $[\alpha, \beta]$, $m_i f$ and $M_i f$ are finite, and for $i = 1, 2, ...n$. Let P=$\{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[\alpha, \beta]$, $m_i f = Inf\{f(x)|x \in [x_{i-1}, x_i]\}$ and $M_i f = Sup\{f(x)|x \in [x_{i-1}, x_i]\}$, for each sub-interval $[x_{i-1}, x_n]$, and let $\Delta x_i = x_i - x_{i-1}$, for all $i = 1, 2, ...n$. Then

$$S_u(f; P) = \sum_{i=1}^{n} M_i f \Delta x_i$$

and

$$S_l(f; P) = \sum_{i=1}^{n} m_i f \Delta x_i$$

are called the upper Riemann sum and lower Riemann sum, respectively, of f on $[\alpha, \beta]$ with respect to the partition P.

Remark 3.15. Because, by definition, $m_i f \leq f(x^*) \leq M_i f$ for some sub-interval of the partition P, we have that $S_l(f;P) \leq S(f;P) \leq S_u(f;P)$ for some sub-interval of the partition P.

In order to define a Riemann integral on the partition of the interval $[\alpha, \beta]$, we will need the following result.

Theorem 3.16. *(i) Suppose P and Q are partitions of $[\alpha, \beta]$ and Q is a refinement of P. Then,*

$$S_l(f;P) \leq S_l(f;Q)$$

and

$$S_u(f;P) \geq S_u(f;Q).$$

(ii) If P and Q are partitions of $[\alpha, \beta]$, then

$$S_l(f;P) \leq S_u(f;Q).$$

(iii) Let $S_l(f)$=:$\sup\{S_l(f;P)|P$ is a partition of $[\alpha, \beta]\}$ and $S_u(f)$=:$\inf\{S_u(f;P)|P$ is a partition of $[\alpha, \beta]\}$. Then $S_l(f)$ and $S_u(f)$ exist and

$$S_l(f) \leq S_u(f).$$

Note that $S_l(f)$ is the supremum of the set of lower Riemann sums, and $S_u(f)$ is the infimum of the set of upper Riemann sums.

Proof. Let $P = \{x_0, x_1, x_2, ...x_n\}$, be a partition of $[\alpha, \beta]$ and P_1 be a refinement of P such that P_1 is created by adding one element to P. So $P_1 = \{x_0, x_1, ...x_{j-1}, x', x_j, ...x_{n-1}, x_n\}$, where x' is the element that has been added. By definition

$$S_l(f;P) = \sum_{i=1}^{n} m_i f \Delta x_i = \sum_{i=1}^{j-1} m_i f \Delta x_i + m_j f \Delta x_j + \sum_{i=j+1}^{n} m_i f \Delta x_i$$

and

$$S_l(f;P_1) = \sum_{i=1}^{j-1} m_i f \Delta x_i + m_j' f(x_{j-1} - x') + m_j'' f(x' - x_j) + \sum_{i=j+1}^{n} m_i f \Delta x_i$$

where

$$m_j' f = \inf\{f(x)|x \in [x_{j-1}, x']\}$$

and

$$m_j'' f = \inf\{f(x)|x \in [x', x_j]\}.$$

Note by design, $S_l(f;P)$ and $S_l(f;P_1)$ differ by the number of points that represent the sub-interval $[x_{j-1}, x_j]$, wherein the refinement P_1 contains one extra point x'. Hence, to make comparisons between $S_l(f;P)$ and $S_l(f;P_1)$ we need only to consider $m_j f$ and $m_j' f + m_j'' f$. As $[x_{j-1}, x_j]$ covers both of the sub-intervals $[x_{j-1}, x']$ and $[x', x_j]$, we know that the corresponding y values for those intervals have a similar relationship in that, the set

$$\{f(x)|x \in [x_{j-1}, x']\} \subset \{f(x)|x \in [x_{j-1}, x_j]\}$$

and

$$\{f(x)|x \in [x', x_j]\} \subset \{f(x)|x \in [x_{j-1}, x_j]\}.$$

So by Theorem 3.7 we know that

$$m_j' f \geq m_j f$$

and

$$m''_j f \geq m_j f.$$

So we have,

$$m_j f \Delta x_j = m_j f[x_{j-1} - x_j]$$

$$= m_j f[x_{j-1} - x' + x' - x_j]$$

$$= m_j f[x_{j-1} - x'] + m_j f[x' - x_j]$$

$$\leq m'_j f[x_{j-1} - x'] + m''_j f[x' - x_{j-1}]$$

$$\implies S_l(f;P) \leq S_l(f;P_1).$$

So continuing in this fashion, we can define more refinements of P such that $P \subset P_1 \subset P_2 \subset, ..., \subset P_k$, where each P_i is determined by adding exactly one element to the partition P_{i-1} for $i = 1, 2, ..., k$. Furthermore, letting Q be a refinement of P_k, we have

$$P \subset P_1 \subset P_2 \subset, ..., \subset P_k \subset Q.$$

Then we conclude

$$S_l(f;P) \leq S_l(f;P_1) \leq S_l(f;P_2) \leq ... \leq S_l(f;P_k) \leq S_l(f;Q).$$

In order to show that $S_u(f;P) \geq S_u(f;Q)$ we construct that same type of refinement P_1 of the partition P of $[\alpha, \beta]$ as we did for the previous case. Except here we work with

$$S_u(f;P) = \sum_{i=1}^{n} M_i f \Delta x_i = \sum_{i=1}^{j-1} M_i f \Delta x_i + M_j f \Delta x_j + \sum_{i=j+1}^{n} M_i f \Delta x_i$$

and

$$S_u(f;P_1) = \sum_{i=1}^{j-1} M_i f \Delta x_i + M_j' f(x_{j-1} - x') + M_j'' f(x' - x_j) + \sum_{i=j+1}^{n} M_i f \Delta x_i$$

where

$$M_j' f = \sup\{f(x)|x \in [x_{j-1}, x']\}$$

and

$$M_j'' f = \sup\{f(x)|x \in [x', x_j]\}.$$

Note that in set of sub-intervals we still have that

$$\{f(x)|x \in [x_{j-1}, x']\} \subset \{f(x)|x \in [x_{j-1}, x_j]\}$$

and

$$\{f(x)|x \in [x', x_j] \subset \{f(x)|x \in [x_{j-1}, x_j]\}.$$

Using Theorem 3.7 we obtain

$$M_j' f \leq M_j f$$

and

$$M_j'' f \leq M_j f.$$

So we have

$$M_j f \Delta x_j = M_j f[x_{j-1} - x_j]$$

$$= M_j f[x_{j-1} - x' + x' - x_j]$$

$$= M_j f[x_{j-1} - x'] + M_j f[x' - x_j]$$

$$\geq M_j' f[x_{j-1} - x'] + M_j'' f[x' - x_{j-1}].$$

Therefore, from the definitions of $S_u(f;P)$ and $S_u(f;P_1)$,

$$S_u(f;P) \geq S_u(f;P_1).$$

Now letting Q be some refinement of P and noting that $P_1, P_2, ..., P_k$ such that $P \subset P_1 \subset .... \subset P_k \subset Q$, we get

$$S_u(f;P) \geq S_u(f;P_1) \geq S_u(f;P_2) \geq .. \geq S_u(f;P_k) \geq S_u(f;Q).$$

Thus, $S_l(f;P) \leq S_l(f;Q)$, and $S_u(f;P) \geq S_u(f;Q)$.

(ii) Let P and Q be partitions of $[\alpha, \beta]$. Then $P \cup Q$ is a refinement of both P and Q. From the above result we have that

$$S_l(f;P) \leq S_l(f;(P \cup Q))$$

and

$$S_u(f;(P \cup Q)) \leq S_u(f;P).$$

Notice

$$S_l(f;(P \cup Q)) \leq S_u(f;(P \cup Q))$$

and so

$$S_l(f;P) \leq S_u(f;Q).$$

(iii) Let

$$m = inf\{f(x)|x \in [\alpha, \beta]\},$$

and

$$M = \sup\{f(x)|x \in [\alpha, \beta]\}$$

As f is bounded, m and M are finite. If we look at the length of the whole interval $[\alpha, \beta]$, that is $(\beta - \alpha)$, we can establish a bound for the set of upper Riemann sums and the set of lower Riemann sums. So because $m(\beta - \alpha)$ represents the smallest "$arc - length$" in the interval and $M(\beta - \alpha)$ represents the largest "$arc - length$" value in the whole interval we get

$$m(\beta - \alpha) \leq S_l(f; P) \leq S_u(f; P) \leq M(\beta - \alpha),$$

so that $S_l(f)$ and $S_u(f)$ are finite. Hence, by Theorem 3.8 part(i)

$$S_l(f) \leq S_u(f).$$

□

Referring to Figure 3.7, we can get a visual interpretation of the lower and upper Riemann sums from left to right respectively. (Note we have expanded our closed interval for good visualization.) It should be noted that the general function depicted on the surface of the disk can be taken to be either positive or negative. In order to have a discernable difference between the values of y in terms of positive and negative, we would need to observe which side of the disk the function is being plotted on independent of the disk's rotation. Because the implicit function on the disk is created for theoretical purposes, it is not necessary to define a condition for the sign of these values. Here we are only interested in the magnitude of the y values. The sign of the y values for the integrator is established by the clockwise or counter clockwise direction of rotations of the wheel. Because we are still under the assumption that our function f is arbitrary, the sign of the function is arbitrary. Note that here we are using the upper and lower Riemann sums to approximate the integral of f and each such calculation of f for each sub-interval of the partition is in

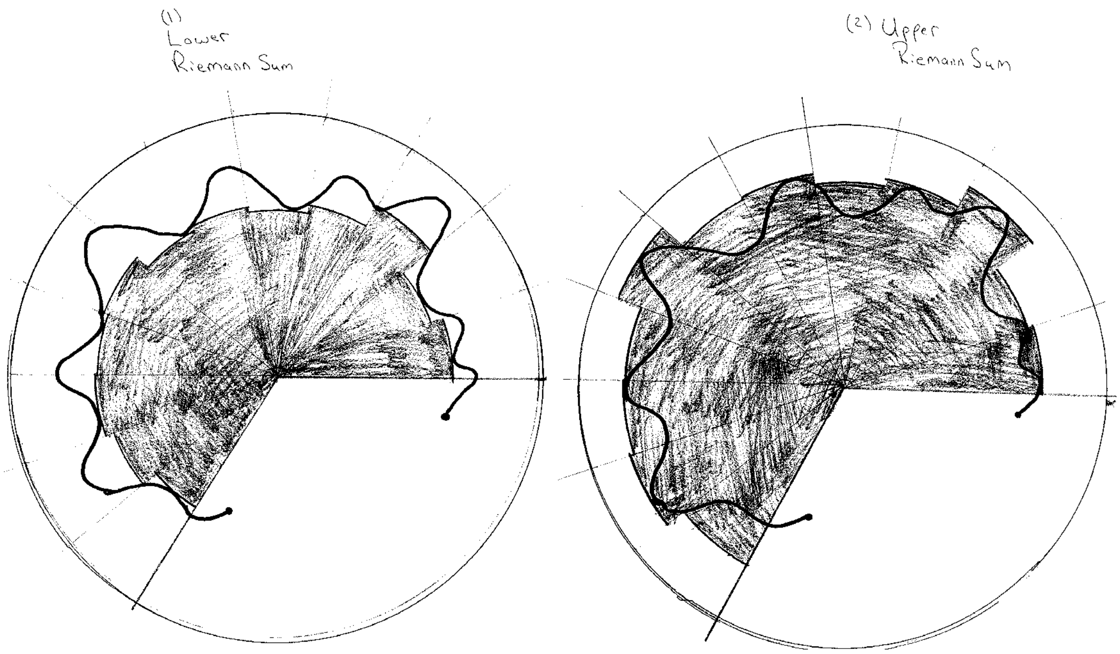

Figure 3.7. Depiction of lower and upper Riemann Sums

the form of an arc length. (Reader beware: The concept of arc-length (curve-length) from a calculus perspective, in terms of polar coordinates, is not that same as our concept of "$arc - length$" here.)

3.4. DEFINING A RIEMANN INTEGRAL

At this point we are ready to define our Riemann integral. From this definition we will now have a way to determine the Riemann integrability of a function.

Definition 3.17. [12] For f, a bounded function on $[\alpha, \beta]$, f is said to be Riemann integrable on $[\alpha, \beta]$ if $S_l(f) = S_u(f)$. Then the Riemann integral of f on $[\alpha, \beta]$ is the common value of $S_l(f)$ and $S_u(f)$, and is denoted by $\int f(x)dx$.

Consequently, to determine whether or not the function f is Riemann integrable, one must assume that the function is bounded so that the Riemann integrability condition is satisfied for the general case.

Theorem 3.18. *[12] A bounded function f on $[\alpha, \beta]$ is Riemann integrable on $[\alpha, \beta]$ if and only if, given $\epsilon > 0$, there is a partition $P(\epsilon)$ of $[\alpha, \beta]$ such that*

$$S_u(f; P(\epsilon)) - S_l(f; P(\epsilon) < \epsilon.$$

Proof. Let P and Q be partitions of $[\alpha, \beta]$, and let Q is a refinement of P. By definition, f is Riemann integrable on $[\alpha, \beta]$ if $S_l(f) = S_u(f)$. That is, the supremum of the set of lower Riemann sums is equal to the infimum of the set of all upper Riemann sums. We know from Theorem 3.16 part(iii) that $S_l(f) \leq S_u(f)$. So in order to show $S_l(f) = S_u(f)$, we need to show given $\epsilon > 0$,

$$S_u(f) - S_l(f) < \epsilon.$$

First, denote

$L = \{S_l(f; Q(\epsilon)) | Q(\epsilon) \text{ is a partition of } [\alpha, \beta]\}$,

and

$U = \{S_u(f; Q(\epsilon)) | Q(\epsilon) \text{ is a partition of } [\alpha, \beta]\}$.

Given $\epsilon > 0$, let $u(\epsilon) \in U$ and $l(\epsilon) \in L$ such that $u(\epsilon) - l(\epsilon) < \epsilon$. But since $u(\epsilon) \leq S_u(f)$, and $l(\epsilon) \geq S_l(f)$, we have that

$$S_u(f) - S_l(f) < u(\epsilon) - l(\epsilon) < \epsilon.$$

On the other hand, if we suppose that $S_l(f) = S_u(f)$, then there exist $u(\epsilon) \in U$ and $l(\epsilon) \in L$ such that

$$S_l(f) \leq l(\epsilon) < S_l(f) + \frac{\epsilon}{2},$$

and

$$S_u(f) \geq u(\epsilon) > S_u(f) - \frac{\epsilon}{2}.$$

Now letting $s = S_u(f) = S_l(f)$, we have

$$s \leq l(\epsilon) < s + \frac{\epsilon}{2},$$

and

$$s \geq u(\epsilon) > s - \frac{\epsilon}{2}.$$

Hence,

$$0 < u(\epsilon) - l(\epsilon) < (s + \frac{\epsilon}{2}) - (s - \frac{\epsilon}{2}) < \epsilon.$$

□

So by defining arbitrarily small partitions of the interval $[\alpha, \beta]$ on the disk, we have proved the our function f satisfies the condition for Riemann integrability. This application of Riemann integrability of a bounded function on a closed interval was chosen so that we could define a Riemann integral on the surface of the disk with as little information about the function f as possible. Now we are convinced that a bounded function on a closed interval is Riemann integrable if and only if there exists some arbitrarily small partition where there is no difference between the upper and lower Riemann sums. So how do we know that our function f specifically satisfies the condition for Riemann integrability? Our $f(t)$, as defined on the surface of the disk, is a position on the surface of the disk that can be partitioned into infinitely small sub-intervals by design and by definition. So this discussion links the theory of integration with the surface of the disk. We now need a way to calculate that integral at all infinitesimal points with no differences between the upper and lower Riemann sums. This mechanism is where the approximation of the integral comes into play. Theoretically, the true integral exists on the surface of the disk. We need a mechanism that super sums up the integral as a function of the rotation of the disk. When we defined our f, we used a pen mount as a reference point to the center of the disk. That pen was affected by its position from the center of the disk and the rotation of the disk as well. So the point at which the pen makes contact with the surface of the disk, at any given portion of a turn of the disk, is the exact point on the plane of the disk at which the discrete Riemann sums become an integral. In other words, this is the point where the upper and lower Riemann sums are equal. We will now mount the wheel properly in place of the pen so that it will be at the exact point where the upper and lower Riemann sums are equal. We know that the wheel's rotation, and accumulation thereof, already represents an arc-length or Riemann sum ($s = r\theta = y\delta x$). Now the implicit curve becomes invisible and the wheel is affected by the variation of the disk carriage about its center with respect to the rotation of the disk. So for any given distance from the center of the disk, the wheel turns through an arc-length for all portions of a turn of the disk. The wheel literally adds up all integral values (consecutive arc-lengths), in "real time," as they are given while the function f itself is being defined on the surface of the disk.

Although this is where our adventure ends, many results in calculus yet remain to be justified. From this application it is now reasonable to assume that the fundamental properties of integration will follow. One of the most obvious results is the continuity of the integral function on the bounded interval $[\alpha, \beta]$ using the Intermediate Value Theorem. One fundamental reason for the function f being continuous is that an operator gets to control the independent variable and there exist arbitrarily small portions of turns within every full turn. This type of analysis of the theory of mechanical integration gives rise to concepts in the time-scales calculus as a mechanical integrator of this type has complete control of the independent variable. Moreover, the DA team has already been solving dynamic equations on time-scales, but that goes beyond the scope of this paper. For the reminder of this paper, it will be left to the reader to make the assumption that the Fundamental Theorem of Calculus holds true, without proof.

4. BASIC OPERATION OF A DIFFERENTIAL ANALYZER

The operations section involves general aspects of setting up the machine for solving simple differential equations. Additionally, views on the educational benefits of using mechanical interrelationships to interpret mathematics are discussed throughout the section. The most difficult aspect of working with a differential analyzer is applying the proper scale factors to various components. Operators of the differential analyzer have the freedom to choose any convention. For example, a convention is established when an operator initially chooses a particular scale factor for a primary component shaft of the machine (i.e. primary disk drive shaft, lead-screw, etc.). Since there exists the freedom of convention, each instance of scaling is not unique. Moreover, scaling a differential equation is also problem dependent. Using a few simple examples, and ultimately leading to a general approach to more complicated problems, the concept of using a differential analyzer to solve differential equations is realized. There are also included in the Appendix many schematic diagrams of the various examples in this section as well as more complex problems not previously mentioned. In this section the theory of mechanical integration is applied, and practical uses of a differential analyzer are revealed.

4.1. QUANTIFICATION AND "UNITY" OF AN INTEGRATOR

An integrator is the first component that should be fully understood before an operator tries to use the machine. For this reason it is the first component to be considered. As mentioned the integrator unit has three movable parts: disk, carriage, and wheel. Respectively, these parts will be called the differential, the integrand,

and the integral, for each of them has its own separate shaft rotation for reference. In the previous section, Theorem 3.18, it was shown that the rotation of the wheel (integral shaft) represents the integral of the wheel's distance from the center of the disk (integrand shaft) with respect to the rotation of the disk (differential shaft), or

$$\text{Turns of Integral Shaft} = \int_{x_0}^{x_n} y(x)dx. \tag{4.1}$$

Moreover, any function determined by the integrand shaft with respect to the differential shaft satisfies the Riemann integrability condition. (Refer to Theorem 3.18.) Theoretically, an integrator will integrate any function that can be interpreted in terms of shaft rotations. However, in practice, producing a perfect integral is contingent upon several necessary mechanical parameters. First there must be absolutely no slippage occurring between wheel and disk. This is achieved by the addition of a torque amplifier and careful considerations of the movable parts of the wheel. Another aspect is the gear trains that are necessary for the movement of the carriage and the rotation of the disk. In theory the disk may turn once for every turn of the differential shaft and likewise the lead-screw can be turned once for each turn of the integrand shaft (or the carriage is displaced by one shaft rotation). In this case, displacing the dependent variable of a function on the disk by one unit corresponds to turning each input shaft one time. However, in practice this is not the case because it's simply not feasible to rotate a heavy disk through a 1:1 gear train and to have a threaded rod whose pitch is one thread per inch. So these two components must be geared down between rotation shaft and movable part. In the case of the disk, there is a reduction gear in place between the differential shaft and the disk gear drive; we denote this by K. In the case of the carriage displacement, there are two things to consider: a reduction gear and the pitch of the lead-screw. These two parameters will be combined and denoted by P. Another necessary parameter for the quantification of the integrator is the radius of the wheel, which we will denote by a.

Our goal now is to take the theoretical result and apply it to a realistic situation in general practice. We need to determine the actual number of turns of the integral shaft given the two inputs of differential and integrand shafts. The simplest place to start is with a fixed displacement of the integrand shaft. The question "at what displacement from the center of the disk does the wheel's edge need to be in order to give a 1:1 correspondence from differential shaft to integral shaft?" is an essential one. From a mathematical point of view, this distance can be interpreted as the *Unity* of an integrator, but, generally speaking, this distance is simply where one turn of the differential shaft results in one turn of the integral shaft. Note that, in the discrete case (when the integrand shaft is fixed), the integrator is a simple gear, and the ratio of turns for the integral shaft is in terms of the two radii, the radius of a concentric circle on the disk (or equivalently the displacement of the carriage) to the radius of the wheel. Just to be clear, one can interpret one turn of the disk as $2\pi y$ and one turn of the wheel as $2\pi a$ with ratio $\frac{2\pi y}{2\pi a}$. Note the factor 2π reduces and the ratio becomes $\frac{y}{a}$. In order to have a 1:1 correspondence, the radius of that particular concentric circle on the disk needs to be equal to the radius of the wheel. So using our notation, let

$$\frac{y}{a} = 1,$$

which implies $a = y$, where y is the radius of the concentric circle on the disk. This answers the question of unity from a theoretical standpoint.

In practice, we must have a way to measure the magnitude of y. Because all quantities are to be measured in terms of shaft rotations, the radius y can be measured by the displacement of its movable part, the carriage. Furthermore, because the carriage is essentially driven by a shaft, it makes sense to let its displacement be a shaft rotation. But now, in terms of shaft rotations, we must know exactly how many turns of y yield a 1:1 correspondence for the ratio of disk to wheel. In order to calculate that, we must also take into consideration how many turns of the differential shaft correspond to the number of turns of its movable part, the disk. For the differential shaft x, x-turns of the differential shaft implies the disk will turn $K * x$ times. For example, if $K = 2/5$, then one turn of the differential shaft will result in $2/5$ of one full turn of the disk, as K is a reduction gear. Similarly, for y-number of turns of the integrand shaft y, the carriage will be displaced $P * y$ inches from center (if the pitch of the lead-screw is measured in inches). Note it is not necessary to measure the pitch of the lead-screw in inches. However, it is necessary to maintain consistency when measuring distance on top of the disk, measuring the pitch of the lead-screw, and the radius of the wheel a.

First, we quantify the integrand shaft using the motion of its movable part. When $a = y$ we assume that the integrand shaft y directly corresponds to the displacement of the carriage. Because it does not in practice, we want to measure the radius of the concentric circle on top of the disk by a shaft rotation that connects the rotation of the integrand shaft to the displacement of the carriage. Thus, we now have $a = y * P$, or

$$\frac{y * P}{a} = 1 \tag{4.2}$$

Some care must be taken in the interpretation of this last equation. It states that the rotations of disk and wheel will have a 1:1 correspondence when $y * P = a$. That is, for any portion of a turn of the disk, the wheel will turn the integral shaft that same portion of a turn, as long as the wheel rests at a particular distance from the center of the disk. The particular distance the wheel rests from the center is determined by y. To talk about this relationship strictly in terms of shaft rotations of the whole integrator unit, we must consider what effect the reduction gear for the differential shaft has on the total number of turns of the integral shaft. So assuming the disk turns one full turn, how many times did the differential shaft turn? It is important not to be confused on the convention of gear ratios. A gear ratio is the number of teeth on one gear divided by the number of teeth on another (or radii as opposed to teeth) where the order of the gears depends on whether or not the gear is a reduction gear. The constants K, and P represent reduction gears so their ratios are between $(0, 1)$. From differential shaft to disk, K is a reduction gear, but from disk to differential shaft, as is the case for our purposes now, K is not. Hence, the disk spinning one full turn implies the differential shaft turned $\frac{1}{K}$ times. Keep in mind that the differential shaft turned more times than that of the disk. For example, if $K = 2/5$ then 1 turn of the disk was a result of 2.5 turns of the differential shaft. What we want to have is a 1:1 correspondence between differential shaft and integral shaft. And we can use the above equation to get an expression of how many times the integrand shaft must be turned to get 1:1 correspondence from disk to integral shaft. We need only factor in the reduction gear K into that expression (Equation 4.2). To see how, note that for any number of turns of the disk the equation

$$\frac{y * P}{a} = 1$$

provides a 1:1 correspondence if we want to use the differential shaft as a reference drive instead of the disk. Then

$$\frac{y * P}{a} = \frac{1}{K}. \tag{4.3}$$

Solving for y, the number of turns of the integrand shaft, we get

$$y = \frac{a}{K * P}. \tag{4.4}$$

Therefore, the integrand shaft needs to be turned $\frac{a}{K*P}$ times in order to have a 1:1 correspondence between differential shaft and integral shaft.

This analysis is important because we have established a point on top of the disk that may be referenced as one unit. For example, suppose one needs the integral shaft to represent the function $f(x) = x$ (a line with slope equal to one); that mapping is obtained by initially setting the wheel to a position consistent with the right hand side (RHS) of Equation 4.4. If $f(x) = 2x$ is the desired function, then double that distance and that mapping is achieved. This concept is discussed in further detail after a change of variables, from the theoretical constants to variables that take into account specific gear trains on the machine, is made.

Suppose that the wheel sits at an arbitrary, yet still fixed, position from the center of the disk. Note that this distance, in terms of shaft rotations, is represented by $y * P$. A new question is now proposed: "how do we determine the total number of shaft rotations of the integral shaft, given a certain number of differential shaft rotations?" For example, let the differential shaft be turned once; this means the disk is turned K times. And for any turn of the disk, the wheel turns $y * P/a$ times as much as the disk. Hence, for one turn of the differential shaft the wheel turns $(K * P * y)/a$ times. Its easy to see that if the number of turns of y is $a/K * P$, then our expression is,

$$\frac{K * P * \frac{a}{K*P}}{a} = 1.$$

This equation makes sense because that particular value of y was chosen to give a 1:1 correspondence from differential shaft to integral shaft. Moreover, if the differential shaft had turned x-times to start in this example, as opposed to once, then the right hand side of the equation would be equal to x. So we can conclude the analysis of this discrete case by saying the total number of shaft rotations for the integral shaft given a certain number of shaft rotations for the differential shaft x, at a fixed integrand shaft rotation y, is given by

$$xy * \frac{KP}{a}.$$

We have established a method of determining the total number of shaft rotations of the integral shaft when the integrand shaft is fixed. Note that for any number of discrete cases, the individual number in each case may be added up, case-wise, to get a total. However, when the integrand shaft is not fixed or non-constant, we can no longer use simple arithmetic calculations because now the output of the wheel is an integral. But we can use the Riemann integral result proved in the last section. We need only to substitute our newly developed expressions, that are in terms of shaft rotations, into the result from the last section to have an expression for the total shaft rotations of the integral shaft in the continuous case.

Note that for the previous discrete cases our function on top of the surface of the disk was a constant function. For the continuous case, we have generalized this to any function satisfying the Riemann integrability condition.

Recall the result from Section 3, Theorem 3.18, that states, a bounded function f (lead-screw or integrand shaft, with respect to a finite number of turns of the disk) is Riemann integrable, only if infinitely small partitions of the turns of the disk exist such that there is an arbitrarily small difference between the upper and lower Riemann sums for the function f, contained on the disk's surface, with respect to the partitions. The Section 3 conclusion provides the reason for the rotation of the wheel being representative of the integral of the function f. By definition a Riemann integral is a Riemann sum that has the property of equal upper and lower Riemann sums everywhere in the domain by which they are defined. The wheel's rotation is defined in exactly the same way. Because the accumulation of the wheel's rotation is a Riemann Sum, by definition, the wheel represents the integral of y (integrand shaft) with respect to x (differential shaft). In order to measure the output of the wheel in terms of shaft rotations, the expression given for the output of the wheel (Result given in Theorem 3.18) is divided by the radius of the wheel (a); thus, incorporating the needed ratio from variable disk radii to the radius of the wheel (Recall that, in the discrete case, in order to calculate the total number of turns of the wheel one must divide, the turns of the disk times the fixed distance of the wheel's edge to the center of the disk ($y * x$), by the radius of the wheel(a); and note that in this continuous case, $y * x$ is represented by the result from Theorem 3.18).

To be clear, taking the expression for the wheel's output, that is,

$$\int_{x_0}^{x_n} y(x)dx \tag{4.5}$$

and dividing it by the radius of the wheel while making the necessary gear reduction substitutions will transform the integral expression for the wheel's output into an integral expression formally in terms of shaft rotations.

For sake of ambiguity, one should change the variables in (4.5) to y^*, and x^* so that there is not any confusion after the substitution is made. So now y denotes the total shaft rotations for the integrand shaft, x denotes the total shaft rotations for the differential shaft, and the integral shaft is

$$\frac{1}{a}\int_{x_0}^{x_n} y^*(x)dx^*.$$

Let $y^* = yP$, and let $x^* = xK$, where y^* is the displacement of the carriage in terms of shaft rotations, and x^* is the rotation of disk in terms of shaft rotations.

Substitution into the above equation yields

$$\frac{1}{a}\int_{x_0}^{x_n} y(x) * Pd(x * K)$$

which yields

$$\frac{KP}{a}\int_{x_0}^{x_n} y(x)dx. \tag{4.6}$$

Because K, and P are both constants, they may be factored out of the integral expression. Note that we have divided the entire result by the radius of the wheel a.

Hence, the constant $\frac{KP}{a}$ comes out of every integrator. Conceptually, one could simply multiply an output by a gear train to cancel out the constant, but that would involve gearing up by too much. As it is only reasonable to gear up by at most two, for mechanical purposes, multiplying by a gear train to cancel out the scale constant is not practical. However, we can use the concept of *Unity* of the integrator to essentially reduce the constant. Note for future reference we will call $\frac{KP}{a}$ the *Integrator Constant* of an integrator, which is just the multiplicative inverse of the value of *Unity* of an integrator, which is $\frac{a}{KP}$. We have the freedom to choose what value of shaft rotation(s) represents one unit. So if we choose one unit to be $\frac{a}{KP}$, then we can say we are integrating with respect to not only x but $\frac{a}{KP} * x$. So within the differential of the integral expression above we have

$$\frac{KP}{a}\int_{x_0}^{x_n} y(x)d(\frac{a}{KP} * x),$$

which implies the integral shaft is now

$$\int_{x_0}^{x_n} y(x)dx,$$

as $\frac{a}{KP}$ is a constant.

And now this measurement of a/KP can be a standard one unit of measurement, denoted by I. We suggest much caution in scaling in this way, for this is just an option for a particular integrator and is not a means to escape issues of the scaling constant in general. As will be seen later, this type of scaling works only when solving linear differential equations. In general, an operator must always assume that a factor of $\frac{1}{I}$ comes out of every integrator.

On "Art," $K = 2/5$, $P{=}\frac{50}{57}{*}\frac{1}{32}$, and $a = 15/16$. Note that P is a combination of two values due to the reduction gear in the clutch box and the pitch of the lead screw (32 threads per inch). So the integrator constant $I = \frac{a}{KP}$, implies $I = 85.5$; that is, 85.5 shaft rotations can be interpreted as one unit. Unfortunately, $I = 85.5$ only corresponds to primary rods contained in the section of interconnection because of the placement of the counters used to measure rotations on the various components. For instance, on the integrator unit, a counter is mounted on the back part of the frame. Due to its location, the reference rod that essentially turns the counter is directly attached to the lead-screw. The placement of the revolution counters imply that 85.5 shaft rotations of the counter rod does not move the integrator carriage the same amount as 85.5 shaft rotations of the integrand shaft. As the gear train inside of the clutch box is 50:57. When measuring the displacement of the integrator carriage with the counter rod, its rotation is not affected by the gear train in the clutch box. Hence, 85.5 turns of the integrand shaft correspond to $85.5/(57/50) = 75$ turns of the counter rod. That is, the counter rod needs to be turned 75 times in order to displace the carriage exactly one unit from the center. The value of 75 does not imply that the counter will read 75 because the counters are designed to read 10 for every shaft rotation or in increments of $\frac{1}{10}$ of a shaft rotation. Additionally, there is a 3:1 gear train between the actual counter sprocket and the counter rod, so for every one turn of the counter rod the counter reads 10/3 on the display. Hence, for unity, 75 turns of the counter rod implies the counter reads $75 * 10/3 = 250$ on the display (250=1=unity). Because all four integrators, input table, and output table have a clutch box, for purposes of using counter reading to get numerical values, 250 is one unit and it is also denoted by I. Remember the freedom to change still exists if so needed for solving different types of differential equations.

Finding the $Unity$ of an integrator is the first type of quantification that must be calculated when working with any differential analyzer. Formally we will denote the $Unity$ of an integrator as $I = \frac{a}{KP} = 250$ and the $Integrator\ Constant$ as $I^{*} = \frac{1}{I} = \frac{1}{250}$. (Note the integrator constant is not the same as the constant of integration from calculus.) So as not to be confused with convention of gear ratios, it is best to establish a way to denote gear ratios and calculate shaft rotations in the beginning and then stick firmly to your convention. For the Marshall DA Team, the consideration of a "direct drive path" concept was adopted, depending on which gear is being turned and which gear is doing the turning. For example, when calculating $Unity$ for the lead-screw shaft as opposed to the integrand shaft, conceptually it is necessary to divide by the gear train that is the difference between the rotation of the two shafts. Note that in this case, 57/50 is the dividend that accounts for the gear train with ratio 50:57. At first, dividing in this way may seem to be an ambiguity in the convention of a direct drive path; however, the direct drive coming from the lead-screw shaft is the reverse gear train, so the convention is still consistent.

This point of fact may seem to be somewhat trivial, but in practice, a trivial subject to some may not be so to others. The concepts of ratios and proportions are deeply embedded in the exercise of quantifying an integrator just as these concepts are embedded in the elementary levels of mathematics. Using a manipulative, such as an integrator along with several options of gear trains, to clarify the subject with students is an activity that the Marshall DA Team seeks to investigate. Negativity and positivity are additional concepts embedded in the action of an integrator. When the wheel sits in the center of the glass, there is no motion picked up by the wheel as the disk is turned. When the wheel sits on one side of the glass it will rotate clockwise and when it sits opposite side of the glass it will rotate counter-clockwise with respect to the rotation of the disk. Either direction can represent positive for example. The more important fact is that, when the wheel sits on opposite sides the of glass with the center as a reference point, the wheel turns in opposite directions if the rotation of the disk is monotonic. Deciding which side represents positive or negative is up to the operator, notably, when two sets of gear trains are meshed together the first gear train reverses the direction of the direct drive. Then the second gear train reverses the direction of the previous gear; thus, the first gear and the last gear have the same direction of rotation. Moreover, the concept of negating an already negative quantity results in a positive quantity is demonstrated via a physical example.(Negative multiplied by negative results in positive.)

4.2. SOLVING SIMPLE EQUATIONS

A beginning discussion of how to use a differential analyzer to solve differential equations would be best served with a simple example of a first order linear differential equation whose solution is a linear function. It should be clear at this point how to get an integrator to represent a linear function whose slope is consistent with the initial displacement of the integrand shaft, which remains fixed. The dependent variable and independent variable are represented by the shaft rotations of the integrand shaft and the differential shaft, respectively. For the purposes of solving differential equations, our formerly established notation for the two input shafts and the output shaft of an integrator will depend on what the variables in the differential equation are. That is, for the notation of a DE, the inputs and output of an integrator will correspond to the variables used to express the DE. Using the integrator to produce a linear function and using the differential analyzer to solve a differential equation whose solution is a linear function are equivalent procedures.

Definition 4.1. A Differential Equation is an equation that relates a function, $y(t)$, its derivatives, and functions of t. (Dr. Bonita A. Lawrence.)

From an analytical perspective, consider the DE,

$$\frac{dy}{dt} = k,$$

where k is a real number. Solving this equation, in general, is a simple matter of integrating both side of the equation with respect to t. So,

$$\int \frac{dy}{dt} dt = \int k dt,$$

is equivalent to

$$y = kt + c$$

where c is a normal constant of integration. This is the general solution to the DE. If we want a particular solution, consider the initial condition

$$y(0) = 0$$

which would imply that $c = 0$ and

$$y = kt.$$

This is a particular solution for the given the initial condition. The differential equation says, if t is taken to be time, the rate at which y changes with respect to time is equal to some real number, or the rate of change of y at any given time is constant. The derivative is the rate of change and by the rule assigned in the equation the derivative is always constant, meaning it never changes. So when thinking about an integrator unit on a differential analyzer, the integrand shaft governs the different rates, or gear ratios of the wheel with respect to the disk. Hence, the integrand shaft is the derivative in this differential equation. Because the equation says that the derivative never changes, the integrand shaft never changes, which is the same situation as the discrete case described above, when the wheel sits at a fixed position, k, and never moves. In the previous section we showed that the rotation of the wheel represents the integral of some function described by the lead-screw with respect to the turns of the disk. Now we have the same relationship but in terms of shaft rotations. When the integrand shaft represents the derivative of a function, by the Fundamental Theorem of Calculus, the integral shaft represents the function itself. (The antiderivative of the derivative of a function is the function within a constant.) This fundamental relationship between the two inputs and output shafts of an integrator is how the machine analyzes the different rates of change, or derivatives, involved in a differential equation.

Mechanically, setting up the machine to solve this simple DE is very straight forward. All that is required is to take the primary motor drive shaft from the section and allow it to drive the differential shaft of one integrator. This setup will solve the DE by just the use of one integrator. To get a useful visual of what's happening mechanically, an output table is needed in this case, because the derivative is constant and the integrator carriage sits still. However, using an output table is not always necessary, for very useful information may be acquired from simply watching the movement of the integrators. So if an output table is used, where the section is a medium for motion transfer, the integral shaft would essentially drive the ordinate carriage and the primary drive shaft would drive the table in the direction of abscissa as well as the differential shaft. The output table draws like an Etch-a-Sketch, and the resulting plot on the surface of the table will be a straight line whose slope is the same value as the initial displacement of the integrand shaft relative to the concept of *Unity* that was previously established.

The concept of the integrator representing a definite integral when using it to solve DE's is an implicit operation embedded within the action of the integrator. The integrator itself represents a definite integral. For every arbitrary portion of a turn of the differential shaft, the integral shaft turns an amount equal to the value of the integral from the starting point that is, from the lower limit, to the upper limit, which is when the differential shaft is stopped. The total number of shaft rotations of the integral shaft is the value of the definite integral at the point where the differential is stopped. But for all infinitesimal portions of turns in between, the values were calculated. An accumulation of shaft rotations may be taken at any time in between the upper and lower limit, which will correspond to a definite integral value whose upper limit lies in the interval between the the two values. These accumulations as they occur in time, may be tabulated by the output table in the form of a curve, and it is this accumulation of the plot that has the illusion of suggesting the integrator is solving an indefinite integral, because the plot is that of a function. However, when integration occurs on the differential analyzer, it is simply calculating all definite integral values from point a to point b, and those values are accumulated by shaft rotations, wherein the shaft rotations are transformed into a plot via an output table. In fact, the differential analyzer is always working on a closed domain. It is up to the operator to match up that domain so that the plot of the particular solution is in a window of interest.

It is also imperative to note that the initial condition for the differential equation is the initial displacement of the integrand shaft before the independent variable drive (primary section drive) is started. Moreover, the initial conditions must satisfy the differential equation analytically. That is in an analytical sense, substituting the initial values into the differential equation will result in an equation that is true. For a first order DE, one piece of initial data is required to get a particular solution. In the case of the example above, we had information about the initial position of the dependent variable y at $t = 0$, resulting in the determination of the constant of integration. If we were to simply use the information we know about the derivative at $t = 0$, we would not be able to strictly determine the value of the constant c, at least analytically. Note that the constant c is the same as the initial value of y at $t = 0$, which is the y-intercept of the linear function plotted on the table. Because the dependent variable y does not appear explicitly in the DE, substituting the initial values in the DE will not determine the y-intercept unless it's an initial condition itself. However, the differential analyzer will plot a particular solution even if we only consider the initial condition of the derivative at $t = 0$, or $y'(0) = k$. So in this case, the y-intercept of the linear graph that is plotted is just in some arbitrary position on the plotting surface, arbitrary yet still fixed. In most cases it is necessary to initially displace the ordinate carriage of the output table so that its initial displacement satisfies the initial conditions given for the differential equation itself. Although solving this DE is a case where the initial setting of an ordinate carriage is not important, it is still necessary to understand why. For example, suppose that we use the machine, as we have in our above example, where the constant of integration is yet undetermined. Furthermore, let the initial displacement of the ordinate carriage read 250 on its reference counter. By our former calculations, this situation would mean that $y(0) = 1$ (analytically). (But this value is not to be taken literally, because it could just as well be some other value subject to interpretation.) Because a linear function is simply shifted up or down the y-axis for different values of c, the curve may be interpreted with $y(0) = c$. This fact allows freedom of convention (convention of the ordinate carriage's initial displacement along the y-axis) in choosing $y(0)$ when the initial position for the dependent variable is unknown, either explicitly or deterministically, by using the DE along with other initial conditions that are given.

This information is still useful in the analysis of different particular solutions, because it says that in this case no matter how much an initial value of the dependent variable changes the qualitative nature of the solution essentially stays the same. The real change happens when the initial displacement of the integrand shaft, or derivative at $t = 0$, has a different value, changing k, ultimately altering the DE itself. The result is in the form of lines with different slopes. This exercise has been demonstrated to elementary algebra students, and the goal of the exercise is to show a linear function has a constant rate of change, or constant slope. It lays the groundwork to compare different polynomial functions in terms of their derivatives without mentioning a formal definition of the derivative.

The process of setting up this problem in the previous example can be outlined schematically, using a Bush Schematic. Referring to Figure 4.1 the rectangle above represents an integrator along with the two inputs and the output, the one below represents an output table. The line to the left, with the arrow that points to the frame is the integrand shaft, the line in the middle represents the integral shaft, and the line to the right is the differential shaft. For the Output Table, the horizontal shaft drives the table carriage in the abscissa direction, and the vertical shaft drives the ordinate carriage. This convention will be the standard notation when setting up problems on the machine. Figure 4.2 is the schematic diagram for the previous example, $y' = k$. Using just one integrator, the vertical lines going to the integrator are bus-shafts, and the horizontal lines are cross-shafts contained in the section, a concept developed by Vannevar Bush. Note that throughout the text we will be inserting various elements to our schematics, like the output table and the adder units, etc. Starting with the first cross-shaft, the primary motor drive, interpreted as the independent variable t, and its corresponding bus-shaft drives the differential shaft. Moving down the cross shafts, the next in line is labeled $k = y'$ representing a constant slope. Then what comes out of the integral shaft is:

$$\int k dt,$$

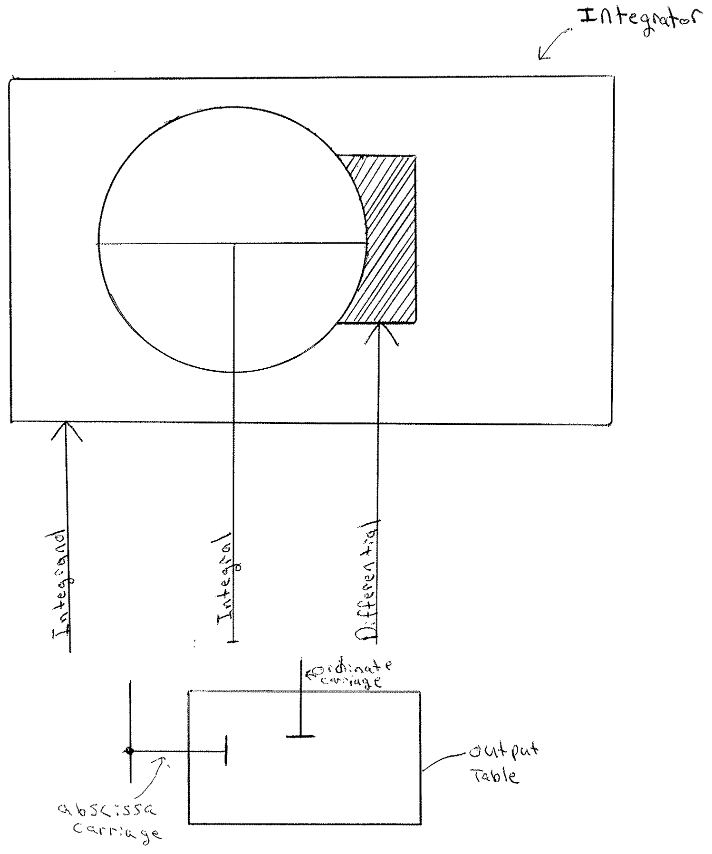

Figure 4.1. Schematic diagram of the basic interconnections of an integrator. The below diagram is the schematic diagram of an output table.

or $kt = y$. Note that we have dropped the use of a definite integral, and have not gained a constant of integration. This is just notation. The integral is sill a definite integral, as described above, but to avoid the confusion of implying that the integral shaft represents one particular value, we denote the accumulation of definite integrals using the notation of an indefinite integral without the added constant of integration. It should be clear that the turns of the integral shaft represents a definite integral from x_0, to x, but the accumulation of such is given as a set of ordered pairs on a

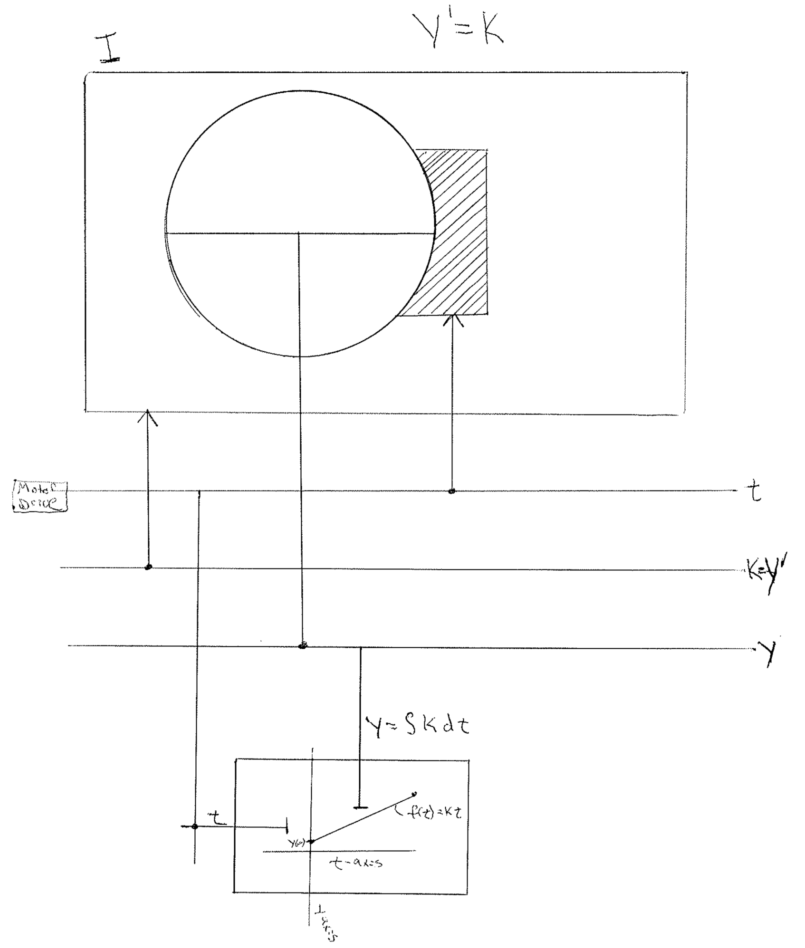

Figure 4.2. Schematic diagram $y' = k$, where the output is a linear function.

closed domain. The function itself is bounded by the interval from (x_0, x), that is from when the independent variable is started until it is stopped. So, from this point on, the notation used to represent an integral shaft will be, in general, the same as an indefinite integral without the constant of integration.

The DE that is being solved is consistent with the set-up previously described. Remember that the integrand shaft can be interpreted as the derivative of the integral shaft by using the Fundamental Theorem of Calculus. So, if we let y' denote the integral shaft then

$$y = \int y' dt$$

and the DE being solved is

$$\frac{dy}{dt} = k,$$

which implies we obtain

$$y(t) = kt$$

on the differential analyzer. Analytically, this would be the case where y at $t = 0$ is 0, or $y(0) = 0$. But for solving on the DA we know that $y(0)$ is not necessary since it does not explicitly appear in the DE, changing the value of $y(0)$ does not change the solution drastically and, if we so desire to start at a particular y-intercept value, we may let any arbitrary position on the y-axis be that initial value, and continue the quantification of this linear example from that standpoint.

Another simple example is the second order DE,

$$\frac{d^2y}{dt^2} = k.$$

For this DE we need two pieces of information to get a particular solution. First let's solve it in general. We may integrate both sides twice to get

$$\int \int \frac{d^2y}{dt^2} dtdt = \int \int kdtdt$$

which yields

$$y = \int (kt + c)dt$$

which gives

$$y = \frac{1}{2}kt^2 + ct + d,$$

where c and d are constants of integration. Furthermore, if we impose the initial condition $y'(0) = 0$ and $y(0) = 0$, we get the particular solution

$$y(t) = \frac{1}{2}kt^2,$$

whose graph is a parabola.

On the differential analyzer, the set up is a simple continuation of the first example (Refer to Figure 4.3 for a Bush schematic.). For one integrator, set it up so that the integrand shaft is constant, send the motion of the linear function that comes out of the integral shaft to the integrand shaft of another integrator. Integrator 2 now has a linear rate of the change. The integral shaft output of the second integrator will be the desired quadratic function. In the schematic diagram, Figure 4.3, the independent variable shaft connects in two places, that is, to the corresponding bus-shafts that drive the differential shafts of both integrators. The input for the first integrator is constant; its output essentially drives the input for the second integrator. The output of Integrator 2 is the desired solution and it will drive the ordinate carriage of the output table. The independent variable, in addition to the two differential shafts, also drives the table carriage of the output table, thus providing a plot of y vs t.

To compare the solution given by the differential analyzer and the general solution, note that. As was the case in the last example, the dependent variable y is not explicitly available in the DE. The only requirement given by the differential equation for the values for y'', y', and y at $t = 0$, is that y'' always equals k. As will be seen later, a relation between the initial conditions of variables is often given by the DE itself. However, in this example, we have $y(0)$ being arbitrary yet fixed, $y'(0)$ being the value of the initial displacement of the integrand shaft of Integrator 2, and $y''(0) = k$. If we let the initial displacement on Integrator 2's integrand shaft

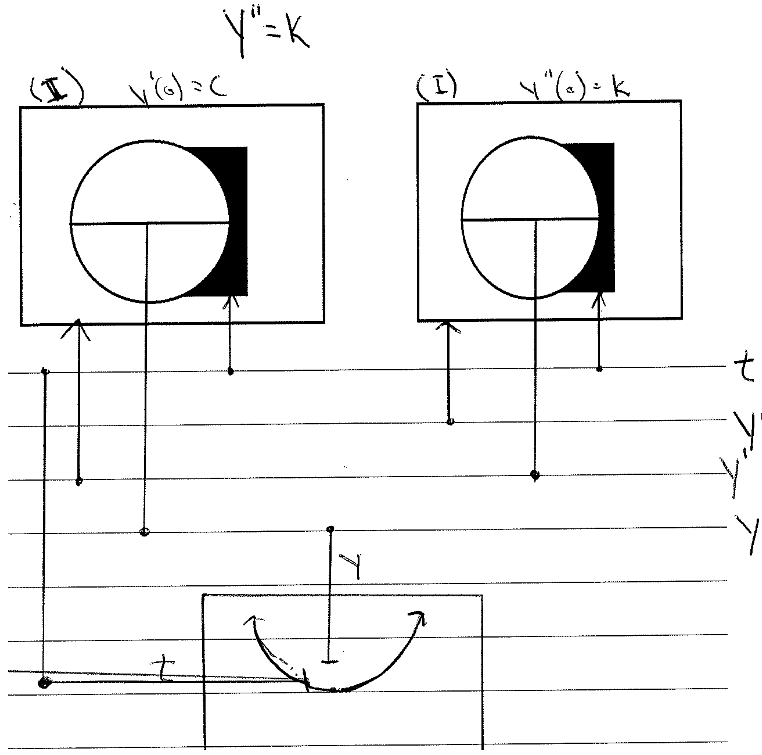

Figure 4.3. Schematic diagram $y'' = k$, where the output is a quadratic function.

be $y'(0) = 0$, and set the ordinate carriage to satisfy $y(0) = 0$ then the plot will be the positive half of an upright parabola, since the solution is $y = \frac{1}{2}kt^2$ on the interval $I_0 = [0, t]$, where t is the value when the independent variable motor is stopped. This is a particular solution of the general solution $y = \frac{1}{2}kt^2 + ct + d$. Note that on the differential analyzer it is not necessary to have an initial condition for y at $t = 0$. The reason for this is the same as in the previous example; y is not explicitly available in the DE. So for any value of y at $t = 0$, the graph is transformed by a vertical shift up or down the y-axis, which we can call whatever we want. So, d is arbitrary yet fixed and k is the initial condition for Integrator 1 to be determined by the DE itself. The value of c in this example is the initial condition for Integrator 2, since $y'(0) = c$. Note that y' is also not available in the DE. However, the given initial data may be useful analytically because the DE has order 2 and we need two pieces of information, chosen from $y''(0)$, $y'(0)$, and $y(0)$, to get a particular solution. It is the case when using a differential analyzer that the particular information needed to solve any DE is the initial positions of the integrand shaft for any given integrator being used in the mechanical calculation of the solution. Because the integrand shaft corresponds to a derivative, or equivalent value, all initial conditions must be at least given for each derivative of the DE. So we need $y'(0)$ to get a particular solution on the differential analyzer, but the DE itself does not require it to be any particular value. In other words, we have the freedom to impose whatever initial condition on y' at $t = 0$ we want. The DE will always be satisfied for our chosen condition, providing us with a unique solution for that particular condition. It is interesting to see how different initial $y'(0)$ values affect the shape of the parabola. To complete this kind of graphical analysis, one needs to create a standard on the output table that is for an x-axis and y-axis. In this case, it is convenient to draw the axes such that the origin is in the center of the plotting surface because, for a parabola, one might be interested in negative independent variable values. To do this, set the initial conditions of the integrators as previously stated and set the ordinate carriage at the center. Then simply run the independent variable motor in the reverse direction to the desired position. Stop it, reset the ordinate carriage back to the center, and reset the initial conditions of the integrators. Then run the motor in the positive direction, which will give a nice curve for a range of negative and positive independent variable values. Note that the positive and negative sides of the disk should not be changed because that would reverse the entire convention that has been previously established for the initial conditions with respect to their positivity. In many cases, when solving DE's, an operator is interested in positive values, but if needed, negative values are easily obtainable.

As the general solution is already known, it is easy to see the effect of changing c by writing

$$y(t) = \frac{1}{2}kt^2 + ct + d$$

in an equivalent form by completing the square on the right hand side to get

$$y(t) = \frac{k}{2}(t + \frac{c}{k})^2 + (d - \frac{c^2}{2k}).$$

In an analytical sense, we see that the value of c essentially changes the coordinates of the vertex of the parabola, when d and k are fixed. Moreover, if the value of c is positive, the vertex is shifted to the left, and if c is negative the vertex is shifted to the right, for k positive. For negative k, this situation is reversed and the parabola is now concave down. Information about the roots may also be obtained if desired. Because all the constants in the equation correspond to initial conditions on either the integrators or the output table, the same information may also be obtained from using the machine.

The differential analyzer provides a different perspective on the different transformations of the graph because the information is in the context of various rates of change. The first integrator has two important pieces of information, the initial position of the integrand shaft k, and the output or integral shaft. If k is positive, the linear function coming from the output of Integrator 1 is increasing. Mechanically, this means that the rotation of that shaft is in the positive direction. This increasing linear function is the rate of change for Integrator 2 and it will pull the integrator carriage in the positive direction, drawing the wheel ever closer to the positive side of the glass. If the position of the wheel on Integrator 2 is initially on the negative side of the glass, then the wheel will eventually pass through zero. Ultimately, reversing the direction of the wheel results in the ordinate carriage (dependent variable) reversing direction as well. Remember that the integrand shaft will move the carriage and essentially change the position of the wheel with respect to the center of the disk. So if the initial position of Integrator 2's wheel sits on the negative side of the disk, the plot of the solution starts off by decreasing and decelerates in the negative direction until it eventually stops at the center of the glass. This value represents the vertex of the parabola. Also, it is instructive to note that the constantly changing position of the carriage represents the derivative of the output, or the derivative of the quadratic function, that is the solution to the DE. Hence, when the carriage lines the wheel up with the center of the disk, it is the same as setting the derivative equal to zero and the y-value on the output table at that point is the critical point. Take note as well that the position of Integrator 1's carriage represents the second derivative of the quadratic solution. If that fixed position is positive then the graph would be concave up resulting in the critical point in question being a minimum, which only happens at one infinitesimal point. Then the carriage continues past the center to the positive side of the glass and continues to increase until the independent variable drive is stopped. This process is entirely observable, and this is just one example of many possible initial conditions and thus transformations that are available. If we start k negative, then the graph would be concave down and, depending on the other initial condition of Integrator 2, the graph would either be increasing or decreasing. Regardless, the position of the integrator wheel with respect to the center of the disk will be consistent with the first and second derivative tests, depending on where the wheel sits on the on the surface of the disk with respect to the center of the disk.

This situation becomes much more elaborate when the quadratic function is sent to the integrand shaft of a third integrator, which implies that the solution coming from the output of the third integrator is a cubic function. The DE formally used is

$$y''' = k$$

and the schematic diagram may be referenced in the Appendix. Suppose that the initial conditions of the first two integrators are as in the above example, when k is positive and Integrator 2 starts at a negative initial position. Recall that Integrator 2 passes through the center exactly once, which means that Integrator 3 has the potential to pass through the center at most twice, depending on the distance from center relative to the distance the wheel is from center of Integrator 2. Further suppose that the initial position of integrator 3 is half the magnitude of the initial position of Integrator 2, and Integrator 3 rests on the positive side of the glass. When the independent variable drive is started Integrator 3's carriage will begin moving in the negative direction. Integrator 3 will pass through zero before Integrator 2 because of its relative distance from the center the disk compared to that of Integrator 2. When this happens, the ordinate carriage will reverse direction creating a critical point. A few units of time later, Integrator 2 will pass through the center, reversing the direction of Integrator 3, causing it to now be pushed in the positive direction. In the meantime, Integrator 3 has managed to accumulate some considerable magnitude on the negative side. So it will take a few more units of time to get back to the center. When Integrator 3 does get close to the center, it will pass through it one more time, causing the ordinate carriage to reverse its direction yet again, creating another critical point. So there are two critical points total for Integrator 3 and as the independent variable drive continues, Integrator 3 will never pass through the center again. Because Integrator 2 continues on its path in the positive direction, as would be expected with a quadratic function after it has reached its vertex, there are not any other critical points, which would make perfect sense, because a cubic function has at most two critical points. Moreover, the correspondence of the first and second derivative tests with the integrator wheel positions are still consistent with the side of the disk the wheel sits on at any given time. The only difference now is the first derivative is now the position of Integrator 3, the second derivative corresponds to the positioning of Integrator 2 and a new third derivative is referenced on Integrator 1, which is of course the constant k in this case.

There are many more cases, using different sets of initial conditions for the cubic problem and the quadratic problem. Far too many to be discussed here. It is beneficial for an operator to become familiar with these three simple examples for they will provide the building blocks for becoming more and more familiarized with the machine and the particular scaling. Unfortunately, the scaling is problem dependent, but, if one can correlate the mechanical interrelationships with the mathematical principle of these simple algebraic functions, then one can conceive a more practical understanding of how the differential analyzer is beneficial to students and to a novice operator. This exercise of linear to quadratic to cubic functions produced by the multiple integrations of a constant function was designed and refined by Dr. Lawrence and the DA Team. The aim was to create a visual interpretation of the relationship between first, second and third order polynomials in terms of their rates of change so that elementary algebra and first-level calculus students can better understand the nature of these functions. The goal of the exercise was to highlight the fact that a cubic function has a quadratic rate of change, a quadratic function has a linear rate of change, and a linear function has a constant rate of change. Moreover, to do so without formally defining a derivative was an underlying theme, at least for elementary algebra students. Experiments with such exercises has proved to be exciting and fun for both the students and the professor. It was a nice change of pace from the classroom norm and also proved to be very beneficial to the operators.

4.3. SCALING A DIFFERENTIAL EQUATION IN THE SIMPLE LINEAR CASE

One very important test that an operator should be familiar with is the circle test. The objective of the circle test is to get an output table to graph a circle so that an order of error may be established within some interval of the independent variable. More specifically, we need to find an equation whose solution is a circle. Because the circle is not a function we need to find some other relation that is. The differential equation

$$y'' = -y$$

or simple harmonic motion, has a general solution given by

$$y(t) = A_0 \cos(t) + B_0 \sin(t),$$

where A_0 and B_0 are constants. Furthermore, if we impose the initial conditions $y''(0) = 0$ and $y'(0) = 1$, then we get the particular solution

$$y(t) = \sin(t).$$

Because the phase plot of y vs y' is $\sin(t)$ vs $\cos(t)$, we can use the output table to plot the two given functions of t in a parametrical plot. As it is a well known fact that these parametric equations produce a circle, simple harmonic motion is the best candidate equation for the circle test. Specifically, the circle test compares to the graph of the unit circle, where the critical values of the phase plot for $y'' = -y$ corresponds to the four intercepts of the unit circle with the axis in a Cartesian coordinate system. Moreover, a very nice trigonometric exercise for pre-calculus students is to begin by taking the general solution

$$y_1(t) = A_0 \cos(t) + B_0 \sin(t)$$

and differentiate it once yielding

$$y_1'(t) = B_0 \cos(t) - A_0 \sin(t).$$

Using the algebraic relation of a circle

$$x^2 + y^2 = r^2$$

substitute $x = y_1(t)$ and $y = y_1'(t)$ to obtain an expression of the form

$$(\sin(t))^2 + (\cos(t))^2 = \frac{1}{A_0^2 + B_0^2} = r^2.$$

Now one can truly see that this phase plot is still a circle with different radii corresponding to different initial conditions. Any linear combination of $\sin(t)$ and $\cos(t)$ plotted against its rate of change will be a circle where the values of A_0 and B_0 determine the size of the circle's radius. So this exercise can relate concepts of trigonometry with calculus preparation. For this exercise, the differential analyzer provides a visualization of the process when the initial conditions are changed, thus changing the values of A_0, and B_0. The Marshall DA Team has had much success with demonstrations of this type for pre-calculus students, an added bonus of the circle test, which was designed for purposes of mechanical error detection.

Forcing an output table to graph a circle will provide insight (i.e., mechanical accuracy) into the accuracy of two integrators, a half section of interconnection and an output table. If the circle is drawn through 2π unit turns of the independent variable and there is no discernable difference from where the circle begins and ends, an operator is assured that within one period of a solution the machine has produced minimal mechanical errors. Moreover, in general, the two integrator wheels have each passed through the center twice without slippage and without backlash in the gear trains. The process may be continued for any number of periods. If the circle continues to draw over itself, then the components being used offer minimum mechanical errors for the given number of periods. In practice, the error in the circle test will depend on the sophistication of the parts being used. With Meccano, elimination of

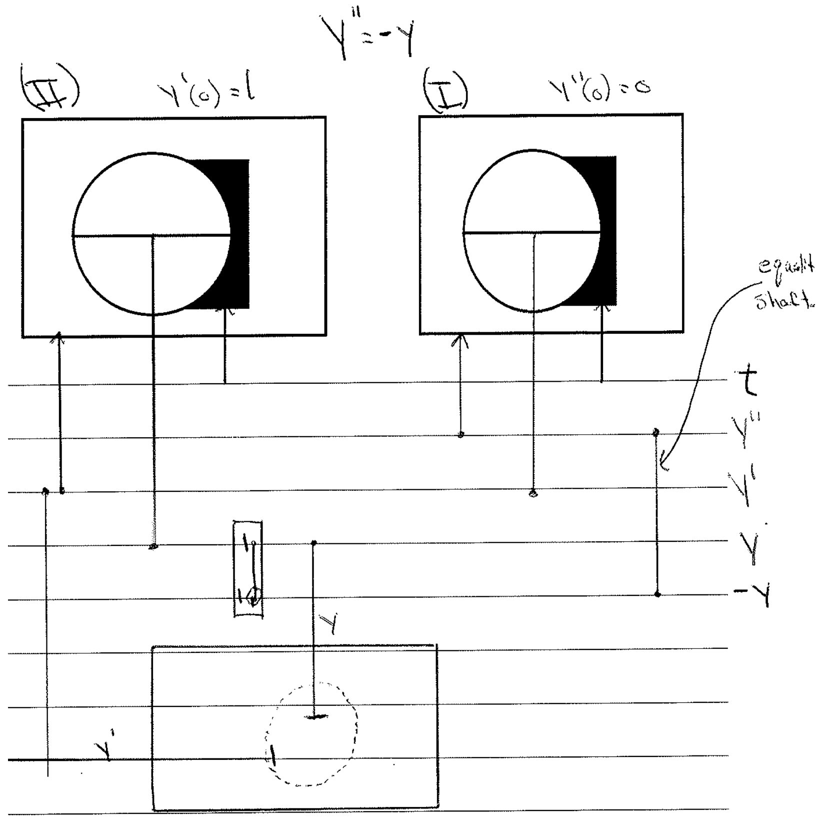

Figure 4.4. Schematic diagram $y'' = -y$, or simple harmonic motion. In this case, the output is parametric plot of a circle.

backlash altogether is not feasible. However, because the errors are mechanical, they may be accounted for as they occur in "real time." The addition of frontlash units greatly reduces the effect of backlash. The use of these units resulted in an error in Bush's machine of 1 part in 10,000. The Manchester machine, built of Meccano, had frontlash units, and the error was measured to be 1 part in 1000 of a unit. For "Art," which does not have frontlash units, the error is a little better than 1 part in 100. That is, a little over two full shaft rotations of the counter for each integrator per period. Figure 4.4 is the schematic diagram for simple harmonic motion. Notice that the values on top of the integrator correspond to the initial values for each integrator at $t = 0$. This diagram is called a *general schematic* because it does not introduce any scale factors. Moreover, any time the scaling of a differential equation is not clear, it is best to create three different schematics: a general schematic, an algebraic schematic and a final schematic with scale factors. In the general schematic, we consider the differential equation without scale factors so that the basic connections for each of the integrators involved can be determined at an early stage. We can see at this point that at least two integrators are required because the DE is second order. For the section of interconnection, cross-shafts are represented by horizontal lines and component bus-shafts are vertical lines. Note that the connection of a component bus-shaft to a cross-shaft in the section is denoted by a darkened dot and an open circle-dot represents a sign change within the direct drive connection. Changing signs schematically is a matter of notation. (Here we are using an open circled dot to denote a sign change.) But in practice changing sign means opposite rotation. So there is no need to consider clockwise or counterclockwise rotations in order to determine whether a shaft has been negated. Simply observe: If the two shafts are spinning in opposite directions, there exists a negation of sign between them. Furthermore, a 1:1 gear train preserves magnitude but reverses direction in practice. It is not always necessary to keep track of all gear train negations schematically. Just to be sure that a quantity is indeed negative, diagrammatically, any such quantity will be represented with a minus sign in the labeling of its cross-shafts in the schematic.

As mentioned earlier, the solving of differential equations with a differential analyzer is a term-wise reduction of derivative process. The idea is to start with the highest order derivative $y^{(n)}$, pass that motion through an integrator to get $y^{(n-1)}$ and repeat this process until the dependent variable is obtained. The differential equation itself will establish equality by explicitly relating the highest order derivative with the other terms in the differential equation. The equality in the machine is made by connecting two shafts together as needed. If there exist any terms other than the dependent variable, its derivatives, and the independent variable, such as nonhomogeneities or nonlinearities, those terms must be generated by other means that will be discussed later. Other nonlinear terms include any type of products, nonlinear or other-wise. Simple harmonic motion is an example with simple terms, but its setup is fundamental to the operation of a differential analyzer.

To create the Bush schematic, first notice that the DE is linear. On the machine this means that all integration is with respect to the same independent variable, t. Hence, the t shaft is the first labeled cross-shaft and it runs along the section connected to the differential shafts of both integrators involved (t is also the primary motor drive). Observe that the second bus-shaft is labeled

$$\frac{d^2y}{dt^2}.$$

Using the second bus-shaft is convenient because this term is the highest order derivative term in the DE, and solving the DE is a termwise process. However, note that we could have just as well called it y. We could solve an equivalent integral equation that describes the same rates of change as the original DE, as

$$\frac{d^2y}{dt^2} = -y$$

is equivalent to

$$y = -\int\int y dt dt.$$

In fact sometimes it is more convenient to do so because this reduces the order of the DE by one, implying the number of integrators needed to solve a particular DE is decreased by one. But in this example we will continue in the general case of using the highest order derivative, as changing the original form does not help.

Starting with the highest order derivative

$$\frac{d^2y}{dt^2},$$

we send this motion into Integrator 1's integrand input shaft. Hence, the output is the integral of that with respect to t, by the Fundamental Theorem of Calculus, the output is

$$\frac{dy}{dt} = y'.$$

This motion is in turn sent to Integrator 2's integrand shaft and gets integrated with respect to t, to yield y. Then from the output of Integrator 2 the motion passes through a 1:1 train of gears reversing the direction of motion of that shaft and essentially becoming a negative value with the same magnitude. From the differential equation itself we know that the second derivative of y must always be equal to the opposite of y. Hence, the cross-shaft labeled $-y$ is directly connected to the cross-shaft labeled y''. This, as the DA Team likes to say, "completes the circuit." The "circuit" represents the statement of equality. That is, representing the differential equation itself. Now, if we so desire, we may plot any two quantities available in the section. Because we want the machine to draw a circle and we know the solution to the differential equation with the given initial conditions is $y(t) = \sin(t)$, we will plot the solution y against its derivative y'. Because, $\frac{d(\sin(t))}{dt} = \cos(t)$ the parametric phase plot therein will force the output table to draw a circle (See Figure 4.4.).

The second schematic for simple harmonic motion is one that includes algebraic scale factors and disregards sign. Because we have a general idea of where the motion needs to be reversed to yield the sign change, we can leave it out here because now we are concerned primarily with the magnitude of the shafts. So referring to Figure 4.5, we see a much more elaborate schematic of the same equation. We start by labeling the independent variable shaft At, for A turns of the t shaft. Next we label the second

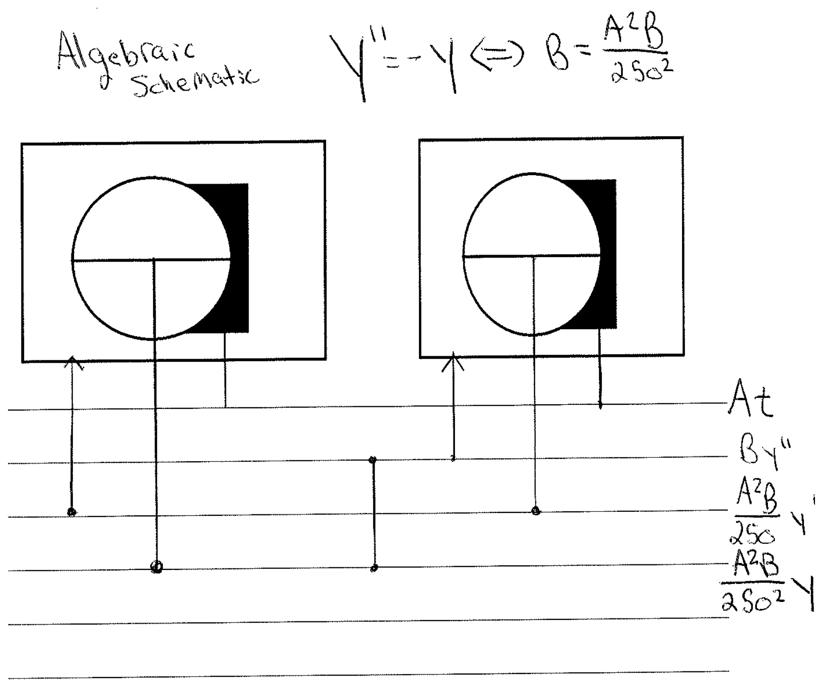

Figure 4.5. Basic algebraic schematic diagram of simple harmonic motion.

derivative shaft as By'' for B turns of that shaft. Formally, A is defined to be the number of unit turns in the independent variable shaft and B is the number of unit turns of the highest order derivative shaft. All other shafts available in the section will be relative to these two shafts. Thinking about the operations mechanically, the first two inputs of Integrator 1 yield an output relative to the constants A and B. Then the two inputs of Integrator 2 are relative to the output of Integrator 1 and the independent variable, so that Integrator 2 yields an output in terms of the constants A and B as well.

In a mathematical context, we have the output of Integrator 1 being

$$\int By''d(At) = \frac{AB}{I}y'.$$

Remember that upon every integration, the integral quantity is multiplied by the integrator constant I^*, or equivalently, divided by the *Unity* of an integrator, I due to the reduction gears of the mechanical integrator itself ($I = a/(KP)$=250). Taking that output of Integrator 1 and integrating it yet again with respect to the independent variable shaft gives

$$\int \frac{AB}{I} y' d(At) = \frac{A^2 B}{I^2} y,$$

because we can pull out the constants A, B and I. Consequently if we want to maintain the equality given by the differential equation, then this output of Integrator 2 must be connected to the shaft originally labeled By''. So we have a new relation given by the equality of the machine that must be maintained and be consistent with the equality of the differential equation we want to solve. That is,

$$By'' = \frac{A^2 B}{I^2}.$$

It is easy to see that the relation for this equation that needs to be maintained must be

$$B = \frac{A^2 B}{I^2}.$$

As the equation is linear and homogenous in y we can, in a sense, reduce B in the equality to yield $A^2 = I^2$ or $A = I$.

With respect to this relation, we have the suggestion to let $A = 250$, since 250 is the counter reading that represents the *Unity* of an integrator. However, reducing B to 1 in the relation is not always the best approach because DE's that are nonhomogenous, and especially those nonlinear, can have more complicated relations. The next section is an attempt to quantify such nonlinear problems. Unfortunately, scaling a differential equation on a differential analyzer is problem dependent. In simple harmonic motion, not only are we working with a simple equation, but all information is known about the solution. Also, there are points of symmetry that allow very simple scaling. For instance, we can gear down the outputs of both integrators by the same amount and still maintain equality within the scale of the algebraic factors. In this case, it is useful to use B in the relation so that we can increase the value of A and still satisfy the condition in the relation. However, in doing so, one will change the scale of *Unity* among the various components of the machine. Let's return to the case of simple scaling, that is, when $A = 250$ and B has been completely eliminated from the relation. Substitution of these values into our expression yields a final schematic, Figure 4.6. This type of scaling will be called *Unity scaling*. (Choosing an A value that inversely corresponds to the integrator constant.) When the motions of the variable shafts are passed through an integrator, the integrator constant reduces with the unit turns of the independent variable. This means that the *Unity* of an output table and other components, as well as the integrators, have the common value of $I = 250$. Notice that if we disregard the choice of B in the final schematic (Figure 4.6) we will obtain a plot of $1 * y$ vs $1 * y'$. There is a slight ambiguity in doing so. By reducing B in an analytical sense, we have inadvertently lost information about the unit turns in the dependent variable y, in a mechanical sense. The issue is: a coefficient of $1y$ in the section yields a value of *Unity* on the output table of 250 shaft rotations being one unit. This is due to the *Unity* of the integrator being 250. In order to maintain consistency between the various components, we will not reduce B in the schematic. We will let $B = 250$, so that now in the schematic we have a plot with coefficients $250y$ vs $250y'$ on the output table (See Figure 4.6). Now we have a nice mathematical comparison between the unit variable on the machine and one unit in a pure mathematical sense (250=1). The reader should be very aware that this nice property only exists for certain classes of linear ODE's (autonomous). So when an algebraic scale factor is reducible, an operator should not disregard its existence. An operator should instead note that the relation of equality in the DE

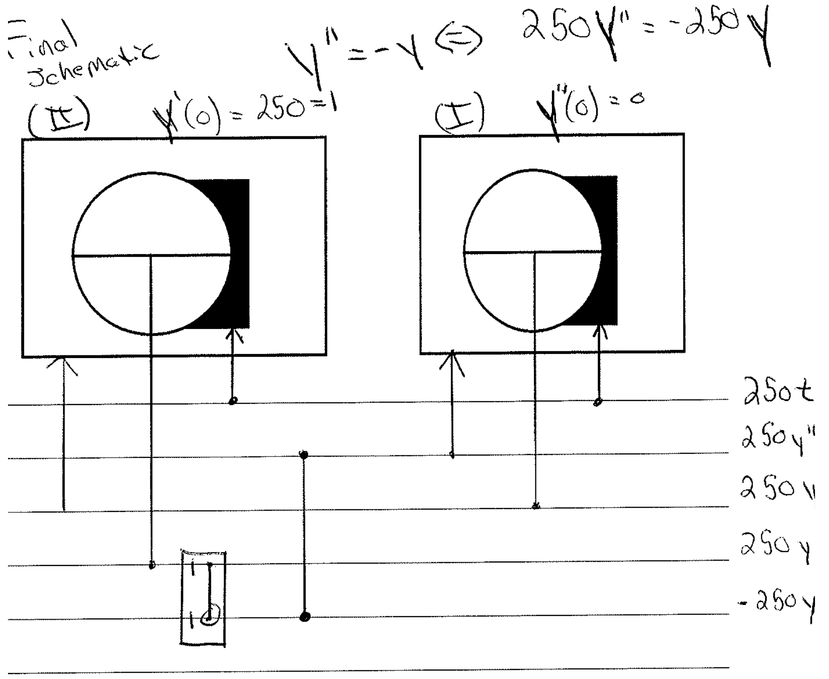

Figure 4.6. Final schematic diagram of simple harmonic motion.

doesn't depend explicitly on that factor. However, within the interconnections of the machine, that factor may still be useful and must be accounted for in the plot and various components such as an integrator.

This usage of the relation with the algebraic scale factors is due to Crank [6]. He gives a very similar example, although, on the Cambridge machine it was necessary to gear down the outputs for mechanical reasons (due to torque amplification). So in his example he basically let $A = 8 * I$ and let $B = I$, to allow the B value to reduce the *Unity* of the integrator and the A value to reduce the gear trains and still satisfy the relation of equality in the machine. So the question arises, which scaling is better? The answer is, whatever scaling provides the maximum use of the full range of the various components, most importantly the integrator disk, while still maintaining the equality of the relation. In the Cambridge machine, it was necessary for this problem to be scaled in such a way for mechanical purposes and maximum range. Although the interesting point of fact in Crank's example is the coefficients of the plot. When this problem is scaled in this way the plot of y vs y' has the coefficients of $4I * y$ vs $4I * y'$. This plot is certainly still a circle for reasons of symmetry. Keep in mind that this is a phase plot, and, as long as both inputs of the output table have the same gearing coefficient, the graph maintains its original properties. But the relevance of these scale factors, with respect to unity, is what is in question. (If the reader is interested in reading the Crank example, (See page 39-49, [6]), it should be noted that his scale factors are in terms of variable shaft rotations and ours are in referenced counter revolution readings.)

Scaling "Art" as Crank did the Cambridge machine in his paper, would yield a plot of $1000y$ vs $1000y'$, $A = 2000$, and $B = 250$. Furthermore having set our initial conditions to $y''(0) = 0$ and $y'(0) = 250$, (numerical values are counter readings here and remember 250=1 analytically), then we get a circle of radius 1000 (counter reading). Analytically speaking, our circle should be the unit circle, so the radius should be one (250=I=1), since the particular solution is $y(t) = sin(t)$. However, our radius is $4 * I = 1000$ because we have expanded the scale of *Unity* on the output table by four times our I value. At this point we should define the difference between scaling down and scaling up. Scaling up is increasing the unit variable coefficients, or zooming in on a window. Scaling down is decreasing the unit variable coefficients, or zooming out on the window. The window is the plotting surface of the output table. Note that scaling up or down is not generally consistent with the convention of gearing up and down, since gearing up and down may take place in various components resulting in different consequences of numerical magnitude.

One could chose to plot $1000y$ vs $1000y'$, to follow Crank's example, but one can just as easily plot off any two shafts that are representative of y and y', respectively. That is, we could have chosen to to plot $500y$ vs $500y'$, resulting in a circle of radius

500. (We actually need this plot because our table range is too small otherwise.) If $500y$ and $500y'$ are available among the various cross-shafts in the section of interconnection we can choose these two. These alternatively scaled algebraic and final schematics are depicted in Figures 4.7 and 4.8, respectively. These algebraic schematics are different due to the fact that each integrator must be sufficiently geared down in order to maintain machine equality. This gearing down by 8 is clear from the newly established relation of equality in Figure 4.7. Substituting $A = 2000$, $B = 250$ and $n_1 = n_2 = 8$ into the relation of equality in Figure 4.7 yields the final schematic Figure 4.8.

There are good reasons for doing a test of this type, not only for analysis of mechanical errors, but for clarification of scaling freedom. For mechanical error analysis, the expanded scale coincides with the results on the output table, which is maximum at a coefficient of 1000 for the circle test. That is, the circle drawn with radius 500 is the biggest circle that can be drawn on an output table of this size. So the test is an extreme case scenario for mechanical errors in the plotting table itself. Moreover, because both integrator outputs pass through several gear trains and are geared down at that, they are, as well, at an extreme case scenario for backlash. Moreover, for the process of scaling, this example provides many cases for different conventions of $Unity$.

Notice that if we take 2000 to be our unit for both independent and dependent variables, then the scale on the output table is scaled up. Also plotting $500y$ vs $500y'$ is beyond the maximum range for the window of observation on the Output Table. An availability for 2000 shaft rotations does not exist on the plotting surface. However, all one needs to do, to a reading on the output table counter, is multiply that reading by 4 to achieve the proper value in terms of one unit. Hence, the plot is of the form $\frac{1}{4}y$ vs $\frac{1}{4}y'$, in an analytical sense. That being said, one could also simply divide a counter reading taken from the output table by 500 in order to obtain an

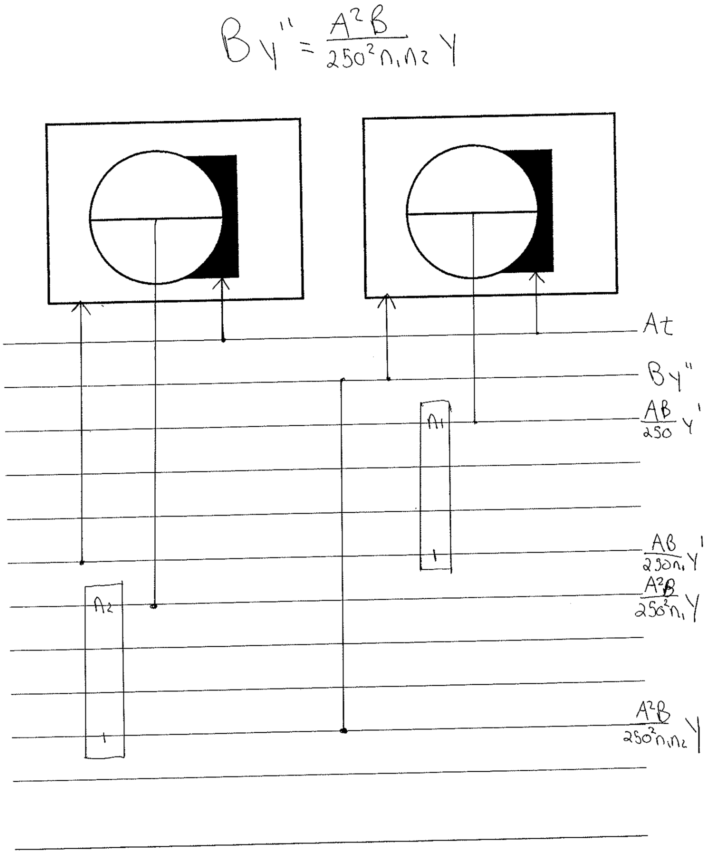

Figure 4.7. Algebraic schematic diagram of simple harmonic motion with the introduction of the gear trains, n_1 and n_2.

analytical value at that point. Moreover, dividing out the coefficient of a particular shaft is best so that only one calculation is needed to get analytical results. So this difference is not severe at this point. But what can be said about the *Unity* of an integrator? For now it remains at the previously established convention $I = 250$. And in this case of *Scale Folding*, we are saying the convention of *Unity* on an integrator is different from the value of a unit in the independent and dependent variable and the *Unity* of the output table. Note that the *Unity* of the independent variable is

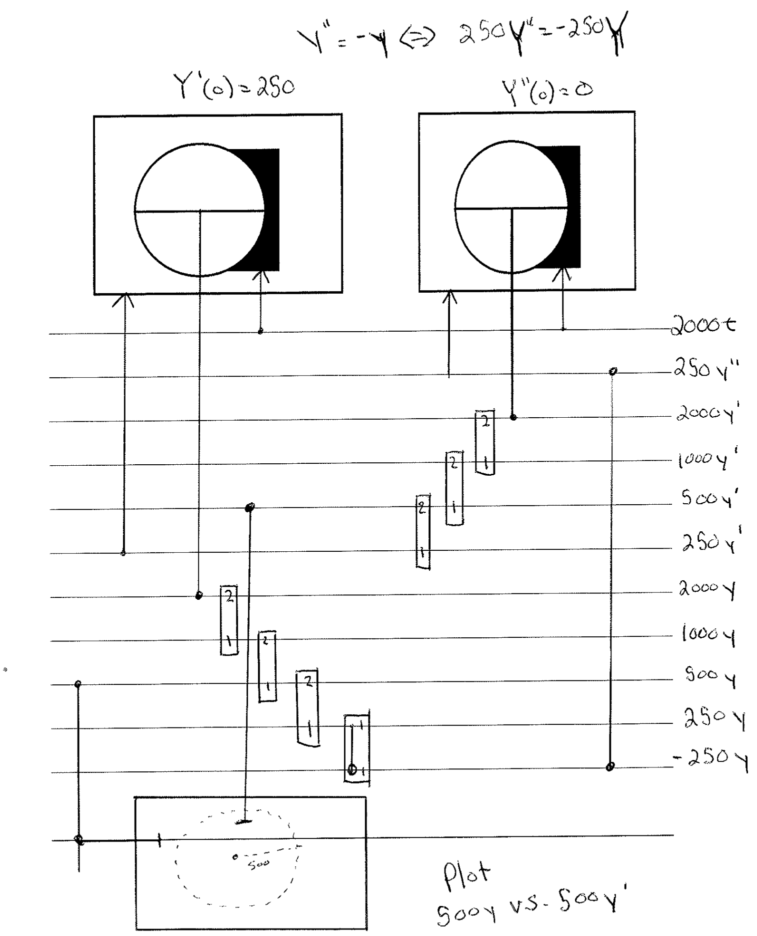

Figure 4.8. Final schematic diagram of simple harmonic motion with added gear trains, where $n_1 = n_2 = 8$.

formally the number of turns that shaft turns to get to one unit. So if that unit variable shaft is geared down independent of the integrator connections, so as not to affect the differential shafts of the integrators, then we have a new representative for the independent variable with a smaller value for one unit.

On the output table we can plot off of this smaller value and take smaller increments of one unit per shaft rotation. Conceptually, gearing down a shaft coefficient is the same for the dependent variable as it is for the independent variable shaft.

(This is what we have done in this last example.) For an integrator it is clear that its *Unity* need not be the same as other component unities. Because the *Unity* of an integrator is defined as the value at which the integrand shaft needs to be to provide a one-to-one correspondence between the differential and integral shafts. Note that the independent variable and differential shaft are not always the same. In the case of linear differential equations, the independent variable shaft turns the differential shafts of the integrators involved. So the value of *Unity* of an integrator isn't necessarily the same as the independent variable shafts unit value A. The *Unity* of the independent variable is established for a specific differential equation that is to be solved. Otherwise it's just a drive shaft. The choice of the scale factor A is made to maintain the equality of the relation described by the differential equation and to fold the scale as needed to provide a good window of observation on the plotting surface of an output table. The *Unity* of the output table is always relative to the value of the unit variable and the *Unity* of an integrator. The key factor is to establish a convention before plotting the solution. Even then some adjustments of the A value and B value may be needed.

One of the best ways to keep track of the different scales on the differential analyzer is to have two separate but equivalent equations: an analytical equation and a machine equation. The analytical equation is the pure mathematical equation, the differential equation. The machine equation is the differential equation in terms of the scaling of the machine. In the above example, we have the analytical equation

$$y'' = -y$$

and the machine equation

$$250y'' = -250y$$

that is, of course, the same; but in the latter of the two, we have a scale factor on both sides of the equation, namely 250, that ensures the differential analyzer is solving the particular differential equation of interest. It is no surprise that the value 250, which is the value of B, and the value of $Unity$ for an integrator appears in the machine equation. For linear homogenous differential equations, the value of B will reduce, and its significance is diminished. This is not to say the algebraic scale factors are not useful because they provide a baseline for the selection of scale convention. Moreover, if B is reduced then the freedom of scaling is simplified to $Unity\ Scaling$ to an extent where the choice of B will always be the same as the $Unity$ value of the integrator and the relation of equality in the differential analyzer is always maintained. But the introduction of such a concept will provide insight to more scaling options such as folding the scale or scaling down an integrator disk. When doing so, remember the scale factor B need not be reduced. A complete algorithmic method of scaling the machine is not available here for differential equations of all classifications, but analysis of this type is the ground work for such generalizations.

4.4. SCALING THE INTEGRATOR

Strictly speaking the $Unity$ of an integrator may be manipulated in two different ways. The first of which, being the most straight forward, is literally change the reduction gears that define the $Unity$ of an integrator. For example, on "ART" we have $K = 2/5$. If we add another reduction gear of say $1/2$ then our new value is $K = 1/5$. By doing this we have changed the $Unity$ of an integrator to $I = 500$, or scaled up by 2. It is often useful to do so for reasons of accuracy. For example, if our initial condition is a small fraction, then we may have available more finite points which will result in better precision accuracy of the wheel measurements. Now granted, all real numbers between 0 and 1 are available on the surface of the disk, because there are infinitely many concentric circles on top of the disk. Furthermore, all values in between are theoretically measured by the wheel. But, in practice, it is always better to utilize the full range of the integrator disk as there might exist large scale inconsistencies within the derivative or integrand shaft of that integrator when not doing so. These inconsistencies could result from the integrand shaft being turned at a rate significantly faster than the speed of the primary drive motor. Analogously, this type of error-effect could be duplicated if the scale on top of the disk is drastically scaled up relative to the shaft coefficient of the primary motor drive. Moreover, in order to feasibly measure some small fraction in an initial condition, a priori or posteriori, it needs to correspond to a counter reading. If the fraction does not correspond to a single rotation of the counter, then the operator is forced to approximate that value by turning the counter through some portion of a digit on the counter.

Using the gearing of the counter, when $I = 250$, a reading of 001 is .3 shaft rotations of the lead-screw. If the *Unity* equals 500, and we want an initial condition of $y'(0) = .25$, this would correspond to the counter reading 125, as opposed to 62.5. So when scaling in this way we can often get precisely measured initial conditions as well, since we are counting by increments of (1/500) instead of (1/250). Furthermore, this concept can be extended to any such fraction where the range of the disk equals any such fraction between 0 and 1. Note that the *Unity* of the integrator in a case such as this would not be available in the range of the disk as its value is well beyond that of 500 (the max range of an integrator disk). This method of changing the *Unity* directly is great for scaling up or increasing the *Unity* value. However, if we want to go in the other direction and scale down to get a smaller *Unity* value, the method is not sufficient for smooth mechanical operation because that would involve gearing up and could overload the torque amplifiers.

For the case of changing the *Unity* to a smaller value we chose a more complex method: absorption of Scale Factors, or *Absorption Scaling*. Note that by defining the scale factor B, the *Unity* of the dependent variable, we can manipulate B to change the *Unity* of an integrator. If the integrand shaft is a variable shaft in y, then changing its value changes the *Unity* of the dependent variable. In our previous example, we have y as a function of t. Ultimately, all variable shafts are functions of t or composite functions of t, but when scaling down the integrator we strictly need to consider the variable that is represented by the integrand shaft. Looking back, our previous example had a machine equation of $250y'' = -250y$, with $A = 2000$, $B = 250$. The value of B corresponds to one unit in the variable y'', and our integrator has a *Unity* value equal to that of $B = 250$. This was by design: we choose $B = 250$ so that there wasn't any discrepancy with the *Unity* of the dependent variable and the *Unity* of the integrator. However, since the B value is directly affecting the turns of the integrand shaft and the integrand shaft itself is a variable in y, specifically, y''. The B value may or may not equal the formerly defined *Unity* of an integrator. If B does not equal the *Unity* of an integrator, then our integrator constant is not completely reduced through the multiple integrations required to solve the differential equation. Due to the fact that the *Unity* of an integrator is independent of the *Unity* of the independent and dependent variables, the *Unity* of the dependent variable be manipulated by a choice of B. In doing so, one must take into account the integrator constant. For each subsequent integration, the integral quantity is multiplied by the integrator constant that has value equal to the multiplicative inverse of the *Unity* of an integrator. As this is the case, in our example of simple scaling, we chose $B = 250$, despite the irrelevance of the B value for that scale. By doing so we can also keep track of the *Unity* of the dependent variable for that scale, which is imperative when working with un-unified scales.

Changing the value of B to scale down an integrator is called *Absorption Scaling* because, in changing B, a factor is in essence absorbed into the *Unity* of an integrator. This absorption factor is $\frac{I}{B}$ and that value is essentially multiplied in the reduction gearing of the integrator to yield a smaller value of *Unity*. Let's say we want to have an initial condition of $y''(0) = 5$. If $I = 250$ then we would have to set the initial displacement of that integrator to a value of 1250, which is not available on the disk (maximum 500). So we let $B = \frac{I}{5} = 50$, to we have a new *Unity* of that integrator, denoted by $I_B = 50$. (Note that in this case the *Unity* of the integrator and the integrator constant are not multiplicative inverses.) With this new value we can obtain an initial condition of $y''(0) = 5$ by setting the initial displacement to 250. Moreover, we can get $y''(0) = 7$ by setting the initial displacement to 350, and we can go all the way to $y''(0) = 10 = 500$.

Note that changing the B value doesn't necessarily mean that the mechanical interconnections within the section will have to be changed. However, what will certainly change is the scale coefficient of each individual cross-shaft within the section, which will in turn affect the output and change the coefficients of the plot. For example, letting $A = 2000$ and $B = 50$ in the general algebraic schematic in Figure 4.7 will yield final schematic diagram Figure 4.9. Take care in noting that our machine equation is

$$50y'' = -50y$$

so that machine equality still satisfies the equality given by the analytical equation. Also notice the each cross-shaft other than the independent variable has a coefficient value 5 times less than the coefficient values of the schematic diagram in Figure 4.8. Additionally, the coefficient of the y' shaft that feeds Integrator 2's integrand shaft is 50. This insures the scale of *Unity* for that integrator is also 50, as is the case for Integrator 1.

In this setup it will be more convenient to plot the coefficient shafts $200t$ vs $200y$. One reason is we can now see the solution against the independent variable t. In doing this we can devise a test that will determine whether our scaling intuition is valid. We know the solution to $y'' = -y$ with $y''(0) = 0$ and $y'(0) = 1$ is bounded

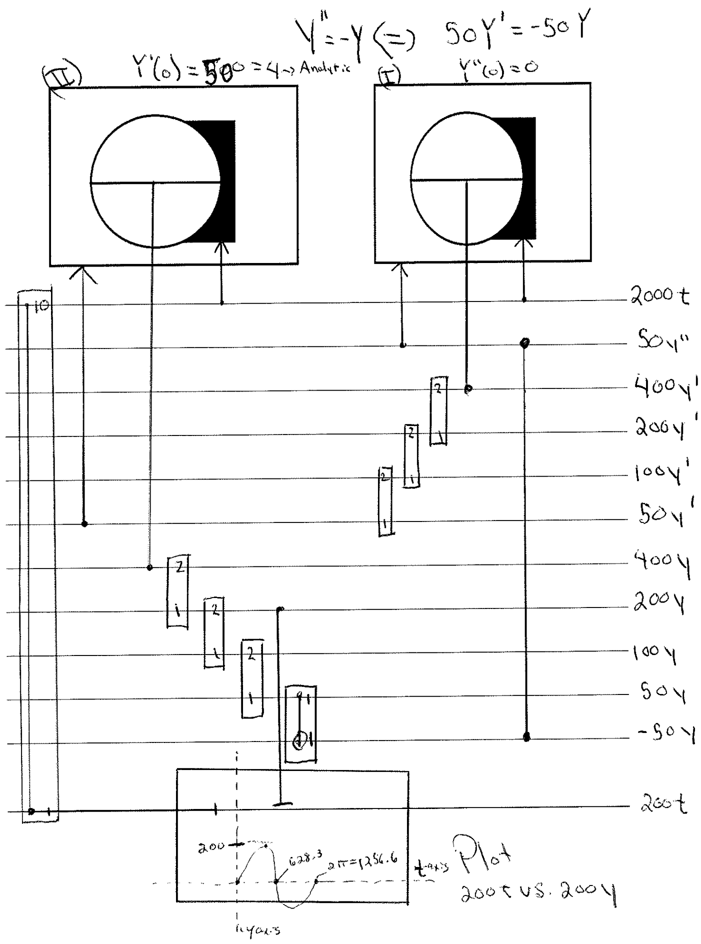

Figure 4.9. Alternate Final schematic diagram of simple harmonic motion.

in y by $[-1, 1]$, and also periodic for every 2π units of t. Furthermore, the distance along the t-axis from 0 to the first t-intercept has the value π. If we take the machine solution and measure the distance from 0 to the next intercept, measure the maximum height and take the ratio of those two values we should get a value approximately equal to π. Note that the machine initial conditions must be set to $y''(0) = 0$ and $y'(0) = 50$ and the plotting coefficient of the t and y shafts must be the same or the test is meaningless. This particular consistency is necessary because we have scaled down both integrators to 50. Now 50 on the integrator is equivalent to one analytically. This test was designed by Dr. Lawrence so that, no matter what scale was chosen, one could always check to see that given the proper initial conditions our scale was consistent with an analytical solution. When the DA Team did this test for this particular setup, we got the value $3.142 \sim \pi$. By using a set of vernier calipers to simply measure the distance on the graph we were able to calculate this ratio. If we had used counter readings, our measurement would have been much more accurate. Nonetheless, this tells us our scaling convention gives the same solution as *Unity Scaling* does.

Another reason for choosing to plot $200t$ vs $200y$ is that the output table has a maximum range of 1000 shaft rotations and values smaller than 200 will offer too small a window of observation, at least for the π test. In other situations we can vary the initial conditions and observe the qualitative effect on the solution. Because we have available initial conditions ranging from 0 to 10 analytically, we must be careful not to overshoot the range on the plotting surface. For example, if we have a machine initial condition of $y'(0) = 500$, that is equivalent to saying we have an analytical initial condition of $y'(0) = 10$. So analytically we know the solution now ranges from -10 to 10. If our *Unity* on the output table is 200 turns per unit, this is $200 * 10 = 2000$. This range poses a problem as the maximum range of plotting surface in the ordinate direction is 1000. In this case our pen will run off the surface of the table (causing potential damage to various components). To avoid running off scale an operator would need to be modest in the choice of *Unity* for the output table. For this example, the choice is simplified because we know the analytical solution and the effect of increasing the magnitude of initial condition. We will need to be consistent with the value $I_B = 50$, the scaled down *Unity* of an integrator, if we want to run initial condition values at most equal to 10. However, if we don't know the solution, one must make a reasonable conjecture about the window of interest because there is this risk of running off scale if an initial condition results in an increase in magnitude of dependent variable beyond the range of 1000 shaft rotations. In most cases, choosing a scale will be clear in at most three tries, due to the observation of the movable parts. One begins to learn how to predict the innate nature of differential equations through mechanical intuition.

The scaling can be even more involved if one so desires to have different values of *Unity* for different integrators. In this case one would need to use a combination of *Absorption Scaling* and directly changing the reduction gear K for some other integrator to scale it back up relative to the first. The final type of scaling is called *OverScaling*. In the next section, we offer a special case of this type of scaling. *OverScaling* occurs when one chooses the value of A in the independent variable such that, after passing through an integrator, the value of the corresponding term of the output is significantly larger than it needs to be to maintain machine equality. That is, a composite number that can be factored to provide not only a scale factor but also another factor to be used within the equality of the section to avoid gearing up. In terms of mechanics, instead of gearing one quantity up we will instead gear all quantities but one down. This type of scaling is very useful and in most cases (besides the special case in the following sub-section) it will be used in combinations with other types of scaling.

Simple Harmonic motion is a good example to use for the concept of scaling because all information can be tested by comparisons to the actual solution which is fully known in this case. Machine equality is the most important concept to remember when scaling the machine. No matter which type of scaling is of interest, *Unity Scaling*, *Scale Folding*, *AbsorptionScaling* or *OverScaling*, machine equality must be satisfied. In terms of plotting coefficients, the choice is entirely a matter of preference to meet the needs of qualitative analysis. A plotting coefficient is merely a window of observation. One doesn't need to use the output table at all if the interests are of a quantitative nature because an operator can always divide a counter reading by its shaft coefficient to get a true analytical value from it if machine equality is satisfied, although it would be very uninteresting not to see the solution revealed before your eyes.

4.5. "T"-SUBSTITUTION

The t-substitution technique was first introduced to the DA Team by Tim Robinson in preparation for the grand-opening of "Art." It was necessary for Tim to utilize this method because, at the time, "Art" couldn't handle a gearing up by two. Since then the servo-torque amplifiers have been redesigned by Tim, and they will drive a gearing up by two although we rarely implement that capability. The method of t-substitution is like *OverScaling* in that we let a choice of scale factor be more than is required and use that factor instead of gearing up. But here we are strictly using the premise of *Unity Scaling*. In doing so we can make inferences about the analytic equation by looking at the machine equation without the ambiguity of different unities within the machine. In terms of algebraic scale factors, we let $A = I$, and maintain that value of *Unity* throughout the independent and dependent variable, the section, the integrators and the output table. Moreover, the machine equality is exactly the same as the the analytic equation because we additionally reduce B in the relation. Now we let the coefficients of all shafts be at *Unity* with the common value of 250. This is the case of *Unity Scaling* and it only works nicely if B is reducible. Because the whole machine is at *Unity*, we can disregard algebraic scale factors and reference the independent variable shaft t with a coefficient of one. Now we want to make a substitution for t; hence, let $t = 2\tau$. Carrying this substitution through one integrator, implies one factor of 2 is pulled out of the integration. Because the machine is at *Unity* the integrator constant reduces, and we are left with 2 times the integral. Thus, we have available 2 times the original integral and that factor may be used to gear up a certain quantity.

Consider the autonomous linear ODE

$$x'' + 2x' + x = 0.$$

To solve this equation on the differential analyzer we need to express the DE in a form that explicitly defines the highest order derivative, that is

$$x'' = -2x' - x$$

and make a general schematic. Figure 4.10 is a general schematic that depicts the basic mechanical interconnections of the machine. Notice that here we have introduced a new symbol for an adder. The symbol looks like a gear train box with a summation symbol within it. (Remember that an open circle represents a sign change in the direct drive.) Also the two direct drives connections in the adder box represent the inputs of the adder unit and the summation symbol represents the output of the adder unit. Note here we are adding two things, so there are two inputs to the adder. We may have as many as we need for multiple additions, in which case, another input is available inside the adder box of the schematic diagram.

So the DE says the second derivative equals the opposite of the sum of twice the first derivative and the dependent variable. We start with t feeding both differential shafts for both integrators, as the DE is linear. Bus-shaft 2, labeled x'', gets integrated with respect to t. The output yields x', then bus-shaft 3 goes through a 1:2 gear train to become $2x'$. Note the convention of the gear train box, 1:2 is a gear up and the reverse of that is a gear down. It was established this way so an operator would know which shaft is spinning faster. For instance, in this case, if bus-shaft 3 spins once then

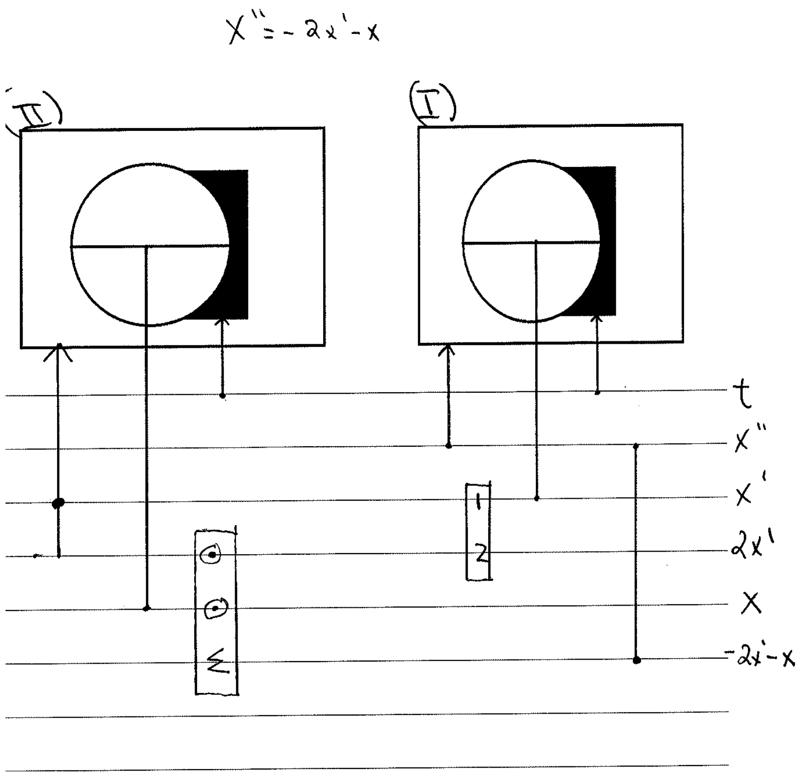

Figure 4.10. General schematic diagram $x'' = -2x' - x$.

bus-shaft 4 spins twice. Note, in addition to going through a gear train, bus-shaft 3 also is fed to the integrand shaft for Integrator 2, thus becoming bus-shaft 5, x. Now an adder box is drawn so as to combine the motion of bus-shaft 4 and bus-shaft 5. Bus-shaft 6 will represent the sum, so where normally a dot would represent its direct drive connection, σ is in its stead. Notice the two open circle dots on bus-shaft 3, 4 and 5 denoting that the two quantities are negated within the direct drive connections of the adder unit. The sum shaft or bus-shaft 6 is labeled $-2x' - x$. Finally, bus-shaft 6 is connected to bus shaft 2 so that we have $x'' = -2x' - x$.

This set-up will work fine if the torque amplifiers are not overloaded. It has been the DA Team's motto to avoid gearing up when possible as gearing up more than a factor of 2 will overload the system and amplify mechanical errors. So we want to solve this system and not have to gear up by 2 in the x' term. We impose the t-substitution by letting the independent variable shaft represent $2t$. The schematic for the t-substitution is given in Figure 4.11. The differential shafts of the two integrators involved are now driven by $2t$. Starting with bus-shaft 2, labeled x'', the outcome $2x'$, or bus-shaft 3 is then connected to the direct drive input of the adder box. Additionally bus-shaft 3 is geared down 2:1 to provide x' on bus-shaft 4. That shaft is sent to the integrand input of Integrator 2, and the outcome is bus shaft 5, labeled $2x$, because we are integrating with respect to $2t$. Bus-shaft 5 gets geared down, 2:1, and becomes x and then x, or bus-shaft 6, is sent to the adder box input. Thus, the sum is created giving bus-shaft 7, labeled $-2x' - x$. Finally, bus-shaft 7 is connected to bus-shaft 2, so that $x'' = -2x' - x$. This method of scaling is the t-substitution and we have successfully represented the equality in the DE without gearing up but instead gearing all quantities but one down.

To get a solution to this particular DE we need initial conditions that satisfy it. So if we want a solution that starts at the origin we need initial conditions such that $x''(0) = -2x'(0)$. Because it will be convenient to have a solution that starts out increasing, we will use $x''(0) = -2$ and $x'(0) = 1$. Analytically this gives the solution te^{-t}, and, if we want to maintain *Unity* within the output table, we need to be sure to plot t vs x, which means we need to gear down the independent variable 2:1 in the plot (See Figure 4.11).

It is interesting to note that the t-substitution is mechanically defined as gearing quantities down instead of gearing one quantity up. So if we take a look at the schematic diagram from a mechanical standpoint and disregard the t-substitution, we can get a differential equation from it. Basically, taking the labeling from the bus-shafts away, and starting with t for the independent variable, we can work through the mechanical integrations to get the DE $x'' = -x' - \frac{1}{4}x$. In a mechanical sense, the interconnections do not change from problem to problem. However, what does

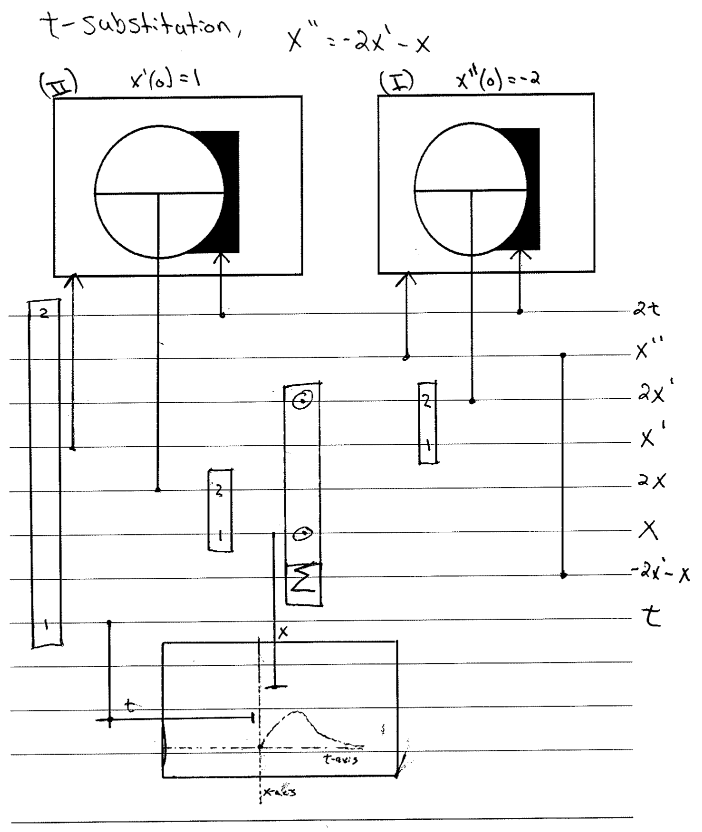

Figure 4.11. Schematic diagram of $x'' = -2x' - x$ with a t-substitution.

change is the set of initial conditions that satisfy each of these. Remember that we previously chose $x''(0) = -2$ and $x'(0) = 1$. For this newly acquired DE those initial conditions do not satisfy it for $x(0) = 0$. Moreover, the two DE's have two different general solutions, due to their auxiliary equations. So, in a mechanical sense, solving $x'' = -x' - \frac{1}{4}x$ with a set of initial conditions that doesn't satisfy its equality, yields a particular solution to a similar yet still different DE whose equality does satisfy those initial conditions. So an operator must be aware of the type of substitutions that were made and always make sure that if a solution to a particular DE is desired, then the chosen initial conditions must satisfy the DE. Otherwise, we are not solving that particular DE; we are solving another equation, whose equality is forced by a set of initial conditions.

4.6. SOLVING A "STIFF" ODE

An example of a differential equation classified as "stiff" is

$$y' = -30y.$$

This particular DE with initial conditions $y'(0) = -10$ and $y(0) = \frac{1}{3}$ is numerically unstable using RK4 with a step-size of .1. A rigorous analysis of this type of numerical instability is beyond the scope of this discussion. Our goal is to solve this DE with the given initial conditions with the differential analyzer and provide a different perspective for finding a solution by a mechanical approximation method.

Clearly, this differential equation is solvable by the separation of variables method. Given the set of initial conditions $y'(0) = -10$ and $y(0) = \frac{1}{3}$ we find the particular solution

$$y(t) = \frac{1}{3}\exp\{-30t\}.$$

This particular solution will be helpful later when comparing quantitative values of the machine solution and providing a numerical argument concerning the accuracy of the machine.

We need to create a general schematic. First take note that it is not feasible to gear up by 30. In this case, when creating a general schematic, the t-substitution is used. In Figure 4.12, label bus shaft 1 as $30t$ and bus shaft 2 as y'. The output of bus shaft 3 is $30y$. We then connect bus shaft 3 to bus shaft 2 and we have $y' = -30y$.

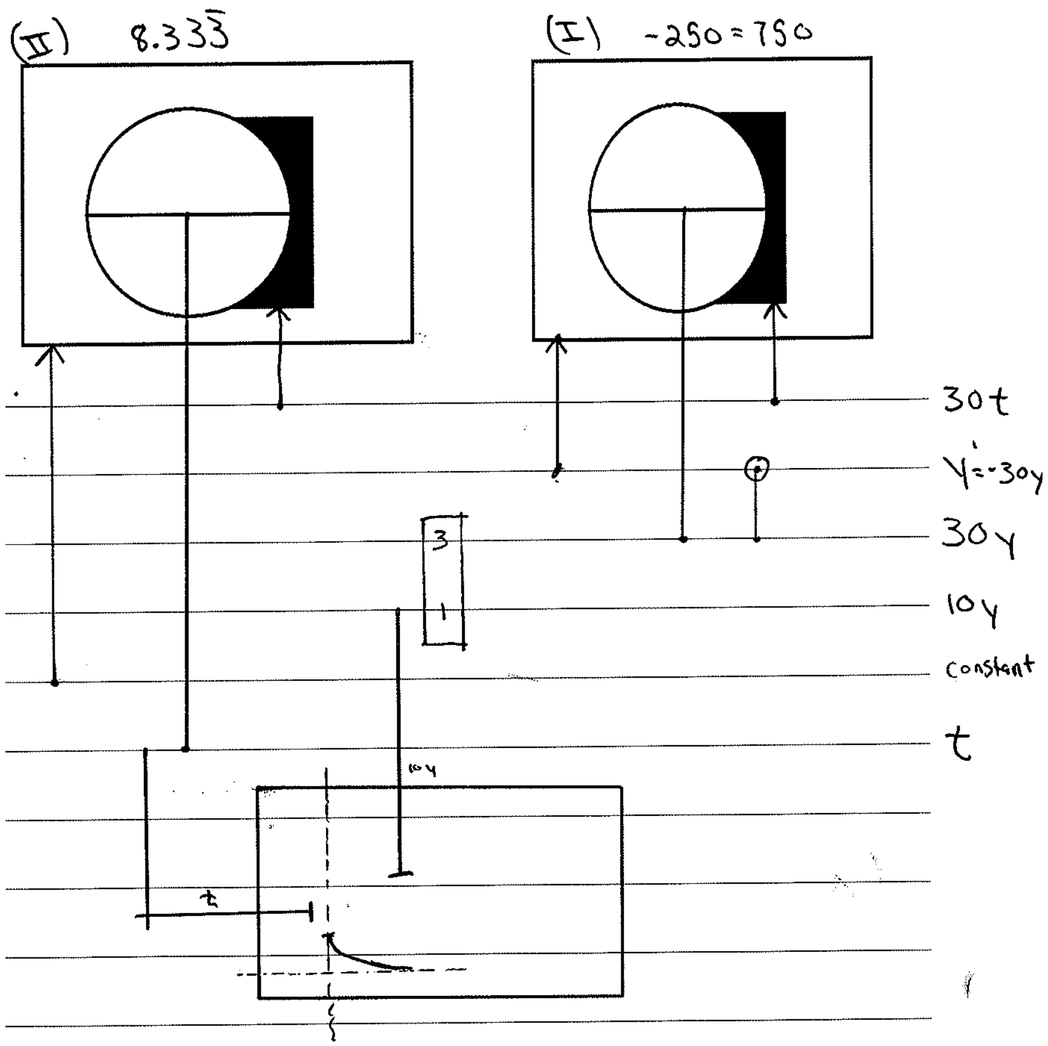

Figure 4.12. General schematic diagram of $y' = -30y$, an example of a "stiff" ODE.

Notice the open circle dot in the direct drive connection of that last shaft, so that we send back a negative quantity into bus shaft 2. The interconnections are easy enough and machine equality is achieved. Now a plot is needed. Because all components are at *Unity*, $I = 250$, gearing down the t shaft by 30, independent of the interconnections of the machine, is necessary. (This is achieved by using Integrator 2 as a reduction gear.) Similarly, for y, a gear down is also necessary so that we may have a plot of t vs y. But notice that we can't get an initial condition of $y'(0) = -10$ with the common *Unity* value $I = 250$. So we need a way to do this and not change our *Unity* scale. To do so, notice we have available in the section $30y$. Because the equation is linear and homogenous in y, we know that a scale on y will result in a similar scaling of y'. So if we plot off $30y$, we would be equivalently plotting just y with an increased initial condition times 30. With that in mind, we set our initial condition to $-250 = 750$ (since the counters are centered at the value of 0 and the numerals reset after a reading of 1000, the reading is 1000-250=750) so that our ratio of integrator disk to wheel is literally 1:1 but negatively correlated. Then we gear the output down by 3, independently of the equality interconnections, so that we have available in the section $10y$. We obtain the plot of t vs $10y$. We can think of this two ways, in an analytical sense; We are plotting ten times the solution with an initial condition of $y'(0) = -1$, or we are plotting the actual solution with and initial condition of $y'(0) = -10$. Either way the analytical solution is the same because the second set of initial conditions yields the solution

$$y(t) = \frac{1}{3}\exp\{-30t\}$$

and the first set with $y'(0) = -1 = -30y(0)$, implies $y(0) = \frac{1}{30}$ which yields a particular solution of $y(t) = \frac{1}{30}\exp\{-30t\}$, but the right hand side multiplied by 10 gives the original solution

$$y(t) = \frac{1}{3}\exp\{-30t\}.$$

So we can use the plot of t vs $10y$ to give an equivalent solution of t vs y with $y'(0) = -10$, when we use the later of the two we can make an additional inference about the DE by looking at the mechanical aspects of the interconnections and disregarding the t-substitution, as we did in the last sub-section. In doing that we derive a similar but different differential equation that satisfies a different set of initial conditions. That is $y' = -y$, with $y'(0) = -1$ whose particular solution is $y(t, 0, -1) = \exp\{-t\}$.

Again we derive this alternate DE by observing the interconnections of the schematic diagram and disregarding the t-substitution. We can look for differences in the former and the later. The main difference is in the plot; note that the interconnections of the schematic diagram do not change from equation to equation, but the plot does. In the latter we don't need to gear down either independent nor dependent variable, but, in the former, we gear down t by 30 and y by 3. So one can conclude that y is simply one third of $y(t, 0, -1)$ at $t = 30\tau$ where τ is the independent variable in the derived equation. So symbolically this looks like

$$y = \frac{1}{3}y(30t, 0, -1)$$

which can easily be verified analytically by letting $y(t, 0, -1) = \exp\{-\tau\}$, as we know the explicit solutions of both DE's. It is not shocking that these two solutions are so closely related, especially after a quick look at the explicit solutions. However, the point is that this kind of information can be obtained without knowing the solution and by simply creating a general schematic diagram of the original DE one can make theoretical generalizations. In fact, just constructing a general schematic is a generalization in itself of a differential equation and in a very simple context. As mentioned previously, assertions as those in this example seem simple because the differential equation is linear and solvable. The idea is to expand on this analysis to include different classes of nonlinear differential equations whose solutions don't have explicit forms. Although here we have simply scratched the surface on classifying different types of differential equations, it should be stated that solving differential equations mechanically certainly provide an alternate perspective into the nature of relating various rates of change in a differential equation.

As for the numerical instability using the RK4 method to solve this “stiff” DE, the gear scaling taking place when solving the DE mechanically can provide some insight. The underlying factor in solving this DE mechanically was gearing down the independent variable by 30. This fact automatically suggests that very important information is happening in increments taken in one thirtieth of one unit of time for the independent variable. This point is even further illustrated given that the solution is exponential decay. So when one implements the RK4 method at a step-size of .1, there is no alternative. Information is lost with a jump that large. With reasonable confidence, one would need at least a step-size of $\frac{1}{30} = .0333....$ to minimally capture all the information in a differential equation with that large of a change in derivative. However, this is simply a conjecture.

Solving differential equations using a differential analyzer doesn’t involve a step-size, nor is there any subsequent round-off error embedded within the operations while the solution is being calculated; which is why the differential analyzer is able to solve this “stiff” DE with very little trouble. However, not employing a discretization method is not to say that a differential analyzer is without error. In practice, the differential analyzer can accumulate considerable error due to backlash when the integrator wheel passes through zero, especially if there are no frontlash units. Furthermore, there will always exist some play in the gear trains and this will increase as the number of gear trains inserted increases. Granted, we try to minimize the mechanical errors as much as possible and we can determine the accuracy of the machine by running various mechanical tests and accounting for those errors as they occur in “real-time.” But in practice the solution is still an approximation. The goal here is not to say that one approximation method is better than the other, because they both have their strengths and weaknesses. Instead, we want to say that continuous analog computing and discrete digital computing are fundamentally different types of approximations. What analog lacks in precision it makes up for in theory and continuous operations. And what digital lacks qualitatively it makes up for in precision and efficiency. Both digital and analog methods have their own degrees of accuracy and justifications therein. No doubt some combination of the two methods is the "golden mean between two extremes" (Aristotle). It is always advantageous for one to consider all available resources when challenged with solving a problem that seems unsolvable.

We shall end our example with a theoretical justification of why the machine solves this particular DE and back up this claim with a plot of the machine solution. From a theoretical point of view we want to show that, as the mechanical errors are minimized, the DA will plot the particular solution to our "stiff" example. So first we will work under the assumption that there are no mechanical errors. That is, there exists absolutely no slippage between wheel and disk, and backlash is completely eliminated with the gear trains when the wheel passes through zero. Since in the last section we proved the function f that is the integrand shaft is always Riemann integrable, so we can construct the solution y of the DE in the form

$$y = \int -30y dt$$

where $y = f(t)$. (Consider the schematic Figure 4.12.) So upon substitution of this expression for y into the differential equation we see that

$$\frac{d}{dt}y = \frac{d}{dt}(-30 \int y(t))dt = -30y$$

is a true statement. We also need to show that the given set of initial conditions also satisfies the differential equation. So we must have that

$$y'(0) = -30y(0)$$

where $y'(0) = -10$ and $y(0) = \frac{1}{3}$. We can easily see that these initial conditions also provide a true statement. Now we can be sure, in a theoretical context, that the machine solution y is indeed the solution to the DE given these assumptions.

This type of mathematical justification can be seen formally, in its original form, in a paper written by James Thomson, and Lord Kelvin in 1887 [15] the inventors of the disk-globe integrator. In their paper, they generalize this concept to include linear homogeneous nth order differential equations and show that their calculation machine will always provide solutions in all cases of this type. On a historical note, Thomson and Lord Kelvin couldn't realize the full extent of their invention because torque amplification was not feasible in that time period. It wasn't until Bush realized how this could be done and extended the idea to build a differential analyzer that found mechanical solutions of problems of a nonlinear nature. Because of the invention of the mechanical torque amplifier by Bush and Niemann, a differential analyzer is able to solve nonlinear problems. Furthermore, with the advent of Servo-mechanisms, due to Arthur Porter, used to amplify torque, the use of differential analyzers was spread over a large range of scientific communities. It's interesting to note that in the days of Porter, Hartree, and Bush, differential analyzers were used primarily by physicists, whose concerns were of a quantitative and practical nature as opposed to a qualitative and theoretical one. At the present time with the implementation of Tim Robinson's servo-torque amplifier, a differential analyzer, namely "Art", now rests in the hands of a group of mathematicians, the Marshall DA Team. So opportunity is now available for a more qualitative and theoretical study of mechanical integration (in a continuous context) to be extended to the nonlinear case.

From the practical and quantitative stand point, nothing accredits the accuracy of an approximation method quite like a solution plot. Additionally, a comparison of the raw data obtained from the approximation and the precise analytical values of the exact solution can be made. To do so, we need to first plot the solution and take counter readings at increments of say 100 on the counter, then find out what analytical value these t-values corresponds to. Simply divide counter readings by the scale constant on the t-shaft, then take those corresponding analytical t-values and evaluate them in the explicit form of the solution, that is

$$y(t) = \frac{1}{3}\exp\{-30t\}.$$

We can see that using the t-substitution is not a good choice for scaling, as the plot covers less than 0.1% of the entire plotting surface, or in other words, it's zoomed out too much. With that fact under consideration we will choose a form of combination scaling with *Scale Folding* and *Asorption Scaling*.

Two separate schematics, the algebraic and the final, are available in Figures 4.13 and 4.14. Upon the introduction of the algebraic scale factors in the schematic, Figure 4.13, we derive the relation

$$A = 30 * n * I$$

where n is the gear train that gears down the output of the integrator. The value of n may also be interpreted as the factor by which the graph is scale folded. We let $n = 6$ because 6 is a factor of 30, and 30 will reduce nicely among the subsequent integrations along with integrator constant factor. Also scaling up by 6 will suffice in finding a desirable window of interest. (It's essentially zoomed in by 6.) So $A = 30 * 6 * 250 = 45000$. Next we need a choice for B. Since we need to scale the initial condition in y, namely $y'(0) = -10$, we should scale B down by a factor of 5 from *Unity*. From those facts, we can see that the value for B should be $250/5 = 50$ so we set the initial position of the integrator to -500 and that will correspond to the analytical value of -10. So letting $A = 45000$ and $B = 50$ in Figure 4.13 yields the final schematic Figure 4.14. Note that in the plot the coefficients are $4500t$ vs $1500y$. This implies we have geared down the t-shaft by 10 independently of the mechanical interconnections that represent equality. So now the initial condition $y(0)$ needs to be set on the output table so that it will correspond to the y-intercept in the solution. We know $y(0) = 1/3$, so, as 1500 represents 1 unit in y on the output table (only), we have that $1500/3 = 500$, which is where the initial y position needs to be at a machine distance of 500 on the counter from the axis origin. Now all components are consistent with the scaling factors, we have a machine equation of

$$50Y' = -1500Y$$

with initial conditions $Y'(0) = -500$ and $Y(0) = 500$.

In practice it is useful to denote the machine equation by Y so that we can separate it from the analytical denotation y. Both notations are consistent with their initial conditions. The solution plot is available in Appendix (ap7). Note that corresponding y-values can be calculated by taking counter readings and dividing each reading by its shaft coefficient. For this plot, several random Y-values were taken for corresponding T-values, then compared to the same value in the analytical solution. We found that the difference between Y-values and y-values, for the corresponding T and t values were approximately 0.001. This would be a compelling argument from data analysis that the mechanical approximation of this solution has order of

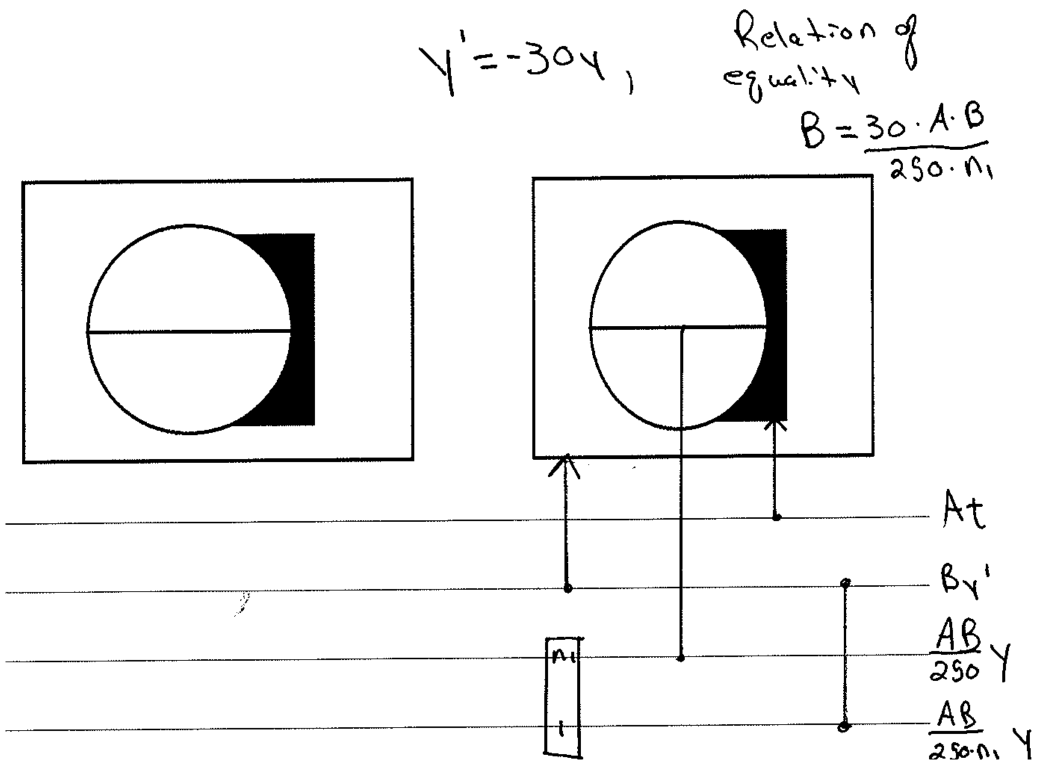

Figure 4.13. Algebraic schematic diagram of the "stiff" ODE. Note: the relation of equality is $B = \frac{30AB}{250n_1}$.

magnitude of error in the thousandths of a decimal place, at least on a closed t-domain of $[0, \frac{1}{2}]$. This is quite accurate considering the "stiffness" of the DE and the fact that "Ar"t is built from Meccano parts without frontlash units. One reason for this accuracy is that the integrator wheel never passes through zero, since the solution is monotonic and decreasing. This also implies that gear backlash is minimized for the same reasons. Additionally, the scale has been expanded so that the counter readings are taken in small increments of corresponding shaft rotations. Moreover, in terms of mechanics, the machine simply operates smoothly when things are geared down.

One need not confuse increments of a counter reading with a step-size; they are entirely different concepts. The difference is that counter readings are not just integer values; they are sets of real numbers. We just like them to be integer values because

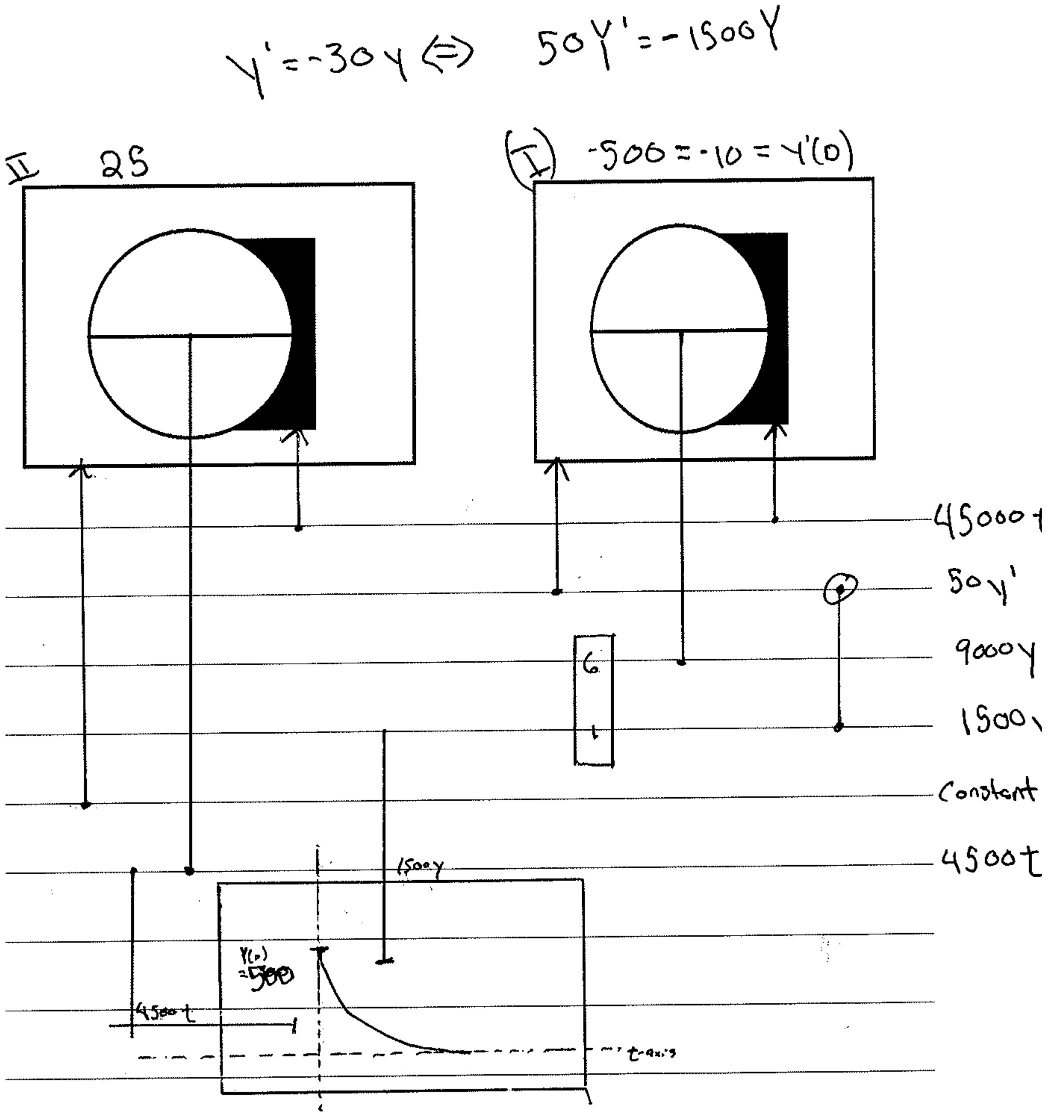

Figure 4.14. Final schematic diagram of $y' = -30y$.

then we don't need to estimate the reading with the naked eye for some portion of a turn in the counter. So, to this end, one could simply gear up the counter to yield better precision, giving the machine more precision accuracy. But gearing up makes the counter turn very fast. A better way is to have a fine adjustment on the counter where increments of portions of counter readings could be better measured. The accuracy alone in the approximation would essentially stay the same because analog computers like the differential analyzer are continuous in nature. The movable parts are what produce error, unlike digital discrete approximations taken at increments with high precision. There is a clear difference between precision and accuracy as usually more of one leads to less of the other. A differential analyzer focuses primarily on the accuracy in the approximation by taking sets of exact measurements in the real numbers. In doing so, there is a loss in precision, at least in practice, because a realistic measurement can only be as accurate as the device used in taking the measurement. The key point of fact is the turning of the wheel as it rests on the surface of the disk can measure values that are unrealistically small because there exist arbitrary small partitions of the surface of the disk through any degree of a portion of a turn. Theoretically, we can always find a refinement of that partition and another refinement of the last refinement, which is the meaning of an unrealistically small measurement; an arbitrarily small refinement of a partition. Because the wheel is literally turned by the disk through friction, provided there is absolutely no slippage, the wheel is forced to roll over all arbitrarily small refinements contained in that portion of a turn. So we know, theoretically, that the wheel's rotation covers all the arbitrarily small measurements in between. One could argue that, theoretically speaking, the surface that is the edge of the wheel can only have a countable number of measurable points on it. Then the distance between any of those two points would have no measurement relative to the wheel's rotation, so the smallest measurement of distance on the edge of the wheel is indeed an arbitrarily small distance relative to what is defined as "the smallest measurement" on the machine. Relative measurements is how the machine is able to obtain very good accuracy with lack of precision.

An Analogy would better illustrate the point. Suppose a carpenter uses a cheap tape measure to find the distance along a wall. After measuring and cutting a board that needs to be nailed along this wall, the carpenter finds that the board is three inches too long. Unknowingly, the carpenter misplaces the cheap tape measure. The carpenter instead uses a very expensive set of vernier calipers, that measures to thousandths of an inch, to make the three-inch cut mark on the board. Upon hanging the board along the wall, the carpenter notices that the board is not of the right length, although a set of very precise vernier calipers was used in the cut. Recall that the carpenter originally used the cheap tape measure to determine the distance along the wall. The reason the board is not of the right length is because the wall measurement is relative to a measurement taken with the tape measure. The carpenter should have found the misplaced tape measure to make the three-inch cut mark regardless of the fact that the calipers are a better measuring device. So accuracy can be preserved independent of precision as long as the measurements being taken are relative to the same scale.

Having lack of precision in practice must be accounted for when making mathematical inferences about the approximation, ultimately leading to conjectures of a more theoretical nature. For example, in the solution of the "stiff" DE there is a point when the wheel actually reaches zero. Analytically this is only possible in the limit as t goes to infinity. In this case, in order to develop the theory, we would need to define the point in which the wheel finally reaches zero as the limit. Theoretically, this is as simple as saying it. But, in practice, when approximating this value for the solution, we would need to consider this value in terms of how precise our measurements are. If our order of accuracy is in the thousandths of a unit, we could overestimate small values of y for large values of t because the counter may not be able to provide a precise enough measure of the arbitrarily small refinements at that point without being significantly geared down. In this case, the value of the limit at a given t-value would have the practical meaning of no further changes in the magnitude of y at the limit value. This interpretation, in a sense, is a practical meaning of converging to zero. So there is some information provided as to how this model would behave as it naturally occurs in time.

This type of argument is strictly from a mechanical perspective with the goal to offer insight into the very abstract nature of a mechanical approximation technique. We can see that a mechanical approximation is of a continuous nature. But with this, in practice, we have a loss of precision. As to the difference between mechanical and numerical approximations, the use of a step-size is the underlying degree of separation. A more rigorous qualitative error analysis would require much more of a numerical argument along with a stronger base of mechanical integration theory, and the journey into that analysis will be saved for a later adventure.

This concludes the operations section. Keep in mind that the schematics in the end of the Appendix include general, algebraic and final schematics, some with little to no scaling mentioned and some with all aspects of scaling completely detailed. Setting up the basic problems mentioned throughout this section is the first step in learning how to use a differential analyzer. One should combine previous knowledge of these basic problems with the new perspective provided by the differential analyzer. By consideration of mechanical tolerances and mathematical fact one can use one's intuition to extrapolate a personal interpretation of how mathematics relates to mechanical interrelationships.

5. SETTING UP A NONLINEAR PROBLEM

This section outlines the setup for several nonlinear problems. Although "Art" is limited in the range of nonlinear problems, this section will provide a means to account for a broad range of nonlinearities that may occur in a differential equation. Additionally, one particular differential equation, namely the nonlinear circuit, is set up with a final schematic. This particular problem has been mentioned in [11] *The Theory of Differential Equations*. The original study of this differential equation is due to Ueda [11]. The goal is to solve this nonlinear problem, which would incorporate all the various components of "Art," and provide insight into solving nonlinear problems on the machine, in general. Furthermore, a successful attempt to run this problem on the machine and create an elaborate phase plot, specifically mentioned in Peterson and Kelley's *Theory of Differential Equations*, was made. All aspects of scaling are mentioned in great detail along with an alternate scale using two more integrators.

5.1. GENERATING NON-HOMOGENEITIES WHEN SOLVING ODE'S MECHANICALLY

To further expand the range of problems solvable by a differential analyzer, we sometimes generate certain terms that may not be available from the subsequent integrations of the highest order derivative. These terms are often non-homogeneities that arise, for example, in a forcing function problem. A forcing function problem is a differential equation, like dampened harmonic motion, where there exists a non-homogeneous term that will contribute significantly to the qualitative nature of the solution. That is, the forcing function will force a dampening effect on the solution curve for example. The example DE used in the description of t-substitution (in the subsection of the previous section) was one such non-homogeneous term that was used as part of a forcing function problem. The problem was created by Dr. Lawrence and Tim Robinson for the Grand Opening of “Art”. The DE was dampened harmonic motion that started from the trivial solution wherein a short lapse of time, the forcing function effect, provided change into the system. Ultimately, the forcing function would converge to zero and provide no further contribution to the system. The result over a significantly large enough (or precision limit point) lapse of “time,” was dampened harmonic motion at an initial condition equal to the point in the solution at which the forcing function “converged” to zero. Afterwards, it was interesting to take the plot of the solution and forcing function, then superimpose the two, so that one can see the net effect that the forcing function had on the autonomous dampened harmonic portion of the DE.

Since then, Dr. Lawrence has designed an exercise for her differential equations courses that she likes to call “Finding a DE For a Given Solution.” It involves the concept of a forcing function. During the creation of the problem, it was necessary to find a differential equation whose solution is the forcing function because, in order to generate a non-homogeneity on the machine, one must solve a differential equation whose solution is the particular non-homogeneity. So the exercise is, “here is a solution, now find a DE that it satisfies.” The method involves taking one or several derivatives of the function in question and algebraically manipulating those expressions into an equation in terms of the derivative(s). For example, consider the function

$$x(t) = t \exp(-t).$$

Taking the first and second derivative yields

$$x'(t) = t(-\exp(-t)) + \exp(-t)$$

and

$$x''(t) = t \exp(-t) - \exp(-t) - \exp(-t).$$

Notice that, if we add the function, its derivative, and its second derivative together, we are left with an expression

$$t \exp(-t) - \exp(-t).$$

As this is the negative of the expression for x', the sum of the function, its second derivative, and twice its first derivative would be zero. So we can extrapolate the DE

$$x'' + 2x' + x = 0$$

from the expressions given for the derivatives.

Now that a DE whose solution is the non-homogenous term has been found, an operator can input that function in one of two ways. The most accurate way is to use extra integrators to generate the non-homogeneity. The process is to create a Bush schematic that involves the differential equation without the non-homogeneity and then create a Bush schematic that that solves a DE whose solution is the non-homogeneity. The final step is to combine these two schematics by letting the solution to the DE, which generates the non-homogeneous term, be available for further interconnection within the section; this will essentially complete the setup of the entire DE. Mechanically, the output solution shaft of the DE that generates the non-homogeneous term will, in turn, drive a main cross-shaft that helps complete the interconnection on the primary DE we are trying to solve. The trick is, at least for non-homogeneous linear equations, is to let the same independent variable shaft drive the corresponding differential shafts of all integrators. Note that some care is needed in solving a DE in this way because of the scaling. An operator must be sure that the simultaneous solution of the non-homogeneity and the autonomous part of the differential equation are scaled similarly. If this is not the case, the two parts being solved could inadvertently be relative to different independent variable scales. Alternatively, one might want to have the two on different scales, which is also possible due to the nature of the non-homogeneity. One needs to keep track of all the scales as a general rule to ensure that an operator doesn't interpret the output improperly.

The second less accurate way to input a non-homogeneous term into the section of interconnection is to use the input table that was described in the construction section. One still needs to find a DE whose solution is the non-homogeneous term. However, in this case one first obtains a plot of the solution on the surface of the input table. Next set up the DE and connect the crankshaft on the input table to the corresponding cross-shaft in the section. Then set all initial conditions including those that would correspond to the input curve. Start the independent drive motor and let the independent variable of the input curve drive the input table carriage. Meanwhile, an operator stationed at the crankshaft turns the crank as needed to keep the pointer reference in target with the curve on the input table. Note that the independent variable of the solution to the DE need not be the same independent variable of the input curve. For example, if we need to generate a non-homogeneity such as e^y, where y is a function of t, then we need to let a cross-shaft that represents y drive the independent variable of the input table. All instances of scaling, when using an input table, are of the same significance as using more integrators. Although the input table is less accurate due to human error, it is still of great importance especially from an educational point of view. The DA team has recently been experimenting with solving differential equations on time scales, and, for this application, the input table will prove to be very useful.

Either of the two ways may be used to solve non-homogeneous differential equations. Two examples are given in figures 5.1 and 5.2. Outlined in Figure 5.1 is the

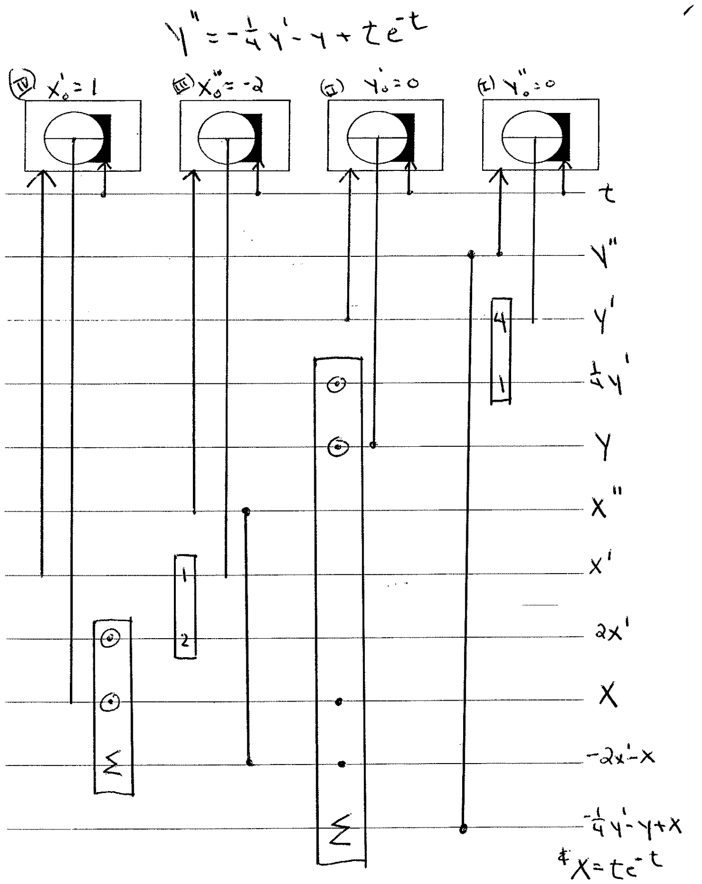

Figure 5.1. General schematic diagram of $y'' = \frac{1}{4}y' - y + t\exp(-t)$.

general schematic for the forcing function problem. In that setup two integrators are being used to generate the non-homogeneity. Figure 5.2 schematically outlines a problem that involves using the input table twice. This problem concerns the calculation of train running times and was taken from Hartree [10]. One of the non-homogeneities is fed using a normal input table. The other non-homogeneous term is fed in the machine as discrete initial settings that correspond to the various positions of the gradient of the terrain.

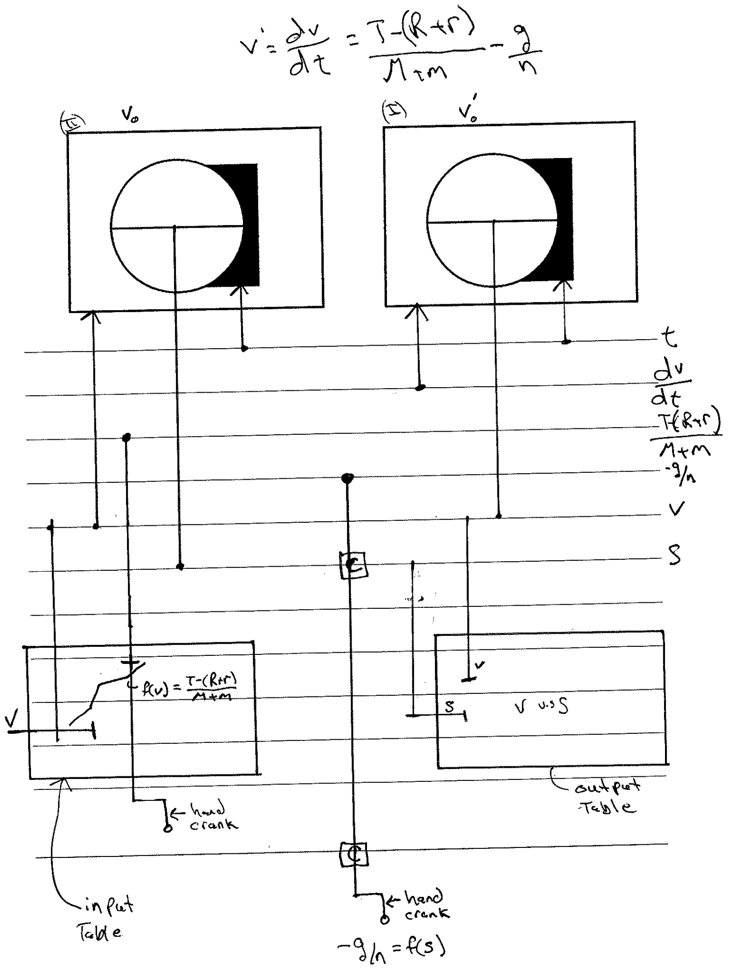

Figure 5.2. General schematic diagram of the Train Running Problem; $v' = \frac{dv}{dt} = \frac{T-(R+r)}{M+m} - \frac{g}{n}$.

Generating non-homogeneities is an integral part of solving more complex problems on a differential analyzer, and using integrators is certainly the most accurate way to do so. However, if an operator is forced to use an input table, human error is introduced. Due to this fact, quantitative information will usually not be very precise. But a qualitative study is possible if the mechanical restrictions are taken into careful consideration.

5.2. GENERATING NONLINEARITIES AND SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

When solving nonlinear, non-homogeneous differential equations one needs to use various combinations of the previously described methods with one most important added feature: allowing the differential shafts of the integrators to be driven by something other than the independent variable drive motor. This is the key difference in solving nonlinear problems in terms of the differential analyzer. The number of differential shafts that will have this property is unfortunately problem dependent, but here we will give some general cases.

For example, if we want to generate a squared term, like y^2, we want to use the y shaft and connect it to both the integrand and differential shafts of an integrator. So we have $\int y dy = \frac{1}{2}y^2$, and thus y^2 will be available for further interconnection. Higher order exponents are similar, and proper choices of scale factors can reduce any undesirable constants (See Figure 5.3 for y^2 and Figure 5.4 for y^3).

We can also integrate the product of two variables by integrating one with respect to the integral of the other (See Figure ap5 in the Appendix).

Moreover, we can multiply two variables on the machine by using a variation of the integration by parts formula. That is

$$\int x dy = xy - \int y dx$$

or equivalently

$$xy = \int x dy + \int y dx.$$

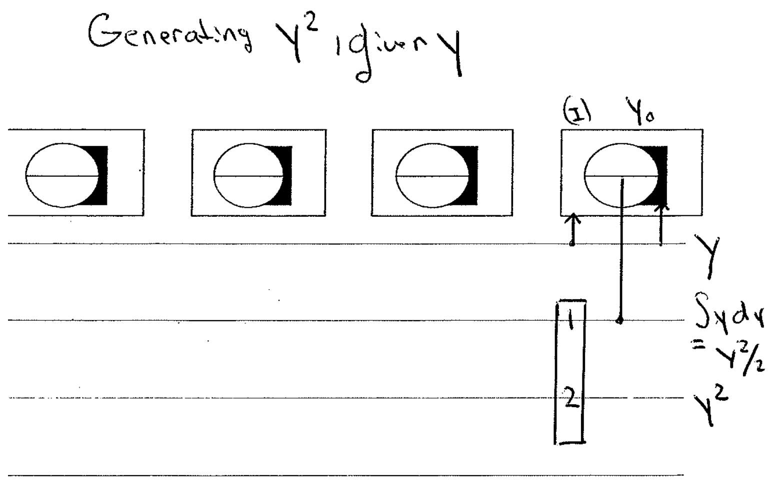

Figure 5.3. Schematic diagram: Generating y^2, given y.

Then mechanically speaking, an operator can use an adder to sum the two integral shafts to yield the product of two variables. This type of variable addition can also be used in combinations if need be (See Figure 5.5 for the schematic diagram of the multiplication of two variables).

The last case can occur when one needs to generate a non-homogeneous and nonlinear composite function. Suppose we need to generate $\sin y$ where y is the dependent variable in the differential equation. So we will need to produce a function of $y(t)$. Producing $\sin t$ is well known so this non-homogeneous term is set up first. Then, independent of that, using subsequent integrations starting with the highest order derivative of y with respect to t yields y. Let the shaft called y drive the differential shafts for the simple harmonic setup you have established. The output in the simple harmonic setup will be a function of y, where y has already been previously established to be a function of t. Hence, we have available a nonlinear composite term generated by the differential analyzer. This type of situation is necessary when solving

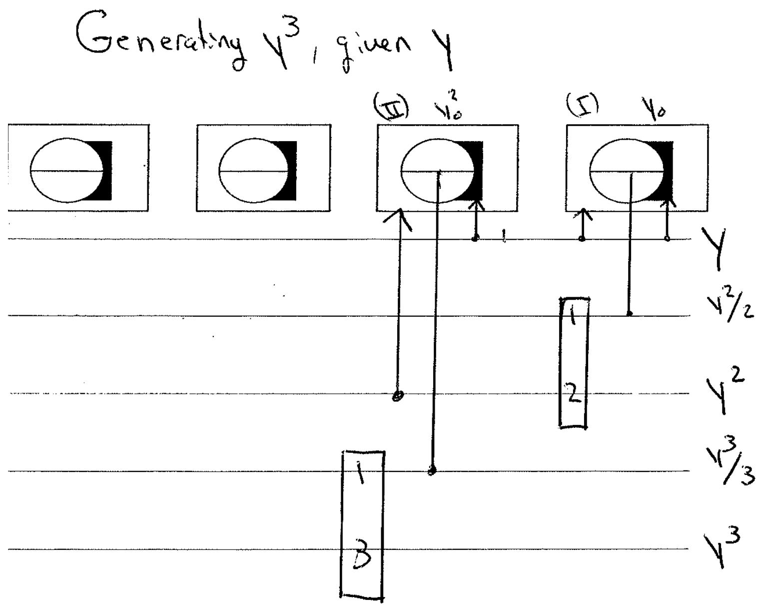

Figure 5.4. Schematic diagram: Generating y^3 given y.

a differential equation such as

$$y'' = \sin(y).$$

(See Figure 5.6 for a general schematic diagram of this DE.)

Some other examples given in the Appendix include the Lamar boundary value problem as described in Crank [6]. Additionally, there are several schematic diagrams for non-homogeneities, nonlinear or otherwise, that were mentioned in Crank (page 91 [6]). It should be mentioned that partial differential equations of certain types have been solved on differential analyzers in the past. Although "ART" has yet to solve a partial differential equation, implementation of the finite differences method in finding solutions to partial differential equations using "Art" is of great interest to

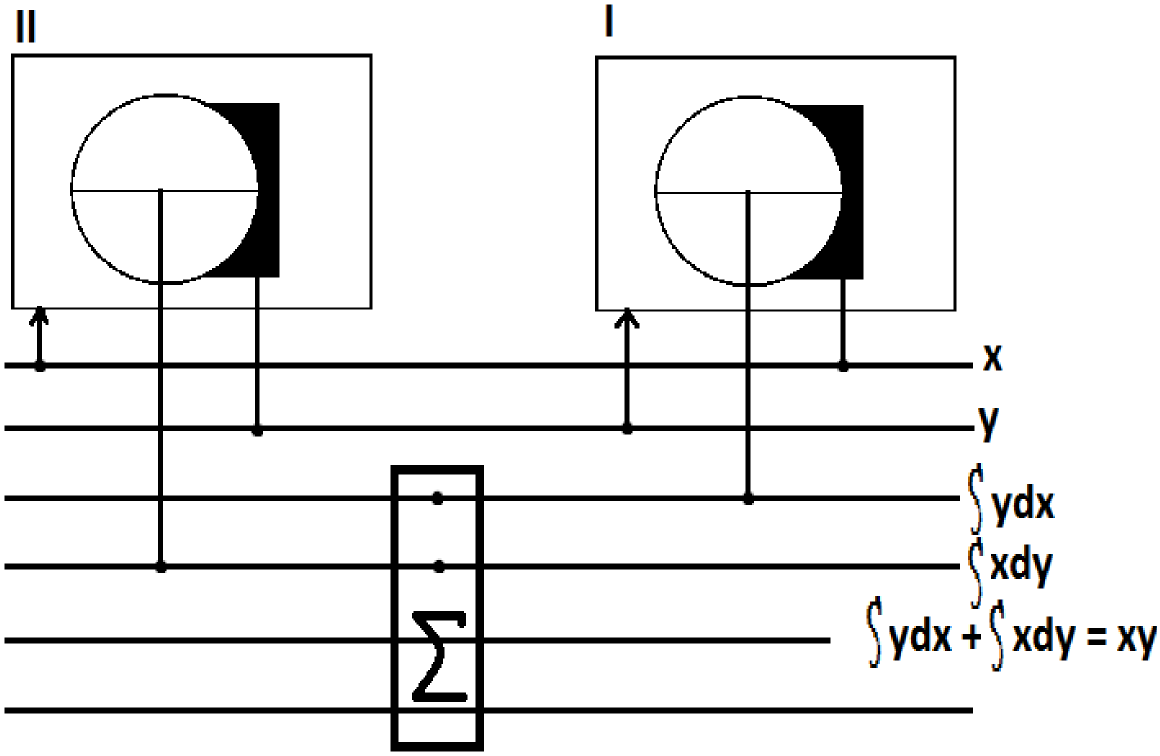

Figure 5.5. Schematic diagram: Generating the product of two variables, xy (Integration by parts).

the DA Team. In the near future, the Marshall DA team foresees many applications to various sub-disciplines of dynamical systems other than ODE's.

5.3. THE NONLINEAR CIRCUIT

The Nonlinear Circuit, or Nonlinear Oscillator, as mentioned in Peterson and Kelly's book [11] is described mathematically by

$$y'' + ry' + y^3 = b\cos(t).$$

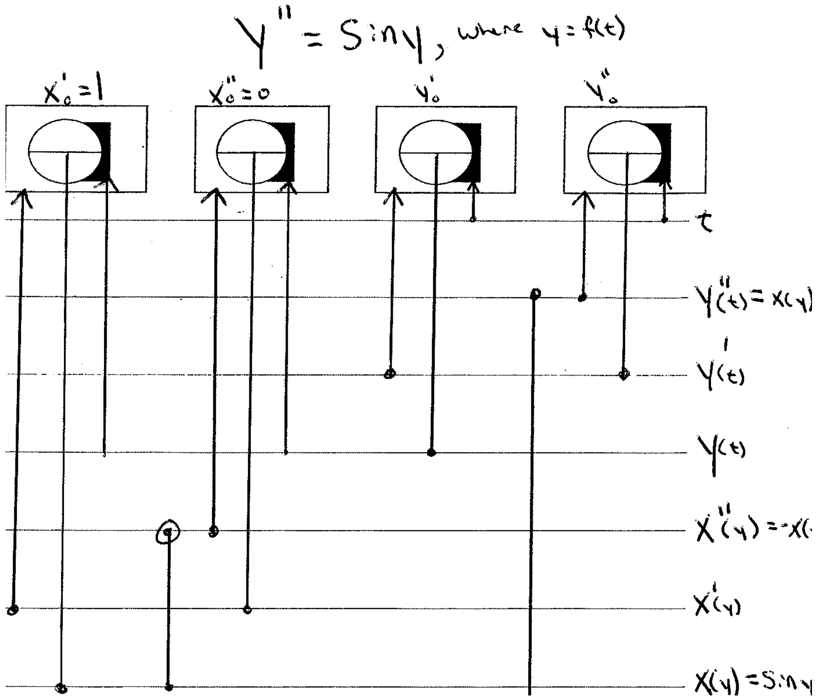

Figure 5.6. General schematic diagram of a nonlinear ODE, $y'' = \sin(y)$.

The aim was to find a simple example that exhibited some "chaotic" characteristics. In this case, the sensitive dependence on initial conditions, or SDOIC, is the classification used for "chaotic" behavior. The DE is a model of a simple circuit with linear resistance y', nonlinear inductance, y^3 and an alternating current source, $\cos(t)$, where y is defined to be the flux in the inductor [11]. Note that the alternating current source is the non-homogeneous term in the differential equation. Historically, the differential equation was solved by first converting it into a three-dimensional system. A numerical method was then implemented to provide phase plots that would ultimately exhibit a period-doubling effect given certain initial conditions. Several phase plots with different initial conditions and parameters are provided in [11]. Also a solution plot against time that specifically exhibits some interesting dynamics is given as well. The point is that some of the phase plots are periodic, and, upon a slight adjustment in parameter, such as changing the b-value from 9 to 9.85, the phase plot produces a period-doubling effect. Similarly, a small change in initial condition, namely changing $y(0) = 1.54$ to $y(0) = 1.55$, results in the solution plot becoming wildly unpredictable at about $t = 40$.

When solving this problem on the differential analyzer there are some limiting factors. Again, the underlying difference in the problems solved in the previous section and this nonlinear problem is that some of the differential shafts are driven by quantities other than the independent variable drive. Unfortunately, this consequence of nonlinearity causes stress on the mechanical operations. For example, because there are no frontlash units, and the disks are of considerable weight, there are significant time-lags in the reversals of the differential shafts when the wheel passes through the center of the disk. That is, when a differential shaft changes direction, some time is passed before the disk will begin to turn in the opposite direction. This reversal also implies that a similar time-lag occurs for the wheel's rotation as it passes through the center of the disk. This time-lag is due to mostly backlash but also due to the fact that the differential shaft is being feed by a servo-motor and not a monotonic drive motor such as the independent variable drive. Additionally, the momentum of the heavy disk tends to cause the servo-motor control to be unstable. This undesirable trait (oscillations of the servo-motor) is due to the weight of the disk combined with the way the servo-motors operate. To try to avoid this instability, the glass disks in the integrators representing nonlinear terms, were replaced by lighter Plexiglass, which helps to lighten the load on the torque amplifiers or servo-motors. However, the time-lags still occur and must be taken into consideration during the phase and solution plots.

The first step in solving this DE on the differential analyzer, as is always the case, is to create a generalized Bush Schematic. First express the DE in terms of its highest order derivative, that is in the form

$$y'' = -ry' - y^3 + b\cos(t).$$

Because this is a second order problem, we will need at least two integrators to generate y, whereupon the first integration yields y' and passing that quantity through another integrator yields y. Note that both of these integrations are with respect to the independent variable t, as y'', y' and y are all linear terms. Also note that we will need two more integrators to generate y^3 and two more integrators to generate $b\cos(t)$ where the term y^3 is a nonlinear term and $b\cos(t)$ is a linear non-homogeneity in the differential equation (Refer to Figure 5.7 for the general schematic).

Notice in Figure 5.7 we use the most precise technique (using integrators) to generate the nonlinear term and the non-homogeneity. Although "ART" only has four integrators, it is beneficial to start the general schematic of such an involved problem by using just integrators. Using only integrators to generate all terms will provide insight into the scaling, since all terms may be dealt with using the same coefficients of the independent variable.

The next step is to create an algebraic schematic to try to find a relation of equality that must be satisfied. In this case, several will be obtained (See Figure 5.8). To do so, start with the assumption that At represents the unit turns of the independent variable and By'' represents the unit turns of the second derivative. One integration of these will yield

$$\int By''d(At) = \frac{AB}{250}y',$$

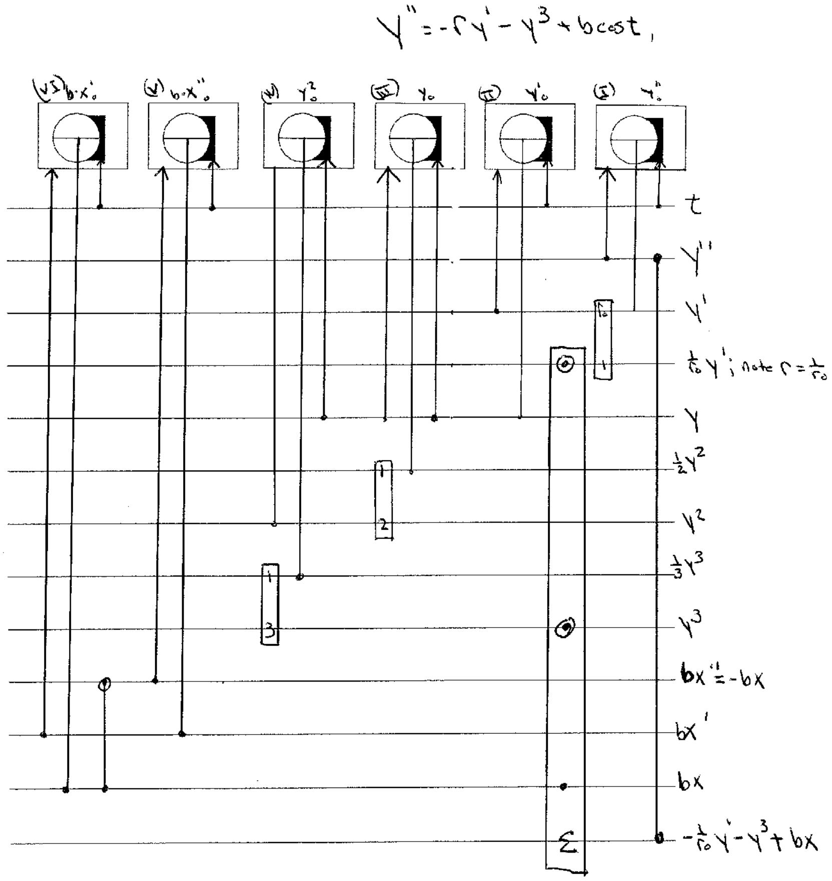

Figure 5.7. General schematic diagram for the nonlinear circuit, $y'' = -ry' - y^3 + b\cos(t)$.

for the output of the first integrator. Next we send that quantity to the second integrator, which becomes

$$\int \frac{AB}{250} y' d(At) = \frac{A^2 B}{250^2} y.$$

This last integration completes the relationship among the linear terms in the setup, given the assumptions. Note that this equation is not a relation of equality; it is simply an equation that relates the constants on y' and y. In order to generate y^3, y^2 is needed first. Hence, let the shaft denoted by y drive both the integrand and differential shaft of Integrator 3. So in terms of the algebraic constants, we have

$$\int \frac{A^2B}{250^2} y d(\frac{A^2B}{250^2} y).$$

If we factor the integrand constants and differential constants out of the integral expression we have

$$\frac{A^4B^2}{250^4} \int y dy = \frac{A^4B^2}{250^5} * \frac{y^2}{2}.$$

So this is the output of Integrator 3 and, since it is representative of a constant times y^2 we can send this quantity to the integrand shaft of Integrator 4 to get a constant multiplied by y^3. To do this we let the output of Integrator 2 drive the differential shaft of Integrator 4. Hence, we are, within a constant, integrating y^2 with respect to y. More formally, Integrator 4's output is given by

$$\int \frac{A^4B^2}{250^5} * \frac{y^2}{2} d(\frac{A^2B}{250^2} y) = \frac{A^6B^3}{2 * 250^8} \frac{y^3}{3}.$$

This integration will generate completely the nonlinearity. As for the non-homogeneity, because it is linear, we may treat its scaling similar to the scaling of the simple harmonic motion examples in the previous section. So we want to let the same independent variable in the primary linear terms drive the independent variable in the non-homogeneity. Hence, let At drive the differential shafts of both Integrators 5 and 6. We want to solve simple harmonic motion on the last two integrators because we know the solution to simple harmonic motion is linear combinations of *sine* and *cosine*. The only difference from the previous algebraic schematics for simple harmonic motion and the schematic of Integrators 5 and 6 is here we let the highest order derivative in the the DE $x'' = -x$ be represented by Cx'', for C unit turns in the variable x''. Upon subsequent integrations using the last two integrators, we obtain the separate relation of equality

$$C = \frac{A^2C}{250^2} \tag{5.1}$$

that must always be satisfied for Integrators 5 and 6, where the output shaft for Integrator 6 is equivalent to the right hand side of this equation. To create a closed system of equality, which is representative of the given DE, within the machine, we must add together the outputs of Integrators 1,4, and 6. Then we connect the resultant sum to the integrand shaft of the first integrator, which is representative of y'', so that

$$y'' = ry' + y^3 + b\cos(t).$$

Note that the output of Integrator 1 is geared down by r_0 to account for the parameter given in the DE. The parameter b within the linear non-homogeneity may be changed by an initial setting of initial condition within its closed system of equality, that is Integrators 5 and 6. Therefore, if we solve the given DE the machine must always satisfy the three relations of equality

$$B = \frac{AB}{r_0 * 250} \tag{5.2}$$

$$B = \frac{A^6B^3}{2 * 3 * 250^8} \tag{5.3}$$

$$B = \frac{A^2C}{250^2}. \tag{5.4}$$

This last result (Equations: 5.2, 5.3,and 5.4) follows from the fact that each relation represents a coefficient in the right hand side of our DE. Although these relations complete the system of equality, the scale factors must be chosen such that specific parameters given within the differential equation correspond to the LHS of Equations 5.2, 5.3, and 5.4. Here we want to solve for two given particular solutions mentioned above. Fortunately, both cases include the parameter $r = .1$, and, because the choice

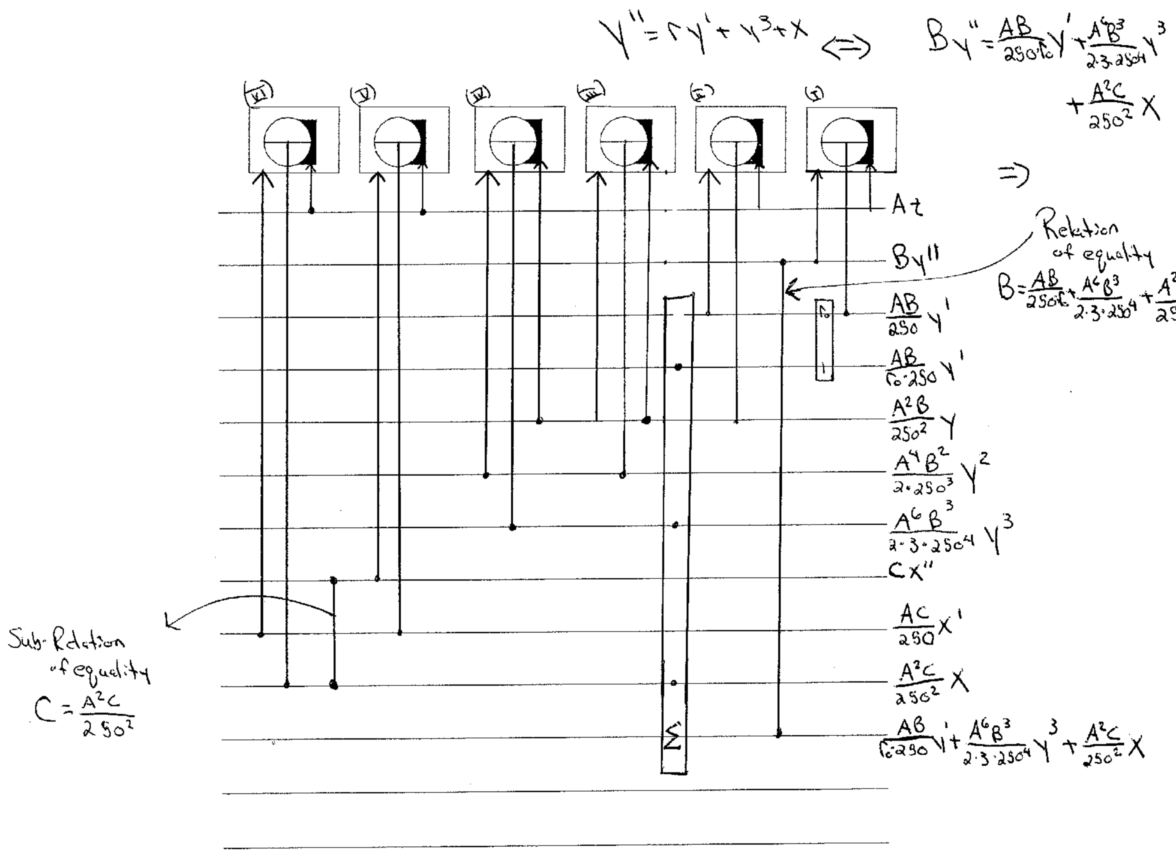

Figure 5.8. Algebraic schematic diagram for the nonlinear circuit. Note: the relation of equality $B = \frac{AB}{250^6} + \frac{A^6B^3}{2*3*250^8} + \frac{A^2C}{250^2}$ and the sub-relation of equality $C = \frac{A^2C}{250^2}$.

of b in the non-homogeneity ranges from 9 to 12, we can account for this parameter by an adjustment of initial condition on Integrators 5 and 6. So, to this end, for the choice of C we must have an initial condition of at least 12 available in the range on top of the integrator disks for Integrators 5 and 6. A maximum range of 500 shaft rotations are available on top of any disk. Because $(500/12) = 41.667$ a choice of $C = 40$ would be sufficient to provide an analytical initial condition of 12. That is, if $C = 40$ then 40 represents 1 unit on integrand shaft number 5, and since $40*12 = 480$, this choice of C would thus be available within the max range of an integrator disk.

Making this substitution into Equations 5.2, 5.3 and 5.4 yields

$$B = \frac{AB}{r_0 * 250} \tag{5.5}$$

$$B = \frac{A^6 B^3}{2 * 3 * 250^8} \tag{5.6}$$

$$B = \frac{40 * A^2}{250^2}, \tag{5.7}$$

where these last three equations represents the new refined set of relations that must be satisfied for maintaining machine equality.

In order to find a suitable choice of A and B, one could implement several methods. Algebraically speaking, manipulating the expression so that A and B are integers seems like a daunting task. But first notice that the choice of the term in the RHS of the relation with r_0 needs to ultimately have a coefficient of .1 in an analytical sense. This coefficient (0.1) is necessary because that term in the relation represents the coefficient of the variable y' in the differential equation. The question arises, "What's the difference between an analytical coefficient and a coefficient represented by the machine?" The answer is the choice in the scaling factor B. The choice of B essentially decides the *Unity* of the machine equation, because all subsequent integrations are taken to be relative to that initial choice. This fact is clear when expressing the differential equation explicitly in the form of the highest order derivative or equivalently expressing the relation explicitly defining B. Hence, if B is one unit then the following must be true for machine equality in the context of the specific parameters in the differential equation: Equations 5.5, 5.6 and 5.7 become

$$\frac{B}{10} = \frac{AB}{r_0 * 250} \tag{5.8}$$

$$B = \frac{A^6 B^3}{2 * 3 * 250^8} \tag{5.9}$$

$$12 * B = \frac{40 * A^2}{250^2}. \tag{5.10}$$

Reducing (5.8) yields $A = \frac{250 r_0}{10} = 25 r_0$, reducing (5.10) yields $A^2 = 18750 * B$. The hardest part is choosing A and B such that (5.8),(5.9) and (5.10) are satisfied. (We may add a gear train on each output that represents the RHS of each relation.) Notice that (5.8), and (5.10) may be related to provide an expression of r_0 in terms of B. However, in consideration of (5.9), we find that the choice of A needs to be sufficiently large so that the RHS of (5.9) is greater than one. Unfortunately, the relation provided by (5.8) and (5.10) yields an unsuitable choice of A in (5.9). The RHS of (5.9) must be greater than one due to the fact that we simply cannot gear up the output of Integrator 4, such that (5.9) is satisfied for $B > 1$. And we must have $B > 1$ because we need to maximize the range on Integrator 1's disk. Note that even a choice of $B = 1$ offers that the second derivative has a magnitude of 500, which is probably underestimated any way. Therefore a choice of $B < 1$ would be a drastic decision for the estimation of the magnitude of the second derivative.

It seems to be the case that (5.9) involves the most complicated algebra, so we will insert gear trains in the input of both integrators that correspond to the relation(5.9), which will allow more freedom in (5.9). To further simplify our analysis we can also work with Integrators 3 and 4 separately from the rest of the relations. That is, instead of carrying out the subsequent integrations starting from Integrator 1, we may instead start from Integrator 3 by labeling the output of Integrator 2 as Dy (note $D = \frac{A^2 B}{250^2}$). Starting from this new perspective we may create a "sub-schematic" that simply outlines the nonlinear terms (See Figure 5.9). Starting with the first bus-shaft, labeled Dy in next figure, we gear that down three separate times, first by n_1, then by n_2, and finally by n_3. Again, this is so that the inputs of Integrator 3 and 4 will have an extra gear for better algebraic manipulation. The output of Integrator 3 is

$$\frac{1}{250}\int \frac{Dy}{n_2} d(\frac{Dy}{n_1}) = \frac{D^2}{2 * 250 n_1 n_2} y^2.$$

Note that the extra factor of 2 is due to the result of integration. We now gear down the output of Integrator 3 by n_4 so that the integrand shaft will incorporate another gear option. The output of Integrator 4 is

$$\frac{1}{250}\int \frac{D^2}{2 * 250 n_1 n_2} y^2 d(\frac{D}{n_3}) = \frac{D^3}{3 * 2 * 250^2 n_1 n_2 n_3 n_4} y^3.$$

Because this is the output of Integrator 4, thus representative of the machine coefficient of Y^3, we know this quantity must be equal to B, or equivalently,

$$\frac{D^3}{3 * 2 * 250^2 * n_1 * n_2 * n_3 * n_4} = B, \tag{5.11}$$

which is a new relation of the nonlinear term that must be satisfied. Note that if we allow $D = 250$ our task is simplified. With, $D = 250$, we have

$$B = \frac{250}{6 n_1 n_2 n_3 n_4}$$

which implies

$$B * 6 n_1 n_2 n_3 n_4 = 250.$$

It will help to express $250 = 2 * 5^3$.

Once we have simplified the choice if B, we can back up and think about the choice for A. We will return to (5.11) shortly. Remember we determined earlier that A needs to be sufficiently large for mechanical purposes. We also know now that the

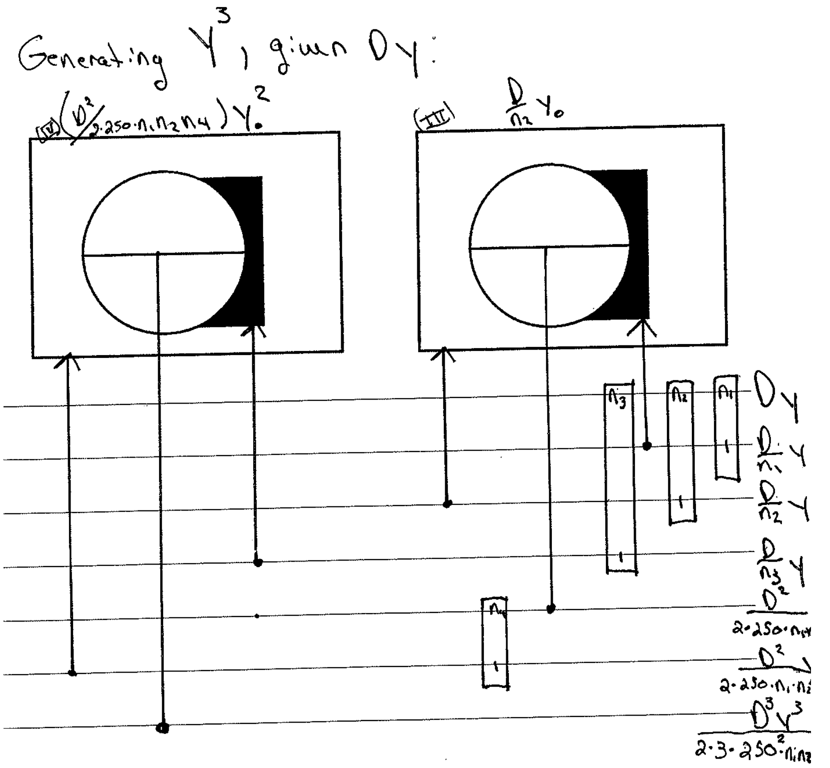

Figure 5.9. Sub-Schematic diagram that generates the nonlinear term in the nonlinear circuit.

output of Integrator 2 should be 250 as

$$250 = D = \frac{A^2 B}{250^2}$$

where the far RHS of this equation is the output of Integrator 2. Hence,

$$\frac{A^2 B}{250^2} = 250. \tag{5.12}$$

At this point we should consider our initial conditions as they are given in the analytical sense. The particular solutions we want to run have initial conditions $y(0) = 1.54$ and,$y(0) = 1.55$,$y'(0) = 0$, for the parameter $b = 12$ and,$b = 9.85, 9,$

respectively. So there are two different values of y'' given the two sets of initial conditions.

Case 1. $y(0) = 1.54$ and, $y(0) = 1.55$, given $y'(0) = 0$ for both, and $b = 12$, implies $y''(0) = 12 - 1.54^3 = 8.347736$, and $y''(0) = 12 - 1.55^3 = 8.276125$.

Case 2. $y(0) = 1.54$, $y'(0) = 0$, and $b = 9$ and ,$b = 9.85$, implies $y''(0) = 9 - 1.54^3 = 5.347736$, and $y''(0) = 9.85 - 1.54^3 = 6.197736$

In either case, the initial condition on y' is between 5 and 10. Now we need to make a reasonable conjecture about the magnitude of the second derivative. Because we know very little about the solution, we will estimate that the second derivative is bounded by 100. That is, we assume that the second derivative is limited to 100 units of magnitude. If we want a range of 100 units on the integrator disk, we need to let the shaft coefficient of the second derivative be $500/100 = 5$. This so happens to be our choice of B. Our task is not quite finished yet because letting $B = 5$ in (5.12) yields an irrational number for A. But we can implement the same kind of compensation of scale factor via a gear scaling method that we used in the nonlinear relation (5.11), which will force A to be sufficiently large and be an integer value. Consider the relation (5.12)

$$\frac{A^2 * 5}{250^2} = 250$$

which implies

$$A^2 = \frac{250^3}{5} = \frac{2^3 * 5^9}{5} = 2^3 * 5^8.$$

If we can multiply the RHS by 2 then we will have a perfect square. Figure 5.10 depicts a refined algebraic schematic for Integrators 1 and 2 only. Here we have the output of Integrator 1 being geared down separately from the gear train r_0. That is, we have a new coefficient expression for y, which now has a factor of n_0 in its denominator. Note the output of integrator 2, with the newly inserted gear train, is

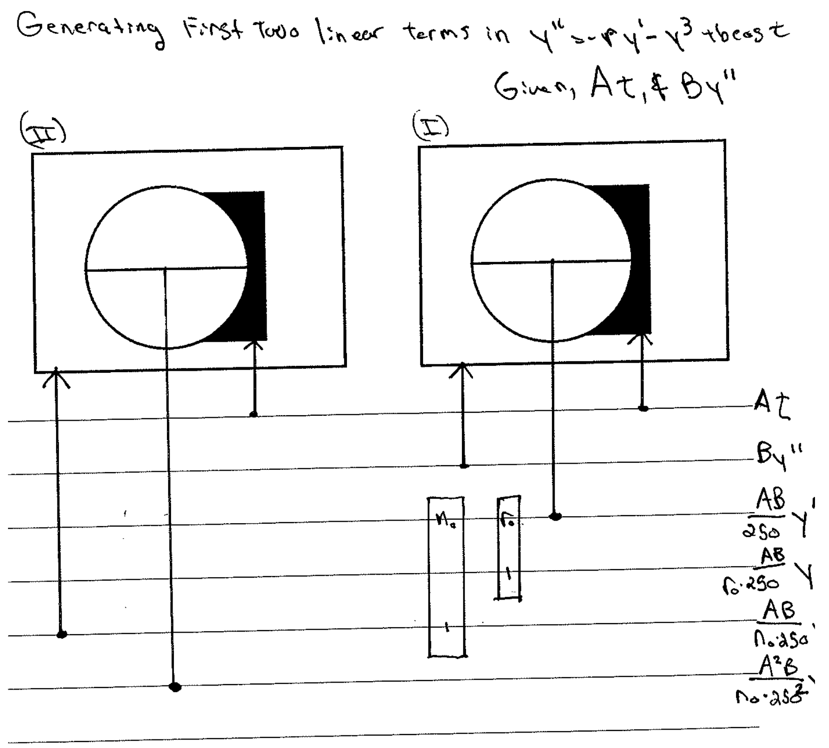

Figure 5.10. Sub-Schematic diagram that generates the first two linear terms in the nonlinear circuit.

still defined to be equivalent to $D = 250$. Hence, (5.12) becomes

$$D = \frac{A^2 * 5}{n_0 * 250^2} = 250. \tag{5.13}$$

Finally, letting $n_0 = 2$ in (5.13) implies, $A = 2500$.

Returning to (5.11) we will need to choose n_1, n_2, n_3, n_4, so that $5*2*3n_1n_2n_3n_4 = 2*5*5*5$ for $B = 5$, which implies $3n_1n_2n_3n_4 = 5*5$. Among the many solutions, we need to take into consideration the scale of integrand shafts for Integrators 3 and 4. The scale must be small enough to include the full magnitude of y and y^2. Integrand shaft 4 should have a range of 50 units as it represents the square of y and therefore could get really large. This range would suggest a shaft coefficient of 10. As for integrand shaft 3, it needs the coefficient 25 to provide 20 units of range in magnitude. If we refer back to schematic Figure 5.9, we can use the coefficient expressions on bus shaft 3 and bus shaft 6 to yield the two relations

$$\frac{D}{n_2} = 25 \tag{5.14}$$

and

$$\frac{D^2}{2 * 250 * n_1 * n_2 * n_4} = 10. \tag{5.15}$$

We now have that as $D = 250$, then $n_2 = 10$ and $n_1 * n_4 = \frac{5}{4}$. All that is left is to fully scale the nonlinear term is to find the value of n_3 in (5.11). To do so, we will choose $D = 250$, $B = 5$, $n_1 = 5$, $n_2 = 10$, $n_4 = \frac{1}{4}$, based on our most recent information. Thus, $n_3 = \frac{3}{2}$.

The last piece of the puzzle is to scale the linear non-homogeneity, that is Integrators 5 and 6. Note that the only dependence this system has to the other terms, or rather subsequent integrations, is that of the independent variable that has a shaft coefficient of $A = 2500$. Moreover, Integrators 5 and 6 have their own closed system of equality that can be dealt with very simply. In fact, one only needs to make sure that a shaft coefficient of 12*B is available from the solution $x(t) = 12\cos(t)$ in the section of interconnection, essentially gearing down the shaft representative of x by some amount. This interconnection will have to be done independently, so as not to to affect the closed system of equality in Integrators 5 and 6. Because throughout this analysis of scale choice we have established several different relations, it is helpful to create a finalized algebraic schematic that will incorporate all added variable gear trains throughout this discussion (See Figure 5.11).

Notice in Figure 5.11 we have added two more gear trains, m_1 and m_2, to allow more options for the non-homogenous term. When working with the algebraic

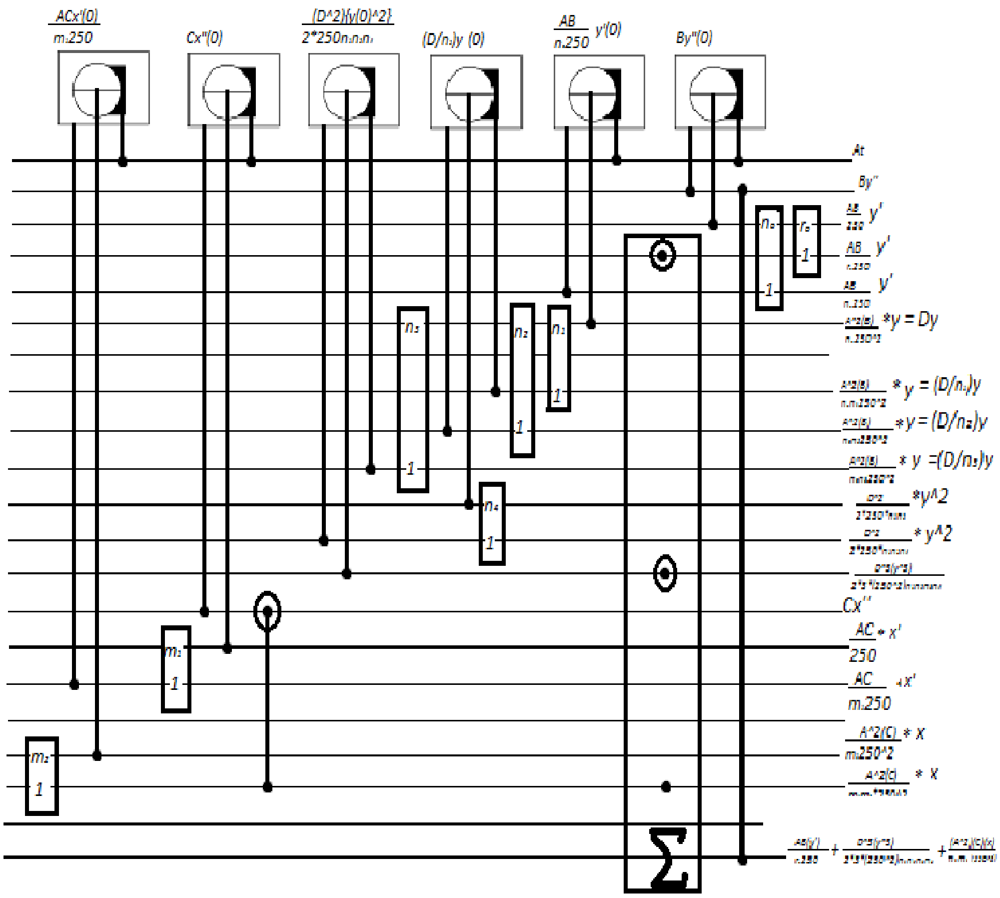

Figure 5.11. A finalized version of an Algebraic Schematic diagram for the nonlinear circuit, $y'' = -r_0 y' - y^3 + b\cos(t)$.

schematic it is not always necessary to go into as much detail as we did in Figure 5.11. For the most part, an operator will begin to notice a pattern emerging from several relations established due to the subsequent integrations we did. Note that the choice of D was simply an educated guess. We found that if $D = 250$ then terms will reduce to a reasonable form, and, after some study, an operator will usually need to simply try a few values for the choice of A and B. This is not to say the algebraic schematic is not useful because it does provide insight for the first choice. Without thinking about the DE in terms of an algebraic schematic, an operator is "diving in blindfolded" so to speak. For some problems, perhaps, the scaling is clear and this rigorous analysis is not necessary. But creating an algebraic schematic can save valuable time in a trial and error process.

If we substitute $A = 2500$, $B = 5$, $C = 40$, $D = 250$, $r_0 = 100$, $n_0 = 2$, $n_1 = 1$, $n_2 = \frac{10}{3}$, $n_3 = 1$, $n_4 = \frac{5}{2}$ and $m_1 = m_2 = 10$ into the schematic diagram, Figure 5.11, we get the final schematic diagram Figure 5.12. Notice that the choice of the gear trains n_i, has changed. We found that gearing up anywhere in this chain of gears was not sufficient for smooth mechanical operation. Also take note that in doing so we have sacrificed the *Unity* value for integrand shaft 3 and integrand shaft 4. Remember that we had determined a reasonable range for integrand shafts 3 and 4 in consideration of the analytical magnitude of the those values. With these new choices we now have integrand shaft 3 with a scaling coefficient of 75 (as opposed to 50) and integrand shaft 4 with a scaling coefficient of 15 (as opposed to 10). Because the values of *Unity* for integrand shafts 3 and 4, have gone up, we have essentially reduced the maximum range in magnitude. Now integrand shaft 3 has a range of $500/75 = 6.6666$, and integrand shaft 4 has a range of $500/33.333$. Analytically speaking, if y is bounded by 5 in magnitude then y^2 will not be more than 25. Analogously, in terms of the machine, if Y does not exceed $5 * 75 = 375$ (that is integrand shaft 3, which is represented by the position of the wheel with respect to the center of disk on Integrator 3), then Y^2 will not exceed $25 * 15 = 375$ (that is integrand shaft 4, which is represented by the position of wheel with respect to the center of disk on integrator 4). We can see if y is bounded by 5 then this is an appropriate range. However, $y < 5$ is an assumption and could very well be false. If

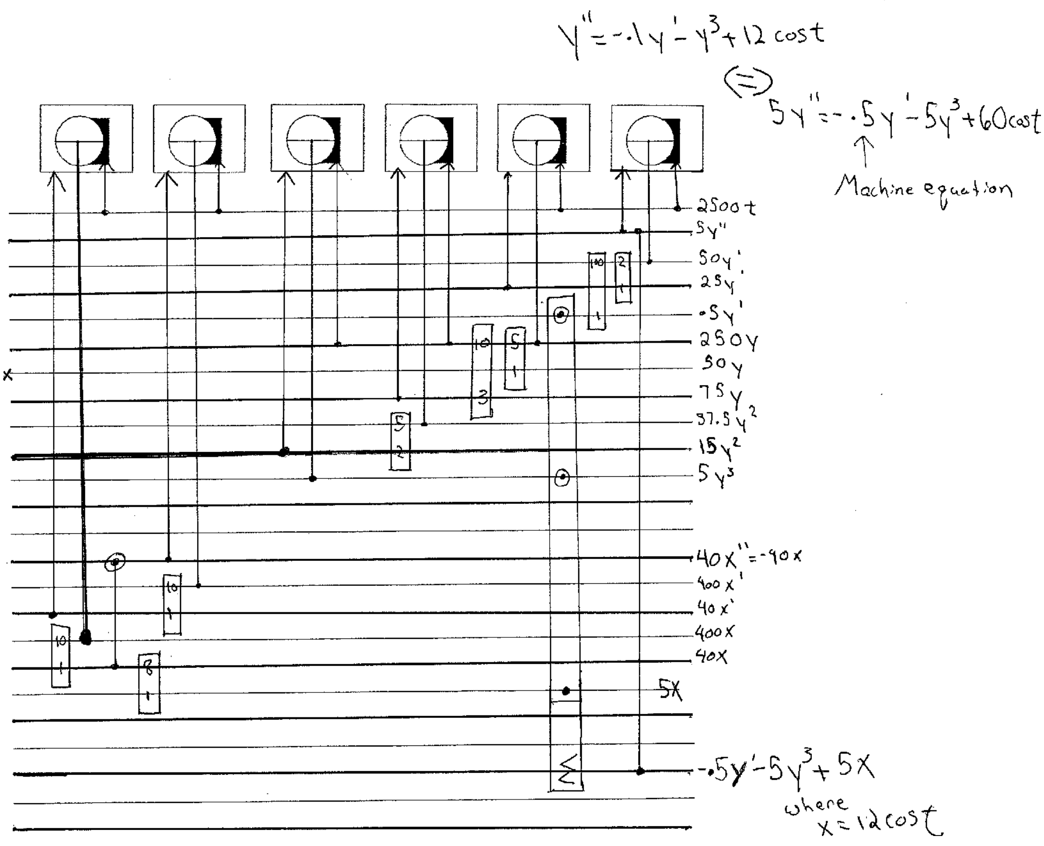

Figure 5.12. Final schematic diagram of the nonlinear circuit using six integrators.

this is the case then Integrator 4 will run out of range first, and we will need to scale down the *Unity* in Integrator 4's integrand shaft. Also take note that the bus-shafts in Figure 5.11 do not exactly correspond to the bus-shafts in Figure 5.12. This is due to the fact that some the gear trains in Figure 5.11 are chosen to be 1:1 and there is no need to mention them in Figure 5.12.

When drawing a final schematic, one can notice a simpler pattern that emerges from each subsequent integration. As our analysis led us to choices for A,B,C, and D, we can start with these values and work through the integrations, similar to the algebraic approach. We start with the integral of $5Y''$ with respect to $2500t$.

We already know that the result is $50Y'$, but we can use a "short-cut" to get each subsequent shaft coefficient value. Take the coefficient of the integrand shaft, multiply it by the coefficient of the differential shaft, and then divide that product by the *Unity* of that integrator. The result will be the coefficient of the integral shaft. This value is the coefficient of the variable, represented by a bus-shaft, which is a result of the integration. Note that this "short-cut" is simply the algebraic method "spelled out" in words. The nice property that makes the calculations easier is that each subsequent integration starts fresh with a reduced value on the integral shaft. We can start from the output of Integrator 2 (labeled 250y in Figure 5.12) and carry out the calculations of the coefficients for the nonlinear terms, as they were the most difficult. First, note that the shaft coefficients on integrand shaft 3 and integrand shaft 4 are $\frac{250}{\frac{10}{3}} = 75$ and $\frac{37.5}{\frac{5}{2}} = 15$, respectively. Now we pass the first pair of coefficients through Integrator 3, yielding $\frac{250*75}{2*250} = 37.5$. Remember that the factor of 2 is in the denominator because

$$\int y dy = \frac{y^2}{2}.$$

Sending that motion through Integrator 4 we get $\frac{15*250}{250*3} = 5$. Similarly the factor of 3 is due to the fact that

$$\int y^2 dy = \frac{y^3}{3}.$$

Working with the final schematic in this way will provide a simpler way to account for any mechanical difficulty the set up may have. This "short-cut" is to be done after the algebraic schematic has been, at least minimally, worked through. Without this preparation operators will find themselves playing what could be a very long guessing game.

At this point we should discuss the linear non-homogeneity. First, we chose, $C = 40, m_1 = m_2 = 10$. Among the coefficients passing through Integrator 5, we have $\frac{40*2500}{250} = 400$. We then divide that shaft coefficient by 10 to yield 40. For Integrator

6, we have $\frac{40*2500}{250} = 400$. Again dividing that shaft coefficient by 10 to get 40. This next connection is very important. The shaft representing X, with a coefficient of 40, needs to make the equality connection to the shaft representing X'' with a coefficient of 40 and with a sign change, such that, $40X'' = -40X$. This Machine equation is not new, but notice that independent of the sub-equality connection to that the shaft labeled $40X$ is a gear reduction of 8 that gives $5X$. Moreover, it is the shaft labeled $5X$ in Figure 5.12 that is connected to the adder that completes the primary equality connection for the differential equation. The solution X, which is fully known by design, is equivalent to $x(t) = 12cos(t)$. So, although a shaft coefficient of 60 for X could obtained otherwise with the proper gear trains, we don't need to have $60X$ for the machine shaft coefficient because a factor of 12 is already imbedded within the solution x. Remember that all one must do to get an analytical value that represents a shaft is divide by the machine shaft coefficient. There is an algebraic correlation from analytical shaft representations to machine shaft coefficients, which is, in the case of X having a coefficient of 40:

$$\frac{X}{40} = x$$

where $x(t) = \frac{12}{40}\cos(t)$. Hence, if we have 5 for the shaft coefficient of X, then $x(t) = 12\cos(t)$, which will now be consistent with our analytically DE. In this case, having $5X$ for a machine shaft coefficient, the primary machine equation is

$$5Y'' = -.5Y' - 5Y^3 + 5X$$

or equivalently,

$$5Y'' = -.5Y' - 5Y^3 + 5 * 12cos(t).$$

When we divide the machine equation by 5 we get our original differential equation, and this completes our quantification of scaling an ODE to a "machine" ODE.

Unfortunately, "Art" is only equipped with 4 integrators, so at first we considered using an input table to account for the non-homogeneity. As previously discussed, a separate DE is set up and solved, a priori. The schematic diagram and the input curve may be seen in Figures 5.13 and 5.14, respectively. In addition to human error, the input curve also has other undesirable qualities. The peaks in the input curve are somewhat non-symmetric, due to the extreme backlash in the 10:1 gear trains necessary for the proper scaling. Because it was originally thought that the independent variable needed to be driven with a coefficient of 2500, we needed the 10:1 gears to maintain consistency with the machine equation $Unity$, which is the scale factor B. To avoid accumulation of increasing amplitude, an operator restarted the input curve (i.e., initial conditions) after every period which helped to maintain a consistent cosine curve. The effect of an operator resetting the initial conditions after one period in the solution is the imperfections in the peaks of curve that can be seen in Figure 5.13. There is no doubt in my mind that the addition of frontlash units will completely eliminate this problem. Furthermore, in order to observe the solution at $t = 40$, the time variable was geared down by a factor of 96. By gearing down we were able to observe more periods, and thus would avoid resetting the input table many times during the calculation of the solution. To account for this gearing we let a coefficient of $\frac{2500}{96}$, in the independent variable t, drive the input table. Although gearing the independent variable down drastically has not been formally tested for accuracy, it is my belief that do so amplifies already existing human error. Moreover, operating the input table became a very laborious task.

Another serious problem was that the torque amplifiers were extremely overloaded when driving the integrators representing the nonlinear terms. Moreover, overloading the torque amplifiers contributed to accumulation of error, because of

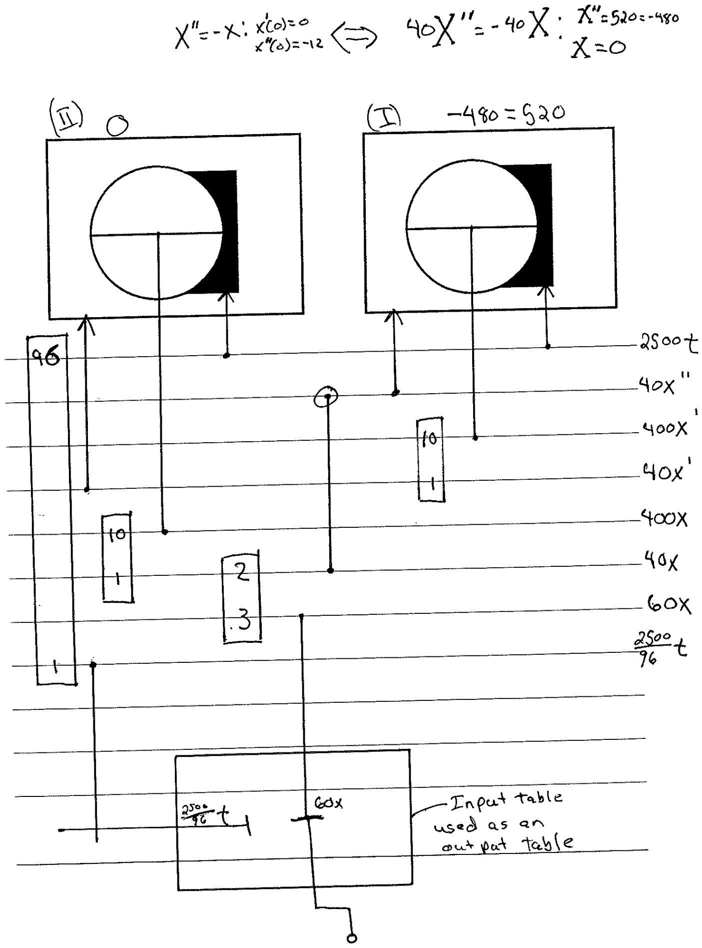

Figure 5.13. Final schematic diagram that generates the non-homogeneous term in the nonlinear circuit.

the rapid oscillations in the servo-motor, and again amplified backlash in the differential drives of the integrators. To make a long story short, the integrators kept running out of range in spite of a long list of reduction gearing. So despite the errors that we knew to exist, the solution should have still exhibited some of the minimum qualitative properties that were known to us from numerical calculation.

Instead of using this input curve, we instead reevaluated the setup entirely, in consideration of the algebraic scale factors obtained from the beginning of our

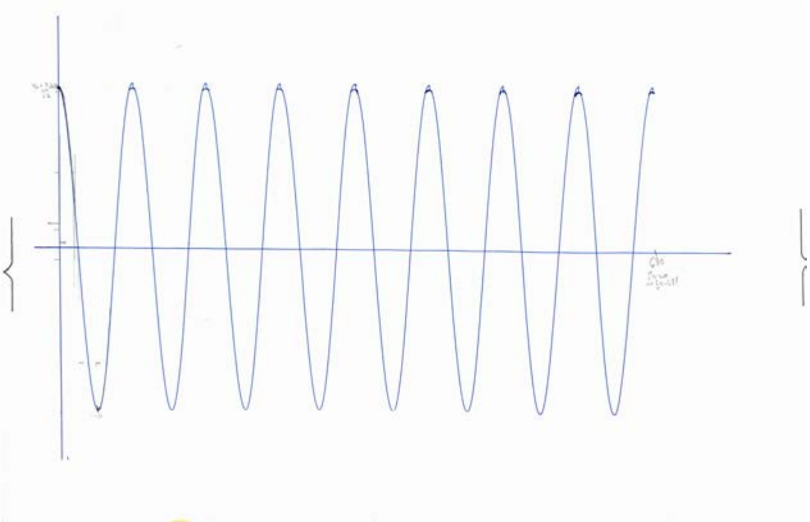

Figure 5.14. First plot of the original input curve, or non-homogeneous term (i.e. $12\cos(t)$.

work. It was thought that, if we could generate the nonlinear term by using the Input Table, we could overcome many mechanical difficulties and let the machine provide the work necessary in generating the linear non-homogeneity. This method was similar to the concept of generating a composite function, previously mentioned, where the nonlinear term was created via function composition by using the Input Table. First, we need a cubic function (refer to Figure 5.15). We can solve the DE

$$z''' = 6,$$

to generate $z = t^3$. (Refer to the Appendix Figure Ap.1 for a general account of how to set up this problem.) The scaling in Figure 5.14 is very important. We know that our choice of $D = 250$ was the underlying scale factor in generating the nonlinear term. So we let $\frac{250}{2} * t$, a factor of D, drive the table. We also know that the coefficient of the Y^3-term is 5, so we let $5Z'''$ drive integrand shaft 1. Upon the subsequent integrations the result is $5Z$, where $z = t^3$ (Remember that Z is a machine

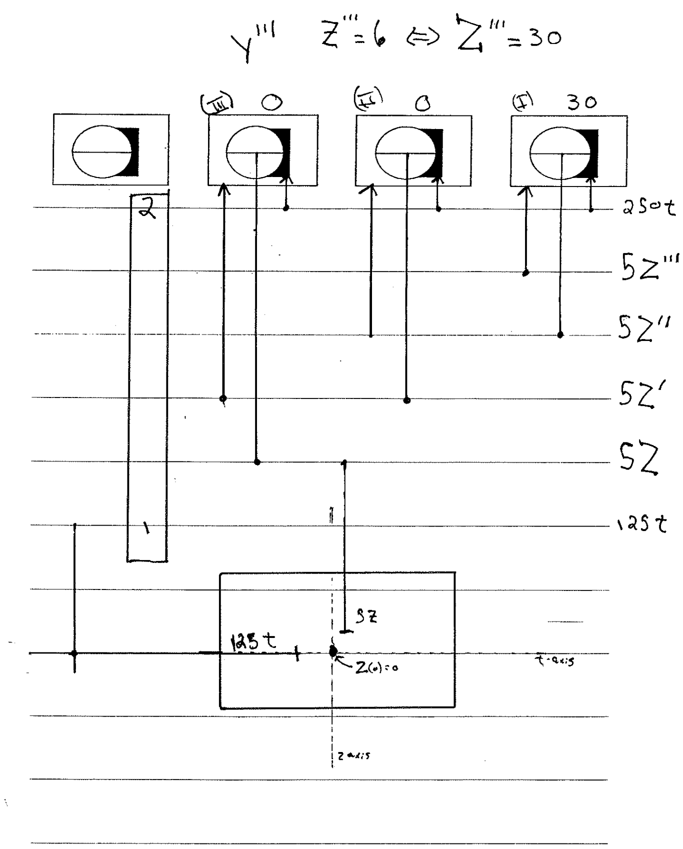

Figure 5.15. Schematic diagram used in the generating of the nonlinear term Y^3

variable and z is an analytical variable, but they are algebraically correlated.) So we obtain a plot of $5Z$ v.s $125t$. The scaling coefficients here are as relevant as the variables they are attached to. What we know is we have a cubic function on a scale of 5 vs 125 where the variable driving the table in the abscissa direction is the variable in the expression for the cubic function. Note the initial conditions on integrator 1 is $5*6 = 30$ (for the machine DE Z"'=30) and the rest are zero. This is so that we have a simple cubic function. (One can verify that the general solution to $z''' = 6$, where

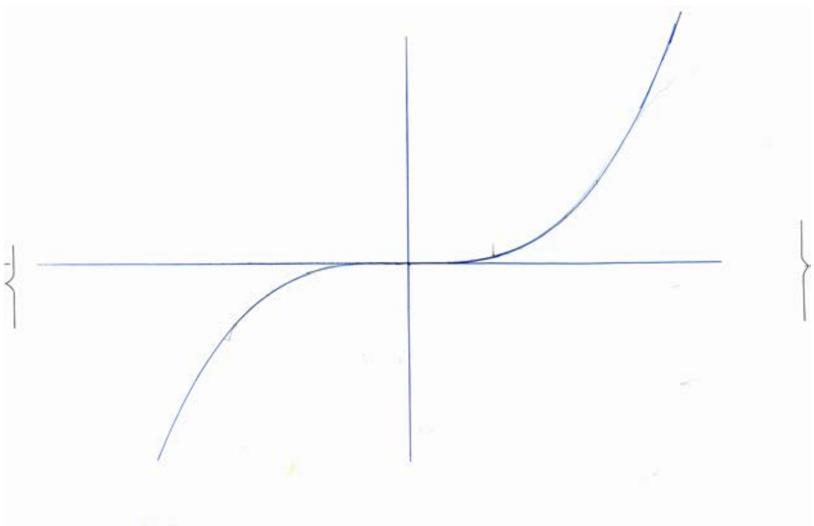

Figure 5.16. A plot of the new input curve for the nonlinear term Y^3. (This was the input used in the calculation of the solution.)

$z = f(t)$, with $z'''(0) = 6$, $z''(0) = z'(0) = z(0) = 0$, is $z = t^3$.) In order to run the curve, an operator must start the ordinate carriage it the origin and then separately run the independent variable in the positive and negative directions for the desired time interval, resetting initial conditions between runs (See Figure 5.16 for a plot of nonlinear input curve).

The trick is that when we set up to run the solution to

$$5Y'' = -.5Y - Y^3 + 60cos(t)$$

on the machine, we need to let the shaft representative of y, that is, Y, with a coefficient of 125 drive the input table in the direction of abscissa. So when an operator turns the hand crank to follow the curve, he or she is inputting $5Z$, where $z = y^3$. Hence, the movement of the shaft that connects the ordinate carriage to the rest of the section is labeled $5Y^3$.

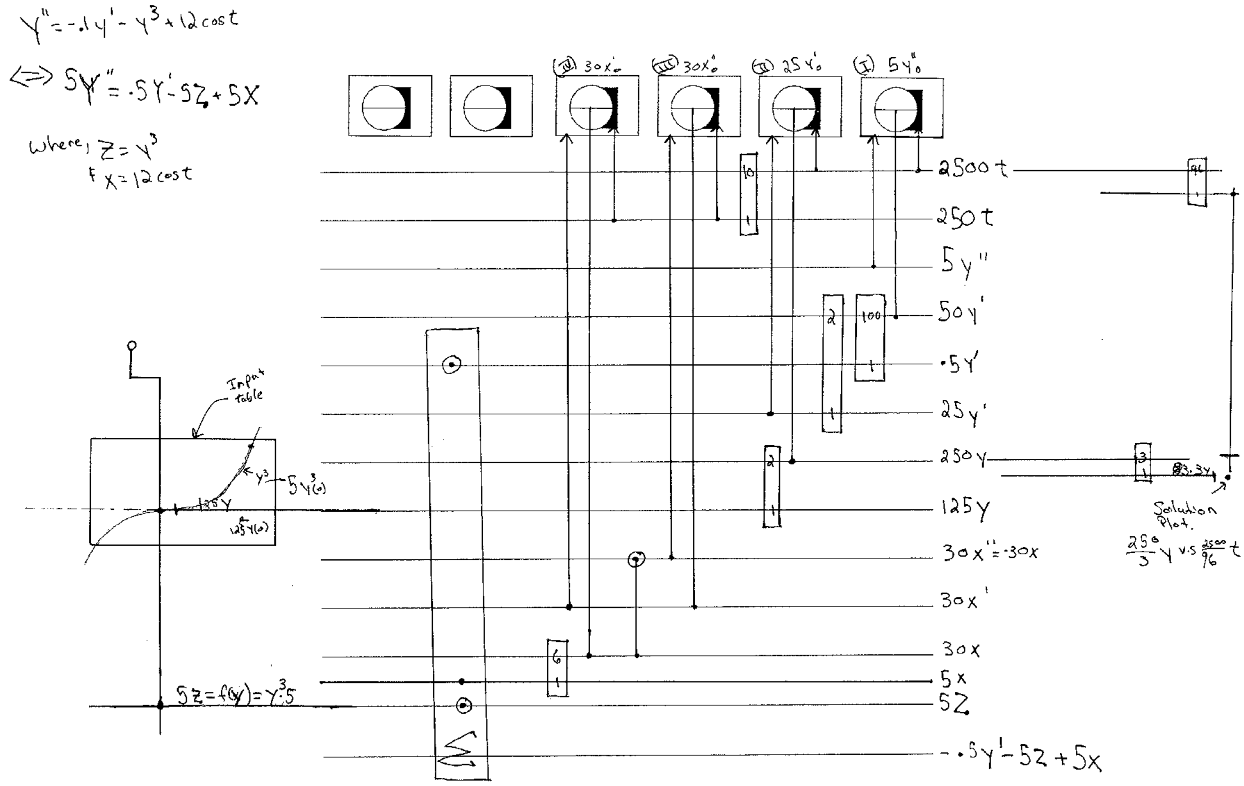

Figure 5.17. Final schematic diagram of $y'' = .1y' - y^3 + 12\cos(t)$. This is the version that was used in the plotting of the solution

Figure 5.17 is the new finalized schematic diagram for

$$5Y'' = -.5Y' - 5Z + 5X$$

where $z = y^3$ and $x(t) = 12cos(t)$, and we are indeed solving

$$y'' = -.1y' - y^3 + 12cos(t),$$

with initial conditions, $y(0) = 1.54, y'(0) = 0$. Figure 5.18 was the first successful solution plot. It was observed that the full range of Integrators 1 and 2 was not being

Figure 5.18. Solution plot of the nonlinear circuit.

used. To expand the range it was decided to let A=1250, B=10, and work through the reduction gearing as given by a final schematic diagram (See Figure 5.19).

In the Schematic diagram (Figure 5.19), notice the new Machine equation

$$10y'' = -1y' - 10y^3 + 10 * (12cos(t)).$$

Also note that we have eliminated all reduction gearing within the section except for the reduction gearing on y'. The gear reduction on y' is 50:1 because the shaft coefficient of the output of Integrator 1 is 50. We want to reduce this to 1 as the *Unity* of the dependent variable has been changed to 10. Despite this one gear reduction, all integrand shafts are being driven with a direct drive coming from the corresponding integrator output. To compensate for the previously described setup with the necessary gear reductions contained within the section of interconnection, we have instead used different independent variable drive constants at various differential shafts in each integrator. This setup greatly reduces the effect of backlash and we have expanded the range. Gearing down a monotonic drive, such as the independent variable, instead of non-monotonic drives, such as shafts being driven by a servo-motor, will provide a much more accurate plot. To provide insight into the accuracy of the solution we will create a phase plot (See Figure 5.20). In Figure 5.20, we have plotted $150y$ vs $75y'$. We ran the machine so that a couple of oscillations of the phase plot could be observed. It was noticed that the phase plot was not periodic. However, in the description of this problem with the given initial conditions, the phase plot was said to settle into a periodic solution after about $t = 300$, [11]. So we ran the machine for approximately 50 oscillations. Then we plotted the result on a fresh sheet of paper (See Figure 5.21). Figure 5.21 does resemble, qualitatively, a similar plot to that of what RK4 calculates for a phase plot in the time interval $300 \leq t \leq 400$ (See Figure 5.22). However, it is clear that the plot will not settle into a periodic orbit on the machine. One reason for this instability is the cumulative error we know to exist in the non-homogeneous term $12\cos(t)$. Furthermore, letting the independent variable run for $t = 300$ becomes a very laborious task for the operator of the input table.

Figure 5.23 is another example of a phase plot that is supposed to settle into a periodic orbit after $t = 300$, and it can be compared to Figure 5.24, which is the same phase plot calculated by RK4.

The best we can do at this point is plot the solution and compare the results to a numerically approximated solution. Figure 5.25 is a numerically approximated solution via the RK4 method. Figure 5.26 is the final solution of $y'' = -.1y' - y^3 + 12\cos(t)$ with $y(0) = 1.55$ and $y'(0) = 0$.

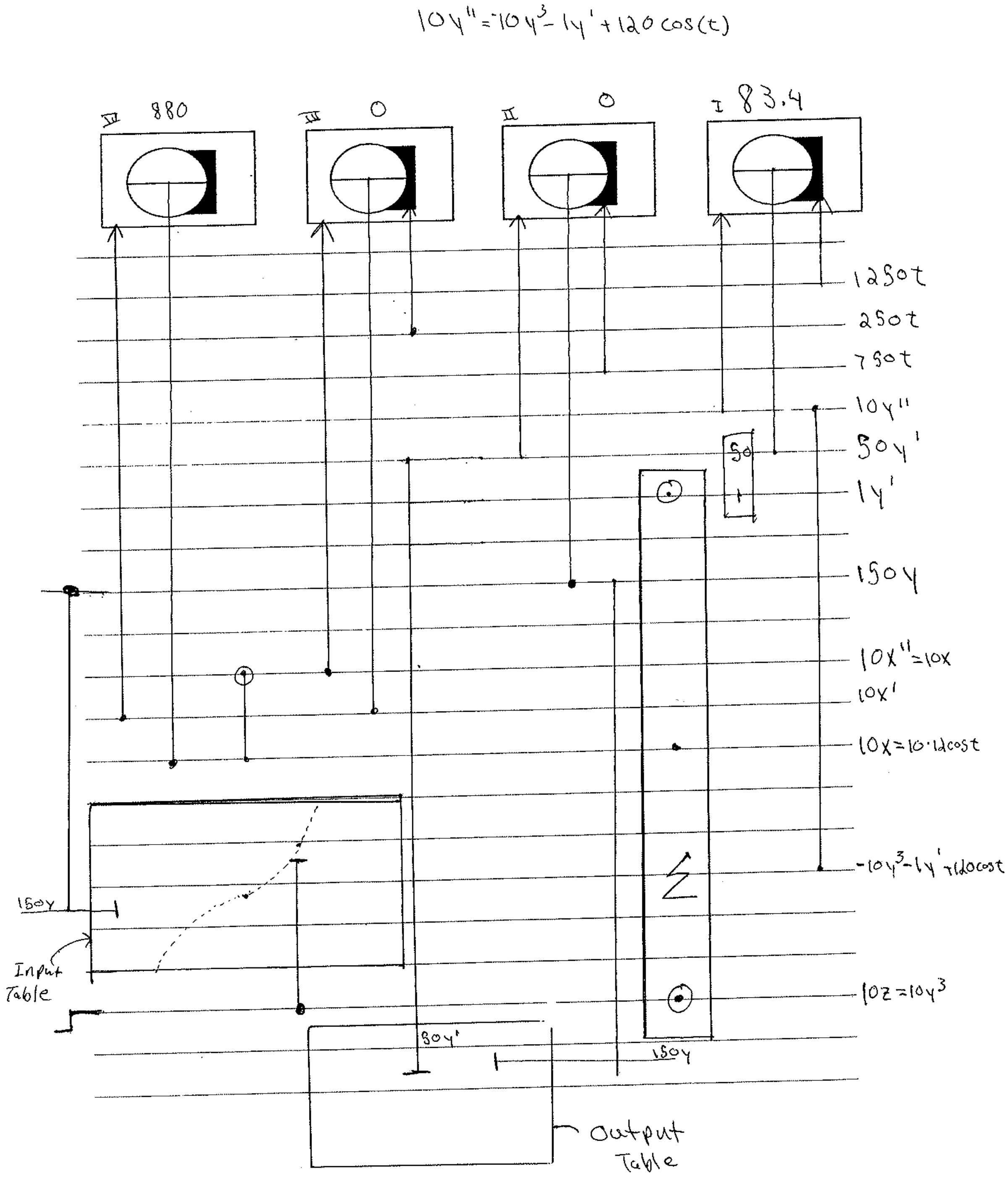

Figure 5.19. A different Final schematic diagram for the nonlinear circuit with a different scale.

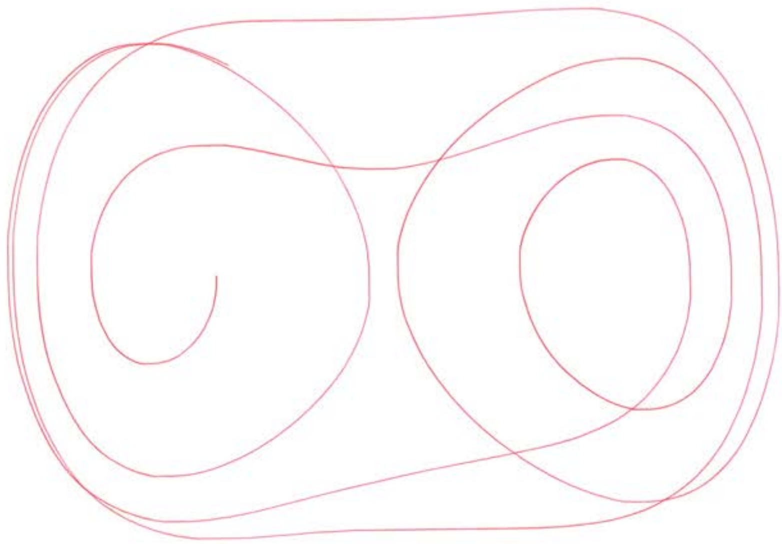

Figure 5.20. This is a phase plot of the nonlinear circuit calculated by "Art", with the parameter $b = 12$, and $y(0) = 1.54$.

With the addition of frontlash units along with two more integrators the nonlinear circuit can be run on the DA autonomously, and the frontlash units will compensate for the accumulation in the non-homogeneous term $12\cos(t)$. With this under consideration, Figure 5.26, produced by the DA, does have the qualitative property

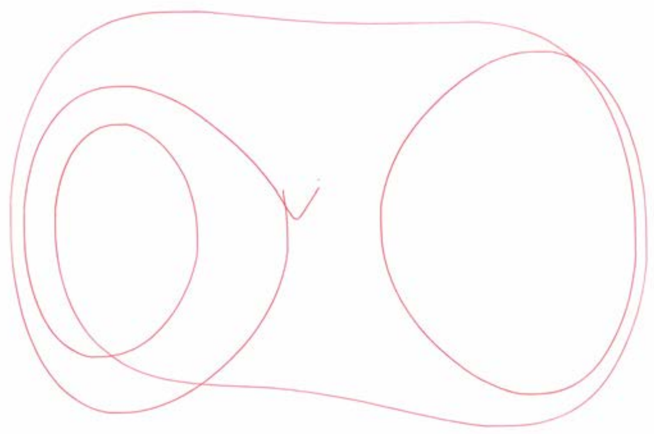

Figure 5.21. This is a phase plot of the nonlinear circuit calculated by “Art” with the parameter $b = 12$, and $y(0) = 1.54$.

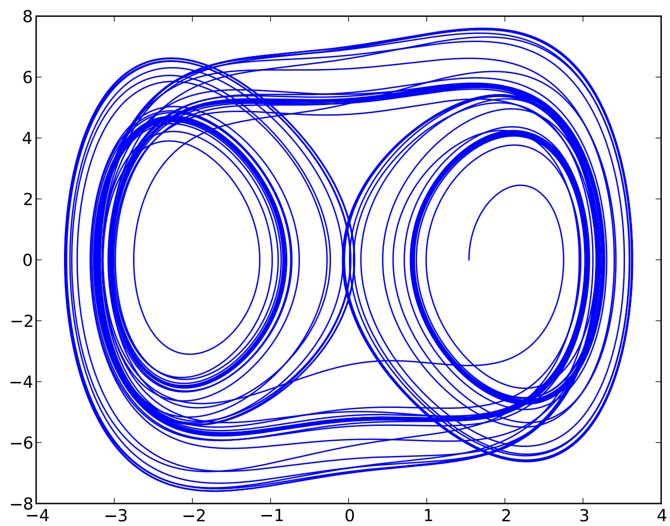

Figure 5.22. This is a phase plot of the nonlinear circuit calculated by RK4, with the parameter $b = 12$, and $y(0) = 1.54$.

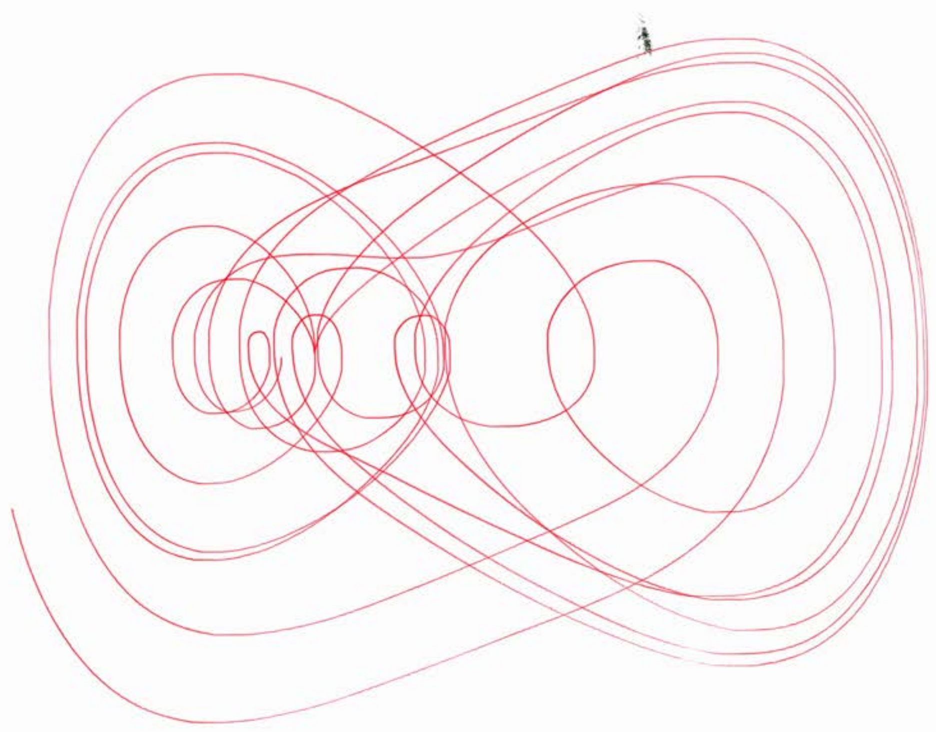

Figure 5.23. Phase Plot produce by “Art” with $b = 9$, and $y(0) = 1.54$.

consistent with the nonlinear circuit’s original assessment; the total disruption of a seemingly periodic orbit at about $t = 40$.

5.4. CONCLUSION

Working with a differential analyzer, as has been been discussed, requires a certain amount of patience. The goal of this section was to demonstrate how to set up nonlinear problems. Dealing with nonlinear terms and generating non-homogeneities

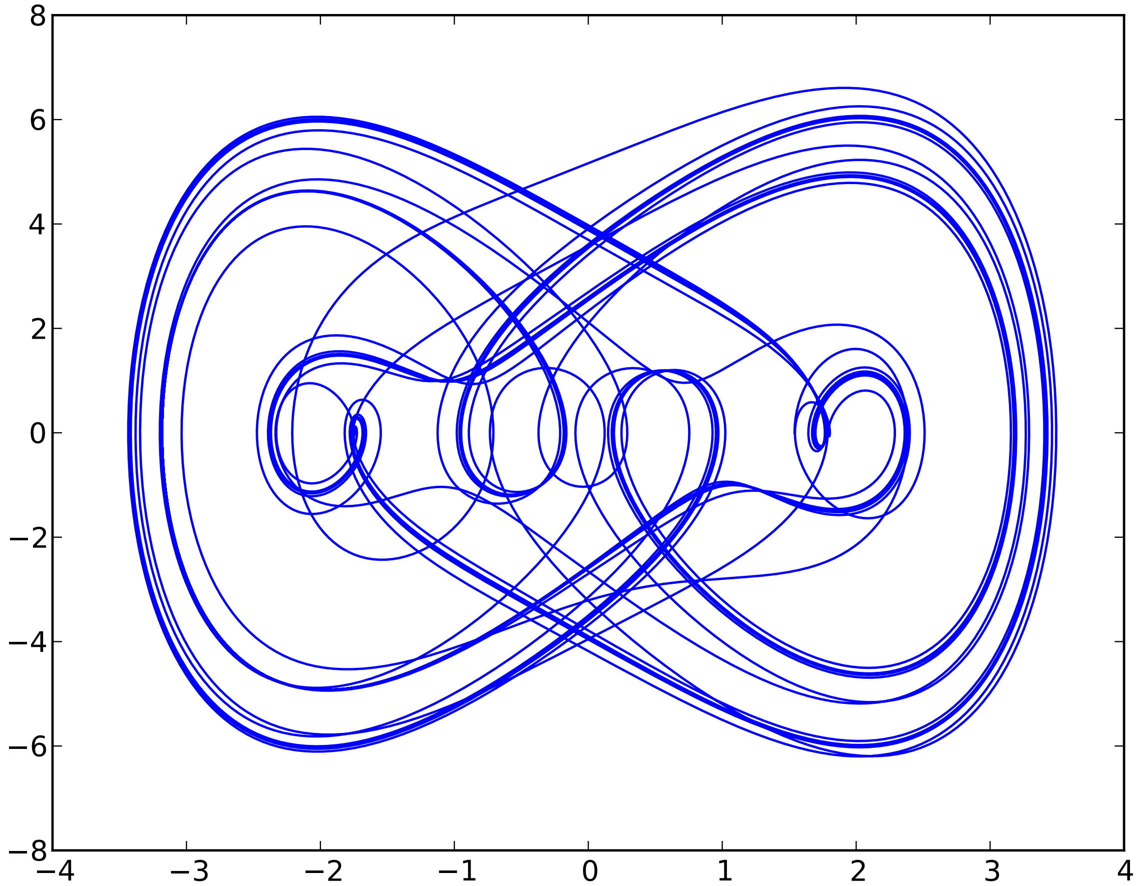

Figure 5.24. Phase Plot produced by RK4 with $b = 9$, and $y(0) = 1.54$.

have been illustrated *via* several general ways. Finally, a complete account of a nonlinear problem was detailed in the fullest regard. Solving this nonlinear problem was an attempt to generalize how to scale a nonlinear differential equation on Marshall's differential analyzer. "Art" is fully equipped for solving the nonlinear circuit. It has been shown that the capabilities of "Art" can include nonlinear problems in the range of problems to be studied.

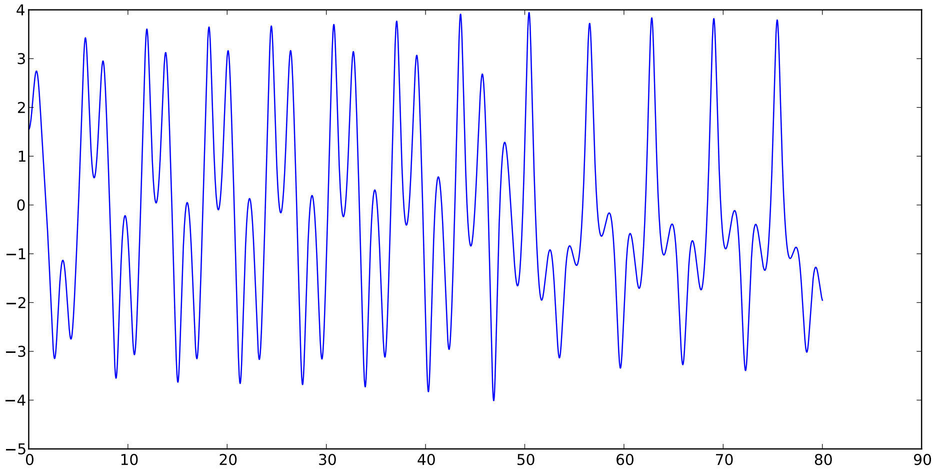

Figure 5.25. Solution to the nonlinear circuit with $y(0) = 1.55$ and $b = 12$, calculated by RK4.

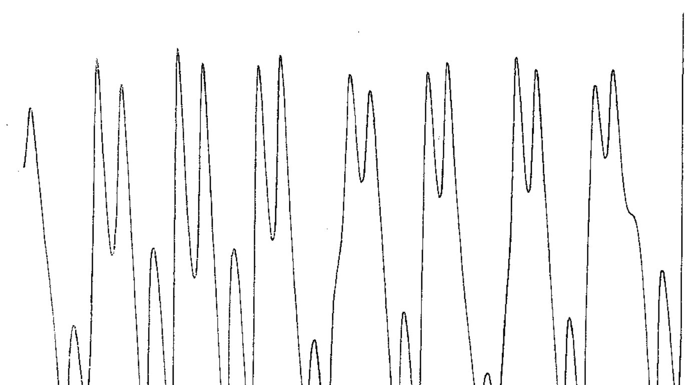

Figure 5.26. Final solution to the nonlinear circuit with $y(0) = 1.55$ and $b = 12$, calculated by "Art".

An interesting account of solving a nonlinear problem with similar properties to the nonlinear circuit was given by Arthur Porter in his memoirs. He said, while remembering working on a problem in chemical control process:

> In retrospect, albeit somewhat tangential, our investigations were a prelude to the chaos theory, which has assumed such great importance in recent years. The Hartree formulation of the problem and how the temperature distribution, step-by-step, was generated on the output table of the differential analyzer was certainly one of the high points in the history of the machine. A very small change in the initial conditions at the boundary of the dielectric had spectacular effects, and indeed, in the ultimate condition gave rise theoretically to the complete breakdown of the dielectric material through a sudden increase in temperature. Many years later there was an analogous effect in meteorology. [14]

Porter is referring to the “butterfly effect,” and so, even in Porter’s day it was certainly possible to observe “chaotic” properties with a differential analyzer.

Using the differential analyzer to solve nonlinear differential equations has proved to be a challenge. To truly analyze complicated behavior on a differential analyzer one must first familiarize oneself with the mechanical action of a differential analyzer. Being able to relate the action of a movable part to mathematical principle is something of an acquired taste. The machine was built using Meccano parts. Using these parts help, for one not so familiar with mechanics, to feel comfortable with the mechanical action of the machine, because the parts are simple (just like Erector set parts). The primary component of a differential analyzer is the integrator unit. We initially took a theoretical approach in the attempt to describe what an integrator does. Using the concept of infinitely small partitions of a Riemann Sum, we proved that any function represented on top of the surface of an integrator is Riemann integrable. With this proof we have provided a foundation to expand the theory of mechanical integration. Because the rotation of the integrator wheel is representative of a Riemann Sum, one can interpret what would be an abstract concept with a physical example. Using mechanical integrators together by means of an interconnection system is, in essence, a differential analyzer. In practice, quantification of scaling is indeed the biggest challenge when using a differential analyzer to solve differential equations. But in the midst of working through the various substitutions, one can, in turn, develop a new perspective on different classes of equations. The first step in knowing how to use the machine is learning how to create a Bush Schematic. In some cases this can be very simple and in others not so much. But viewing the various interconnections of just a general schematic diagram literally provides one with a "map" of the differential equation itself. Likenesses are comparable to various concepts in graph theory in the attempt to further generalize and simplify complex ideas. When one follows through with an abstract idea on "Art" the results are of a most elegant practical nature. By using practical knowledge to provide insight into theoretical assumptions, and vice-versa, one can develop pure mathematical justifications based on very concrete realizations. It has been my experience when working with "Art" that one is transported into a time where computational science is much more involved as opposed to the digital computing power of today. And for this reason the differential analyzer has been lost to time. The Marshall DA team has adopted what seems to be an ancient practice of problem solving and critical thinking, but it is our belief that one can only benefit from doing so.

APPENDIX A

Schematics

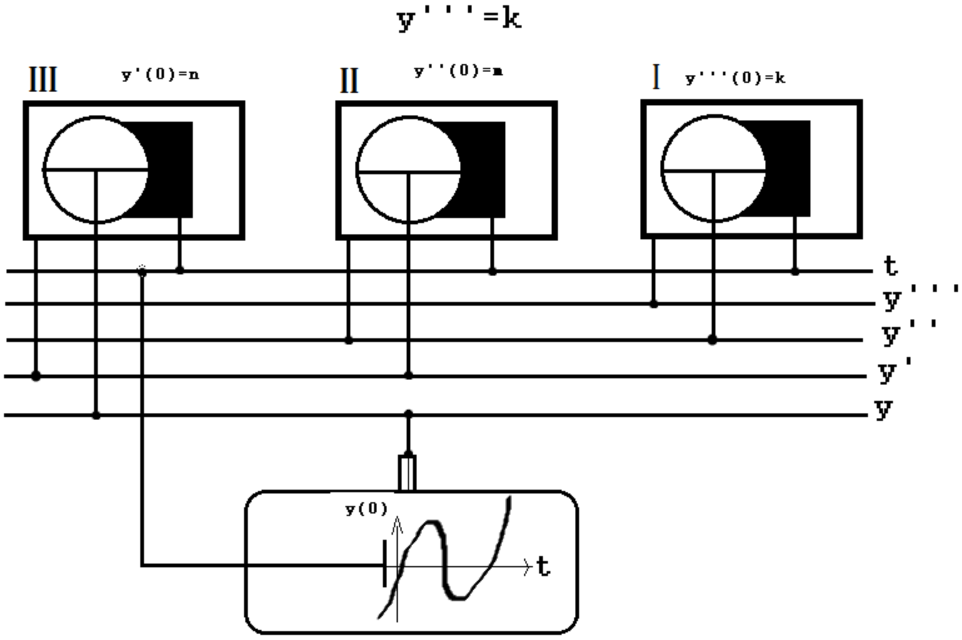

Figure A.1. General schematic diagram for $y''' = k$; the output table will produce a cubic function.

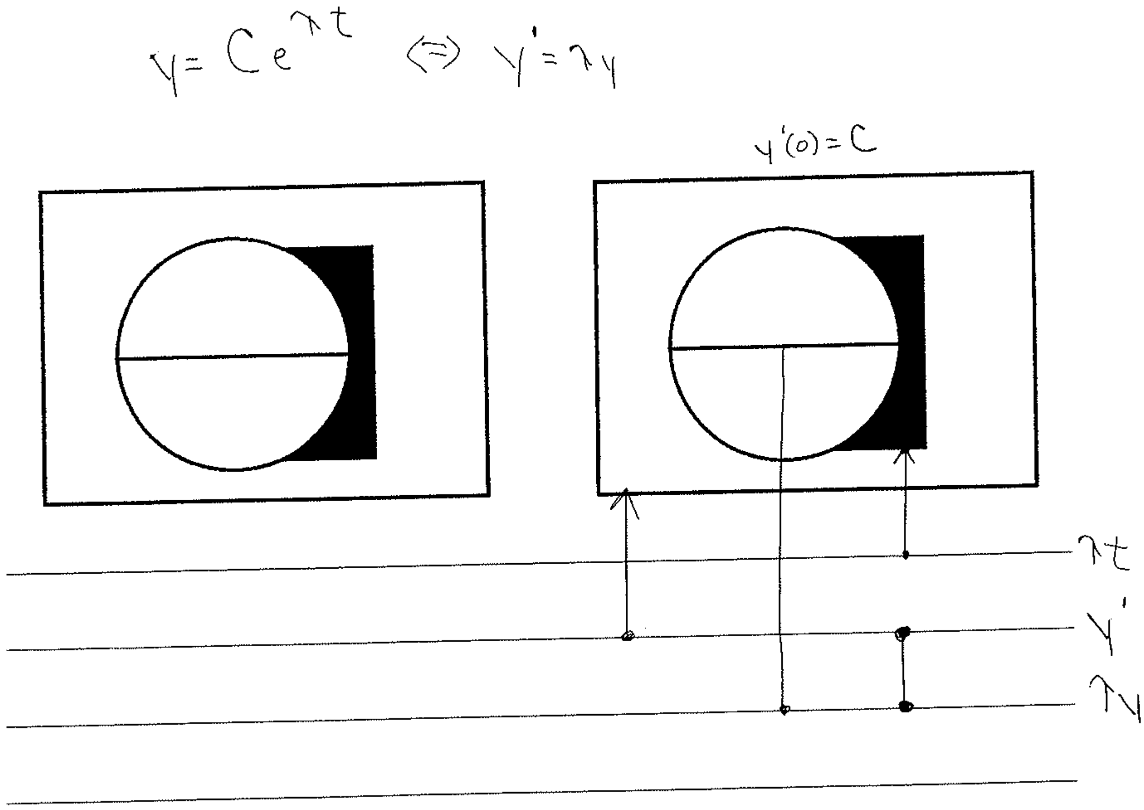

Figure A.2. Schematic diagram: Generating an exponential function.

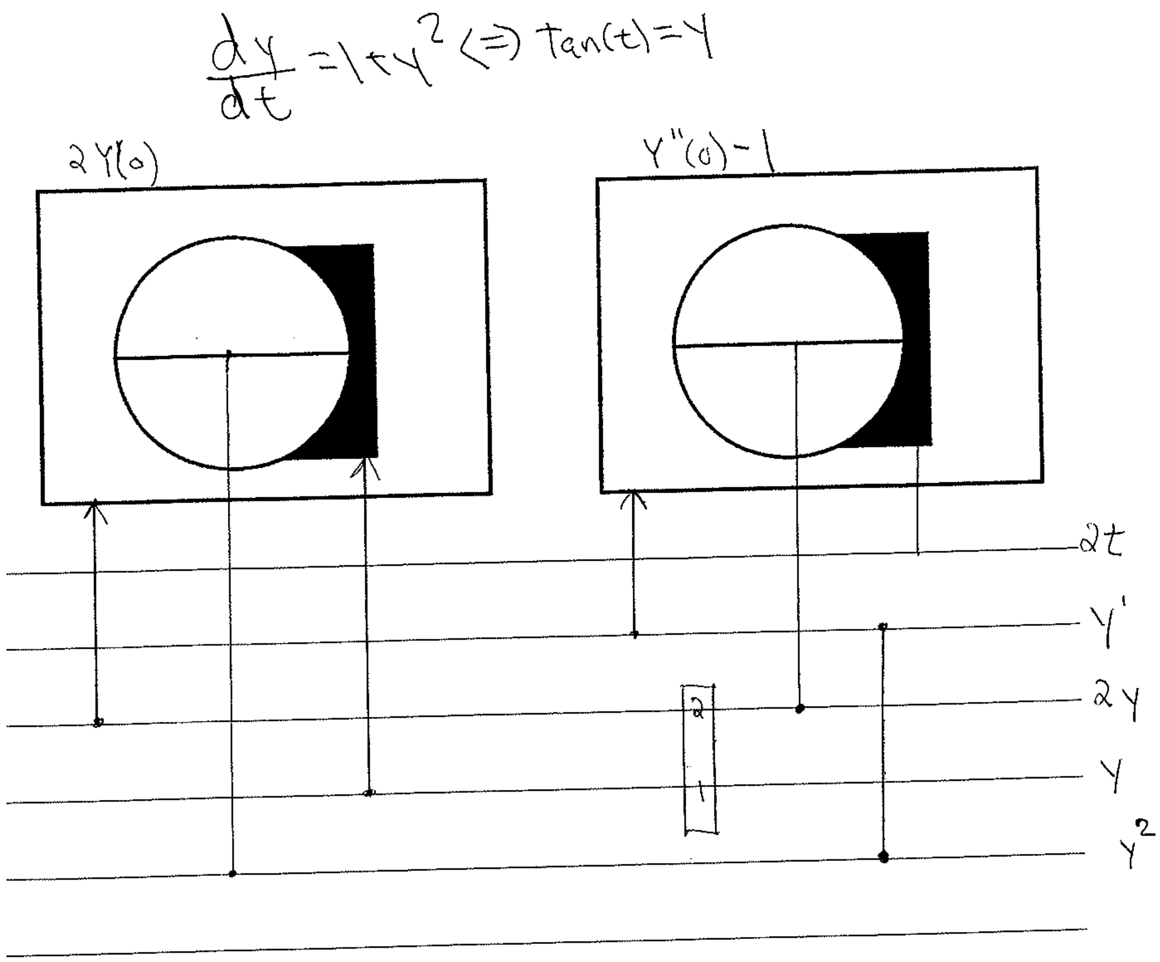

Figure A.3. Schematic diagram: Generating a tangent function.

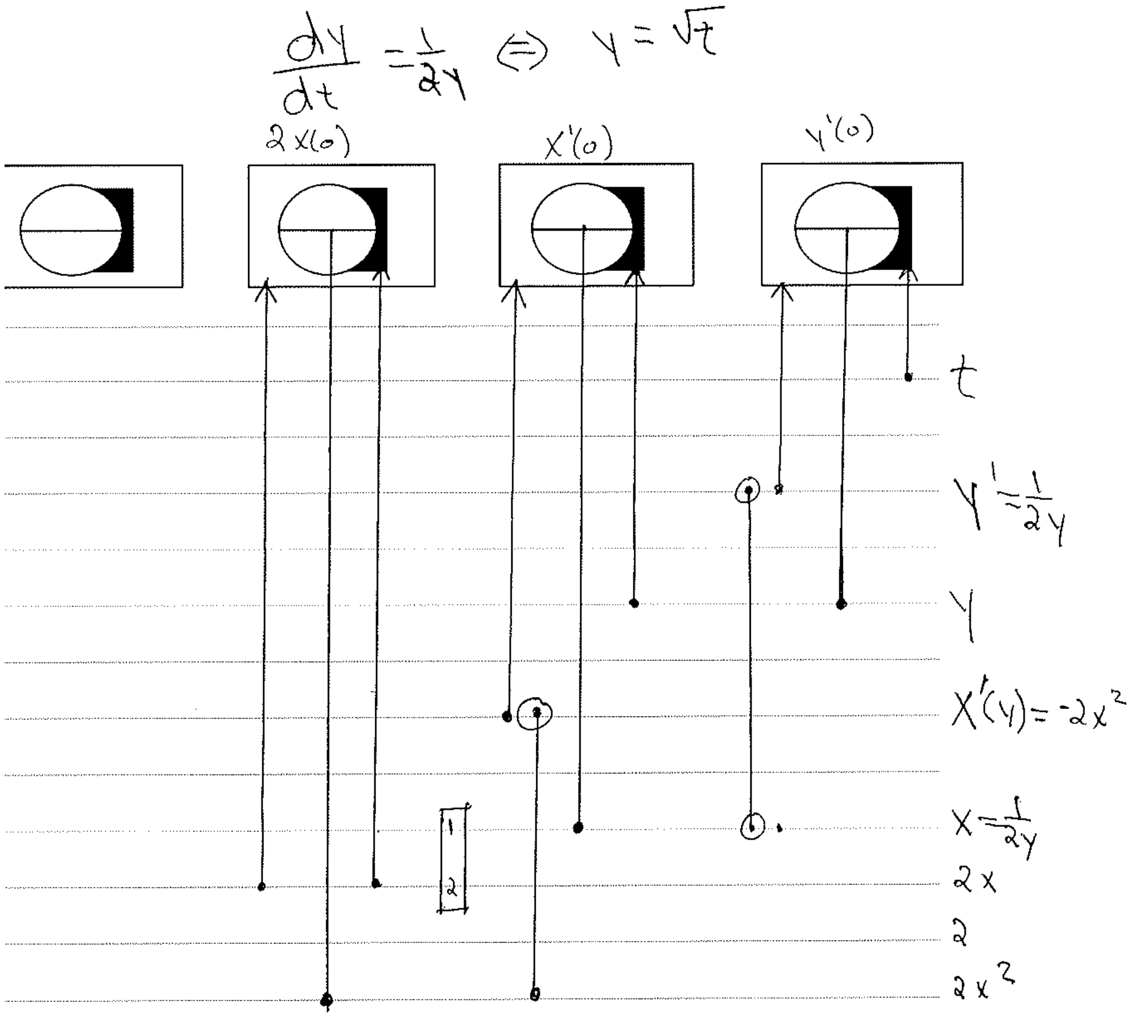

Figure A.4. Schematic diagram: Generating a square root function.

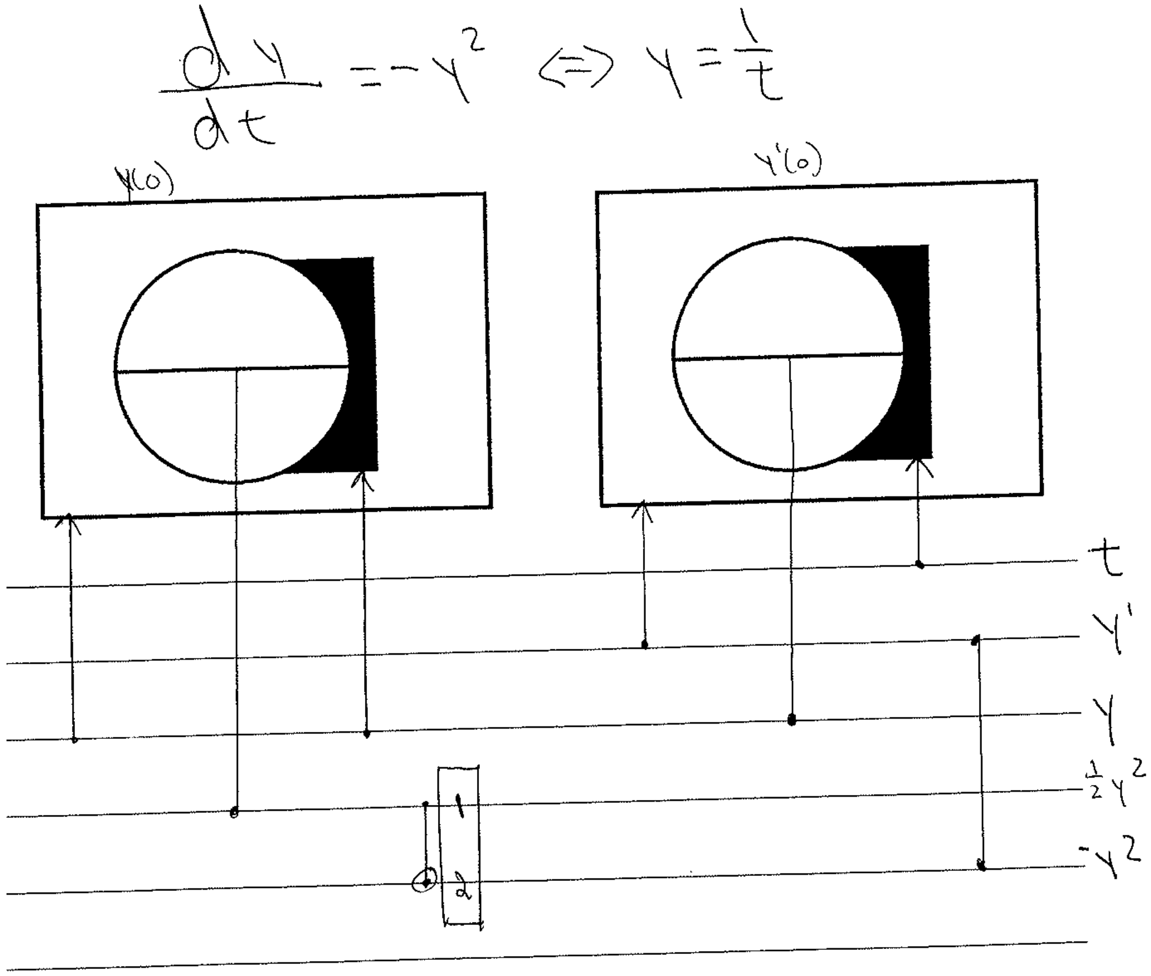

Figure A.5. Schematic diagram: Generating a rational function.

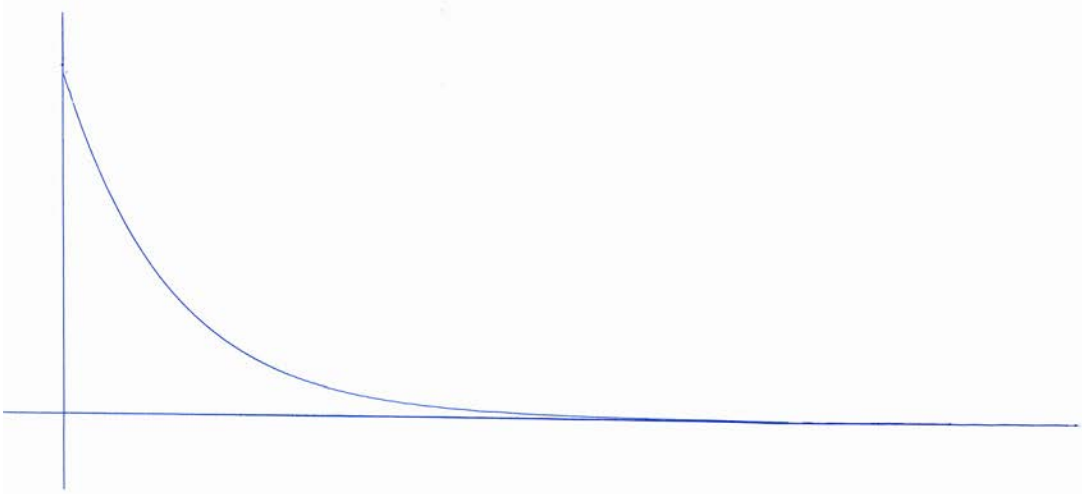

Figure A.6. Solution plot, produced by "Art," of the Stiff ODE described in section 4. $50Y' = 1500Y$, with $Y'(0) = -500$, is equivalent to $y' = 30y$, with $y(0) = 1/3$

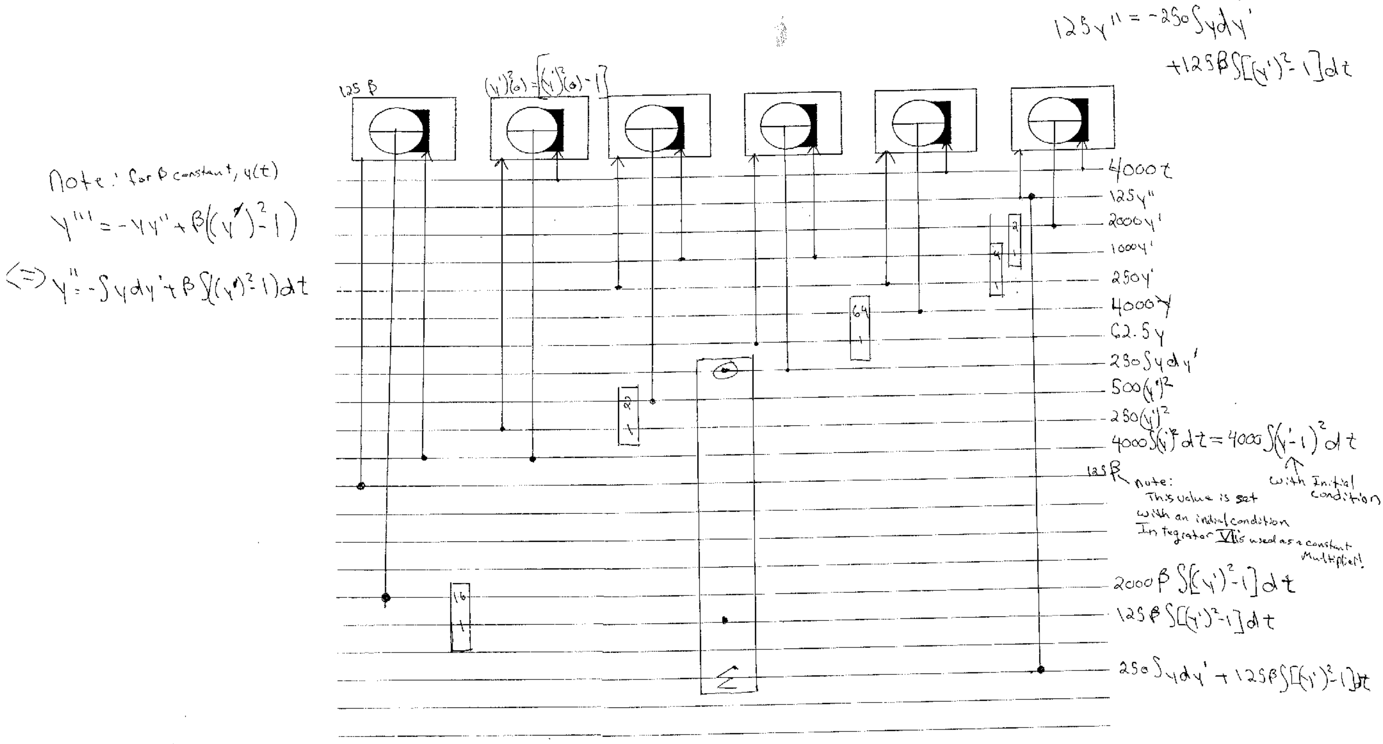

Figure A.7. Final schematic diagram for the Lamar Boundary value problem; $y'' = \int(y)dy' + \beta \int((y')^2 - 1)dt$. Note that β is set as an initial setting on Integrator 6.

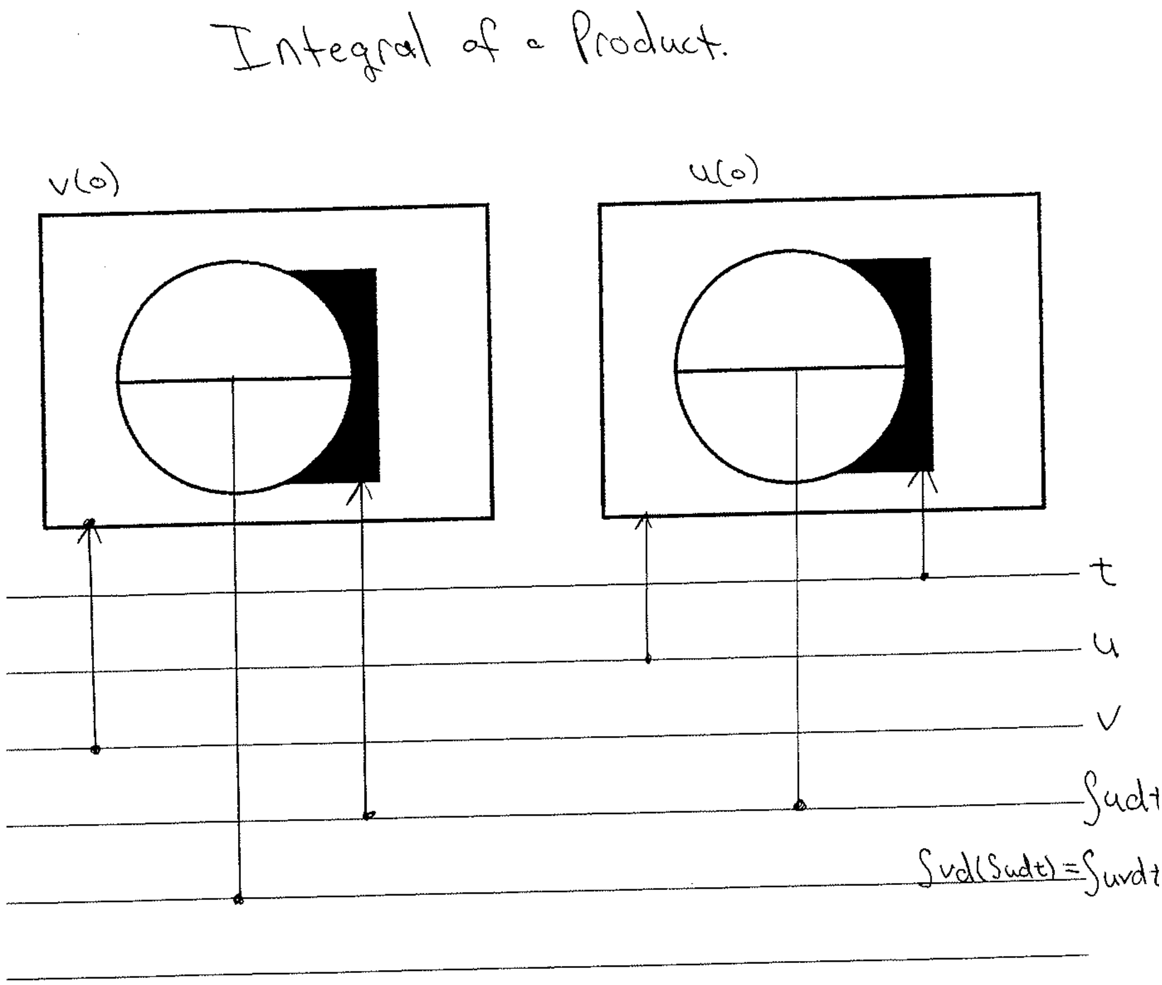

Figure A.8. Schematic diagram: Generating the integral of a product.

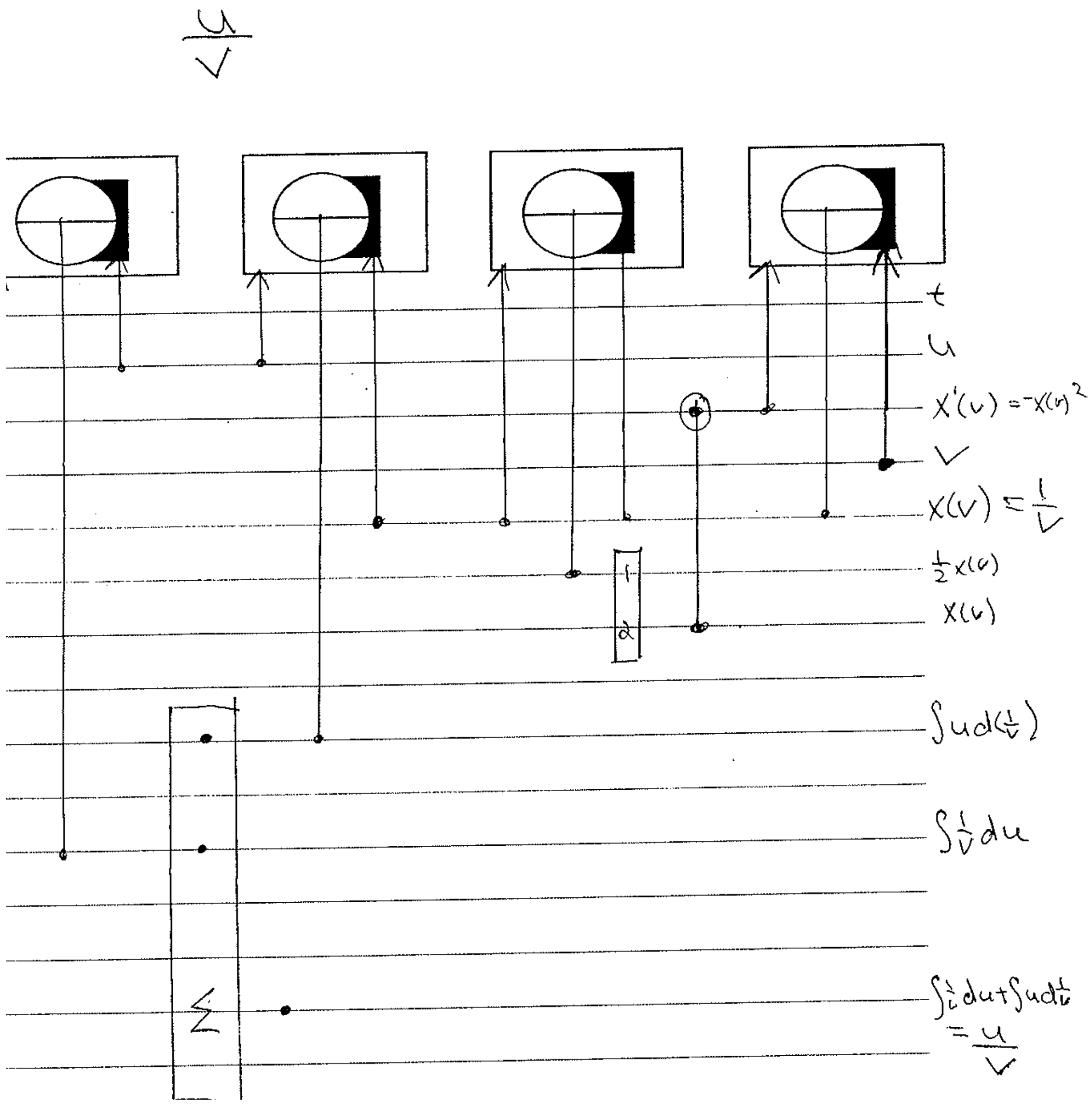

Figure A.9. Schematic diagram: Generating the product of a rational function and another function. (Quotient.)

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