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Solutions to Dynamic Equations on Varying Times Scales

Sher B. Chhetri
chhetri@marshall.edu

SOLUTIONS TO DYNAMIC EQUATIONS ON VARYING TIME SCALES

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by
Sher B. Chhetri

Approved by

Dr. Bonita A. Lawrence, Ph.D., Committee Chairperson
Dr. Basant Karna, Ph.D.
Dr. Alfred Akinsete, Ph.D.

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ABSTRACT

Time Scales calculus, first introduced by Stefan Hilger in 1988, is the unification of the theory of difference equations with that of differential equations unifying differential and integral calculus with the calculus of finite differences. Using the properties of delta derivative, delta anti-derivative and the concept of Hilger's Complex Plane, we discuss the analytical and graphical behavior of particularly chosen first order dynamic equation, $y^{\Delta}(t) = \frac{1}{2}y(t)$ with initial condition $y^{\Delta}(0) = \frac{1}{2}$ and the second order linear homogeneous equation $y^{\Delta\Delta} = -y$ with the initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta}(0) = 1$ on different time scales. We create a sequence of time scales that tends to a chosen time scale $\mathbb{T}$. Each time scale in the sequence is the union of two disconnected closed intervals. In this thesis, our claim is that if we decrease the gap of two closed intervals of the time scales, the solution converges towards solution on the original time scale $\mathbb{T}$.

1. INTRODUCTION

The theory of time scales calculus was first initiated by Stefan Hilger in his PhD dissertation in 1988 in order to unify continuous and discrete analysis. It is a unification of the theory of difference equations with that of differential equations unifying differential and integral calculus with the calculus of finite differences and offering a formal study for hybrid discrete-continuous dynamical systems. If we differentiate a function defined on the real numbers, then the definition of derivative is equal to that of standard differentiation. But if the function is defined on the integers then it is equivalent to the forward difference operator. On the basis of Stefan Hilger's work, Martin Bohner and Allan Peterson published a book Dynamic Equations on Time Scales- An Introduction with Applications, which has made an important contribution in the field of time scales. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, whereas other results seem to be completely different in nature from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice, once for differential equations and once for difference equations. The general idea is to prove a result for a dynamic equation in which the domain of the unknown function is a so-called time scale, an arbitrary nonempty closed subset of the real numbers.

In this thesis, we will discuss the basic terms, related theorems on differentiation (delta-derivative) and integration (delta-antiderivative) on time scales. We also define the exponential function, as introduced by Stefan Hilger, and give several important properties of the delta derivative and the delta antiderivative. The most exciting part of this work will show the solutions of some first and second order linear differential equations on varying time scales solved in the Differential Analyzer Lab at Marshall University, led by Dr. Bonita A. Lawrence. First, we will exhibit the behavior of the exponential function as a solution of a first order dynamic equation and we will present interesting outcomes that we obtained using the Differential Analyzer machine. The main goal here is to observe the results of some first and second order linear dynamic equations on varying time scales.

Most importantly, first we will run the particular first order linear dynamic equation

$$y^{\Delta} = \frac{1}{2}y, \quad y(0) = 1,$$

on the time scale $\mathbb{T} = [0, 6]$. Then we will create a sequence of time scales, $\mathbb{T}_i$, that converges to $\mathbb{T} = [0, 6]$ and analyze the solutions of our dynamic equation on these time scales graphically using the differential analyzer. Each time scale in the sequence will be a union of two closed intervals. The important issue we will discuss is the behavior of the solutions after the jump from one closed interval to the other with solutions obtained from a single initial condition. By gradually decreasing the gap between the two disconnected pieces, the solutions converge point-wise to the solution of our first order dynamic equation on $\mathbb{T} = [0, 6]$. Similarly, we are considering the second order dynamic equation

$$y^{\Delta\Delta} = -y,$$

with the initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta}(0) = 1$. We will give basic definitions, theorems, and a short analytical discussion of this particular second order DE, and we will present the graphical solution on the given time scales obtained in the Differential Analyzer Machine of Marshall University.

2. TIME SCALES CALCULUS

In this section, we give an introduction of the basic terms and some of the interesting properties that we should know before reading the new results obtained in the remaining sections. Let us look at some essential terms and their descriptions.

2.1. BASIC DEFINITIONS

A *time scale*, denoted by $\mathbb{T}$, is an arbitrary closed subset of real numbers, $\mathbb{R}$, where $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. The set of reals, $\mathbb{R}$, the set of integers, $\mathbb{Z}$, the natural numbers, $\mathbb{N}$, and the nonnegative integers $\mathbb{N}_0$ are examples of time scales. The rational numbers $\mathbb{Q}$, complex numbers $\mathbb{C}$ and the open interval $(2, 3)$ are not time scales.

Definition 2.1. (i) For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} :\to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

(ii) For $t \in \mathbb{T}$, we define the *backward jump operator* $\rho : \mathbb{T} :\to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.\blacktriangle$$

An important property should be noted about σ and ρ: We define $\inf \emptyset \equiv \sup \mathbb{T}$, so that $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$. Similarly, we define $\sup \emptyset \equiv \inf \mathbb{T}$, so that $\rho(t) = t$ if t is the minimum of $\mathbb{T}$.

For $t \in \mathbb{T}$, and $\sigma(t) > t$, we say that t is *right-scattered*, and for the case when $\rho(t) < t$, we say that t is *left-scattered*.

Points in a time scale are said to be *isolated* if they are right scattered and left scattered at the same time. A point t in $\mathbb{T}$ is called *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$. In a similar way, the point t in the time scale $\mathbb{T}$ is said to be *left dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$. Points are said to be *dense* if they are right-dense and left-dense at the same time. We also should be curious about the distance from a point t to $\sigma(t)$ which we name as *graininess function*.

Definition 2.2. Define the function $\mu := \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$ which is called the *right-graininess function*. Our assumption throughout this section will be that $\mathbb{T}$ is unbounded above and the graininess function, μ is bounded.▲

Now let us define a set $\mathbb{T}^k$, which plays a very important role in later sections that we obtained from the time scale $\mathbb{T}$:

Definition 2.3. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$. Otherwise, $\mathbb{T}^k = \mathbb{T}$ i.e.;

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}) & \text{if } \sup \mathbb{T} < \infty; \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \blacktriangle \end{cases}$$

We use the symbol f^σ and $fo\sigma$ equivalently in the coming sections. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)), \text{ for all } t \in \mathbb{T},$$

i.e. $f^\sigma = f \circ \sigma$.

Let us look at some examples and determine $\sigma(t), \rho(t)$, which are isolated points and the graininess, $\mu(t)$, for various t values in the given time scale, $\mathbb{T}$.

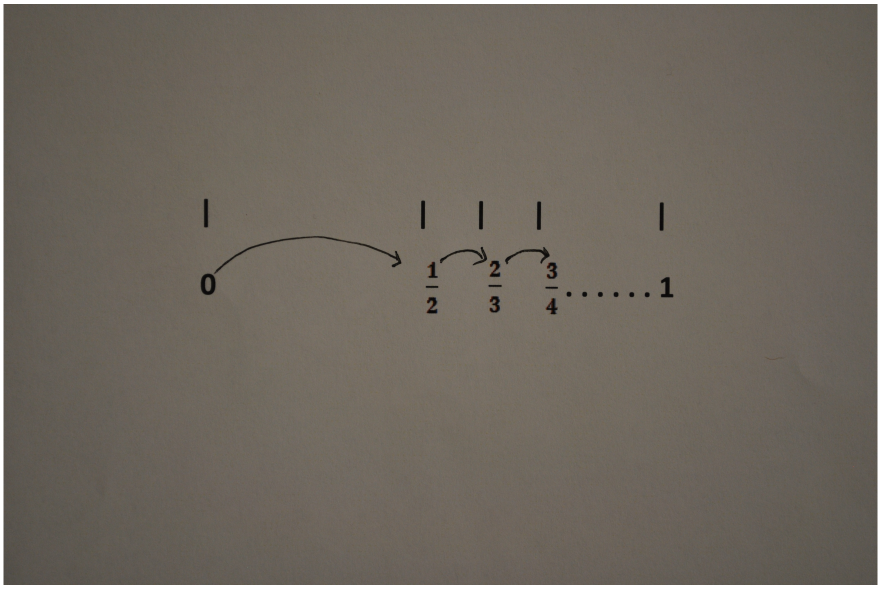

Figure 2.1.

Example 2.4. Let $\mathbb{T} = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$. Then:

(i) For $t = 1 - \frac{1}{n}$, $n = 2, 3, 4, ...$,

$\sigma(t) = \sigma(1 - \frac{1}{n}) = \inf\{s \in \mathbb{T} : s > 1 - \frac{1}{n};\text{ for } n \in \mathbb{N}\} = \frac{1}{2-t}$,

$\rho(t) = \sup\{s \in \mathbb{T} : s < 1 - \frac{1}{n};\text{ for } n \in \mathbb{N}\} = \frac{2t-1}{t}$ and

$\mu(t) = \sigma(t) - t = \frac{(t-1)^2}{2-t}$.

We note that these t values are left-scattered and right-scattered, hence isolated.

(ii) For $t = 0$,

$\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = \frac{1}{2}$,

$\rho(0) = \sup\{s \in \mathbb{T} : s < 0\} = 0$, (since $\sup \emptyset \equiv \inf \mathbb{T}$) and

$\mu(0) = \sigma(0) - 0 = \frac{1}{2}$.

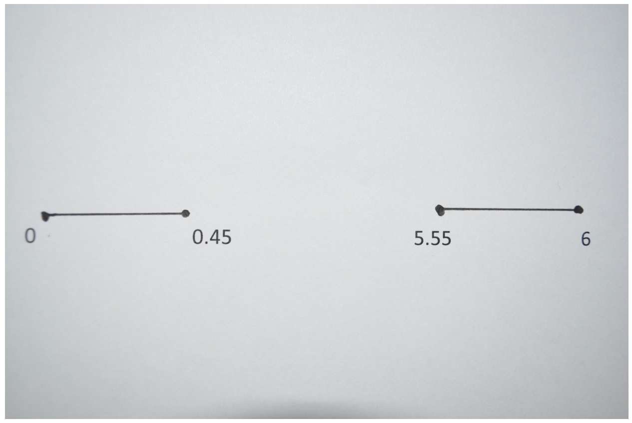

Figure 2.2.

In this case, t is left-dense and right-scattered.

(iii) For $t = 1$,

$\sigma(1) = \inf\{s \in \mathbb{T} : s > 1\} = 1$, (since $\inf \emptyset \equiv \sup \mathbb{T}$),

$\rho(1) = \sup\{s \in \mathbb{T} : s < 1\} = 1$ and

$\mu(1) = \sigma(1) - 1 = 0$.

In this case, t is dense.

Now, we want to talk about an example of a time scale consisting of the union of two closed intervals of $\mathbb{R}$ and compute $\sigma(t), \rho(t)$ and $\mu(t)$ for various $t \in \mathbb{T}$. We use this example in a later sections in our analytical study of the behavior of solutions of a first order dynamic equation.

Example 2.5. Let $\mathbb{T} = [0, 0.45] \cup [5.55, 6]$.

Using the definitions $\inf \emptyset \equiv \sup \mathbb{T}$ and $\sup \emptyset \equiv \inf \mathbb{T}$, we have from Figure 2.2:

(i) For $t \in (0, 0.45) \cup (5.55, 6)$,

$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = t,$

$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = t$ and

$\mu(t) = \sigma(t) - t = t - t = 0.$

For these particular t values, t is dense.

(ii) For $t = 0$,

$\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = 0$

$\rho(0) = \sup\{s \in \mathbb{T} : s < 0\} = 0$ and

$\mu(0) = \sigma(0) - 0 = 0.$

Therefore, $t = 0$ is also a dense point.

(iii) For $t = 0.45$

$\sigma(0.45) = \inf\{s \in \mathbb{T} : s > 0.45\} = 5.55,$

$\rho(0.45) = \sup\{s \in \mathbb{T} : s < 0.45\} = 0.45$ and

$\mu(0.45) = \sigma(0.45) - 0.45 = 5.55 - 0.45 = 5.10.$

In this case, t is right-scattered and left dense.

(iv) For $t = 5.55$,

$\sigma(5.55) = \inf\{s \in \mathbb{T} : s > 5.55\} = 5.55,$

$\rho(5.55) = \sup\{s \in \mathbb{T} : s < 5.55\} = 0.45$ and

$\mu(5.55) = \sigma(t) - t = 5.55 - 5.55 = 0.$

Hence, t is left-scattered and right dense.

(v) For $t = 6$,

$\sigma(6) = \inf\{s \in \mathbb{T} : s > 6\} = 6,$

$\rho(6) = \sup\{s \in \mathbb{T} : s < 6\} = 6$ and

$\mu(6) = 6 - 6 = 0.$

Thus, point t is dense in this case.

Using these basic terms, in the next section we discuss some details about how differentiation works in time scale calculus.

2.2. DIFFERENTIATION

In this section, we give an introduction of differentiation in time scale calculus, including some examples and useful theorems. This will give us an idea of how to do a comparison of the derivative from the usual calculus.

We begin with the formal definition of the derivative from time scale calculus. We call this the *delta derivative*.

Definition 2.6. (Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. We say $f^\Delta(t)$, the delta (or Hilger) derivative of f at t, provided $f^\Delta(t)$ exists , is the number with the property: Given $\epsilon \geq 0$, there exists a neighborhood U of t, that is, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s|,$$

for all $s \in U$.▲

Throughout this work we use the symbol $f^\Delta(t)$ to denote the derivative of f at t and when we refer to the derivative we mean the delta derivative.

As in our usual calculus, the following theorems are very essential theorems on time scales. These theorems, involving differentiation, continuity and limits play an important role in the connection with the properties of integration.

Theorem 2.7. *(Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number.

In this case, we have $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

Example 2.8. If $\mathbb{T} = \mathbb{R}$, $f^{\Delta}(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$, $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$.

Example 2.9. Consider our time scale from Example 2.5, $\mathbb{T} = [0, 0.45] \cup [5.55, 6]$. We want to compute the derivative at points 0.30, 0.45, 5.55 and 5.80.

Using the definition of derivative, for any continuous function f, we have:

(i) Because the point $t = 0.30$ is dense, then using part (*iii*) of Theorem (2.7),

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

So that

$$f^{\Delta}(0.30) = \lim_{s \to 0.30} \frac{f(0.30) - f(s)}{0.30 - s}.$$

(ii) The point $t = 0.45$ is right-dense, we then have

$$\begin{aligned} f^{\Delta}(0.45) &= \frac{f(\sigma(0.45)) - f(0.45)}{\mu(0.45)} \\ &= \frac{f(5.55) - f(0.45)}{5.55 - 0.45} \\ &= \frac{f(5.55) - f(0.45)}{5.10}. \end{aligned}$$

(iii) This is similar to that of part (i), $t = 5.55$ is right dense, so,

$$f^{\Delta}(5.55) = \lim_{s \to 5.55} \frac{f(5.55) - f(s)}{5.55 - s}.$$

(iv) We know that $t = 5.80$ is dense, so that

$$f^{\Delta}(5.80) = \lim_{s \to 5.80} \frac{f(5.80) - f(s)}{5.80 - s}.$$

Note that the part (iv) of the Theorem 2.7 is true for any time scale $\mathbb{T}$. This is an important formula that we frequently use in the later discussions of our work.

The following theorem establishes formulas for the sum, product and quotient of two differentiable functions f and g, the derivative of the product of a function with a constant and the differentiation formula for the inverse of a function.

Theorem 2.10. *(Bohner and Peterson [1]) Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$(\frac{1}{f})^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$(\frac{f}{g})^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Remark 2.11. Let us note some of the important properties of delta differentiation:

(i) If x, y, z and r are delta differentiable at t, then we have

$(xyzr)^{\Delta} = x^{\Delta}yzr + x^{\sigma}y^{\Delta}zr + x^{\sigma}y^{\sigma}z^{\Delta}r + x^{\sigma}y^{\sigma}z^{\sigma}r^{\Delta}$ holds at t.

(ii) The delta derivative of (f^3) is calculated as follows:

$$\begin{aligned}(f^3)^{\Delta} = (f.f^2)^{\Delta} &= f^{\Delta}(f^2) + f^{\sigma}(f^2)^{\Delta} \\ &= f^{\Delta}(f^2) + f^{\sigma}(f^{\Delta}f + f^{\sigma}f^{\Delta}) \\ &= f^{\Delta}(f^2) + f^{\sigma}f^{\Delta}f + f^{\sigma}f^{\sigma}f^{\Delta}.\end{aligned}$$

(iii) If the functions f and g are twice differentiable, the product fg is not always twice differentiable.

(iv) If f and g are twice differentiable and if f^{σ} is differentiable, then we have

$(fg)^{\Delta\Delta} = f^{\Delta\Delta}g + (f^{\Delta\sigma} + f^{\sigma\Delta})g^{\Delta} + (f^{\sigma\sigma})g^{\Delta\Delta}$.

(v) If both $f^{\Delta^{\sigma}}$ and $f^{\sigma\Delta}$ exists, $f^{\Delta^{\sigma}}$ is always not equal to $f^{\sigma\Delta}$.

Now, we want to talk about the chain rule briefly to show that how it works in time scale calculus and also state some chain rules under different conditions as given in Bohner and Peterson [1].

2.3. CHAIN RULES

If $f, g : \mathbb{R} \to \mathbb{R}$, then the chain rule from the usual calculus is given by the relation $(fog)^{'}(t) = f^{'}(g(t))g'(t)$ where g is differentiable at t and if f is differentiable at $g(t)$.

The chain rule that we know from the usual calculus does not hold in general for the time scale. i.e; in general, $(fog)^{\Delta}(t) \neq f^{\Delta}(g(t))g^{\Delta}(t)$.

To prove this, consider an example in whcih the time scale $\mathbb{T} = \mathbb{Z}$ such that $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(t) = t^3$ and $g(t) = t^2$, for all $t \in \mathbb{T}$.

Then, $(fog)(t) = f(t^2) = t^6$ and by definition of delta derivative,

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= \frac{(fog)(\sigma(t)) - (fog)(t)}{\sigma(t) - t} \\
&= \frac{(\sigma(t))^6 - t^6}{1} \\
&= (t+1)^6 - t^6 \\
&= 6t^5 + 15t^4 + 20t^3 + 15t^2 + 6t + 1.
\end{aligned}$$

Also,

$$\begin{aligned}
f^{\Delta}(g(t)) &= \frac{f(\sigma(g(t))) - f(g(t))}{\sigma(t) - t} \\
&= \frac{f(t^2+1)^3 - t^6}{t+1-t} \\
&= t^6 + 3t^4 + 3t^2 + 1 - t^6 \\
&= 3t^4 + 3t^2 + 1.
\end{aligned}$$

And,

$$\begin{aligned}
g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\
&= (t+1)^2 - t^2 \\
&= 2t + 1.
\end{aligned}$$

Now,

$$\begin{aligned}
f^{\Delta}(g(t)).g^{\Delta}(t) &= (3t^4 + 3t^2 + 1)(2t+1) \\
&= 6t^5 + 3t^4 + 6t^3 + 3t^2 + 2t + 1 \\
&\neq (6t^5 + 15t^4 + 20t^3 + 15t^2 + 6t + 1) \\
&= (fog)^{\Delta}(t).
\end{aligned}$$

Hence we have proved that $(f \circ g)^{\Delta}(t) \neq f^{\Delta}(g(t))g^{\Delta}(t)$ for all $t \in \mathbb{T}$.

Next, we want to mention two theorems that offer two different forms for the chain rule. They are important for differentiation on time scales.

The following chain rule is useful on time scales if f and g are continuous functions and defined from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 2.12. *(Bohner and Peterson [1]) Assume* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is* delta differentiable *on* $\mathbb{T}^k$*, and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then there exists* c *in the real interval* $[t, \sigma(t)]$ *with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t)$$

for all $t \in \mathbb{T}$*.*

The following chain rule is useful to compute $(f \circ g)^{\Delta}$ if $g : \mathbb{T} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. This chain rule is first derived by Christian Postzshe in 1998 (which is also mentioned in Stefan Keller's PhD thesis).

Theorem 2.13. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $g : \mathbb{T} \to \mathbb{R}$ *is* delta differentiable*. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t)$$

holds.

In the next section, we will present the formal definition of delta antiderivative as well as its important properties.

2.4. INTEGRATION

To describe classes of functions that are integrable, we need the concept of regulated functions and right-dense continuous functions. Let us define them and state some of the important theorems and properties on delta antiderivative.

Definition 2.14. (Bohner and Peterson [1]) A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *regulated* provided its right-sided limits exists (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left-dense points in $\mathbb{T}$.▲

Definition 2.15. (Bohner and Peterson [1]) A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (or *rd-continuous*) provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.▲

The following important theorem gives us the results concerning continuous functions and right-dense continuity.

Theorem 2.16. *(Bohner and Peterson [1]) Assume* $f : \mathbb{T} \to \mathbb{R}$*, then:*

(i) If f *is continuous, then* f *is rd-continuous.*

(ii) If f *is rd-continuous, then* f *is regulated.*

(iii) The jump operator σ *is rd-continuous.*

(iv) If f *is rd-continuous on* $\mathbb{T}$*, then* f^σ *is rd-continuous on* $\mathbb{T}$*.*

(v) Assume f *is continuous and* $g : \mathbb{T} \to \mathbb{R}$ *is rd-continuous, then the composite function* fog *is rd-continuous.*

(vi) If f *and* g *are rd-continuous on* $\mathbb{T}$*, then the sum* $f + g$ *and the product* $f.g$ *of the functions* f *and* g *are rd-continuous on* $\mathbb{T}$*.*

Before defining our delta antiderivative, we need the concept of pre-differentiable function and the existence of pre-antiderivatives.

Definition 2.17. (Bohner and Peterson [1]) A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *pre-differentiable* in the region D, if $\mathbb{T}^k \backslash D$, $D \subset \mathbb{T}^k$ is countable and has no right-scattered points of $\mathbb{T}$, and f is differentiable for all $t \in D$.▲

The following theorem shows the existence of pre-antiderivative.

Theorem 2.18. *(Bohner and Peterson [1]) Let* f *be a regulated function. Then there exists a function* F *which is pre-differentiable with region of differentiation* D *such that* $F^\Delta(t) = f(t)$ *holds for all* t *in* D*.*

With the existence of pre-differentiable, $F(t)$, we can define the indefinite integral of a regulated function f.

Definition 2.19. (Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 2.18 is said to be a *pre-antiderivative* of f and the indefinite integral of a regulated function is given by the expression

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is pre-antiderivative of f.▲

Now we are ready for the formal definition of the delta antiderivative of a function F from the time scale $\mathbb{T}$ to $\mathbb{R}$.

Definition 2.20. (Bohner and Peterson [1]) A function $F : \mathbb{T} \to \mathbb{R}$ is called the *delta antiderivative* of the function $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}$. Here we define the integral of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

for all $t \in \mathbb{T}$.▲

The next theorem gives us some familiar results for integration on time scales for the right-dense continuous functions f and g.

Theorem 2.21. *(Bohner and Peterson [1]) If a, b and $c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, f and g are right-dense continuous functions, then the following results hold:*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int\limits_a^b f(t)\Delta t = \int\limits_a^c f(t)\Delta t + \int\limits_c^b f(t)\Delta t, a < c < b;$

(v) $\int\limits_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int\limits_a^b f^{\Delta}(t)g(t)\Delta t;$

(vi) $\int\limits_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int\limits_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$

(vii) $\int\limits_a^a f(t)\Delta t = 0;$

(viii) *If* $|f(t)| \leq g(t)$ *on* $[a, b)$, *then*
$|\int\limits_a^b f(t)\Delta t| \leq \int\limits_a^b g(t)\Delta t;$

(ix) *If* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int\limits_a^b f(t)\Delta t \geq 0.$

The next theorem gives an important expression for the integration of a rd-continuous function defined in an interval $[t, \sigma(t)]$.

Theorem 2.22. *(Bohner and Peterson [1]) If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t),$$

where μ *is the graininess function and* σ *is the right jump operator.*

The following theorem is a consequence of the previous results of integration on time scales.

Theorem 2.23. *(Bohner and Peterson [1]) Assume* a, $b \in \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$ *is a rd-continuous function. Then the integral has the following properties:*

(i) *If* $\mathbb{T} = \mathbb{R}$, *then* $\int\limits_a^b f(t)\Delta t = \int\limits_a^b f(t)dt,$
where the integral on the right is the usual Reimann integral from calculus.

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b; \\ 0 & \text{if } a = b; \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b. \end{cases}$$

where $\mu(t)$ is the graininess function.

(iii) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if } a < b; \\ 0 & \text{if } a = b; \\ -\sum_{k=\frac{a}{h}}^{\frac{a}{h}-1} f(kh)h & \text{if } a > b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b; \\ 0 & \text{if } a = b; \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases}$$

In the next section, we will give the definition of and the notation for a first and second order linear dynamic equation and discuss the Hilger complex plane. We will present some of the definitions and properties that we need before defining our general exponential function on time scales, the solution of the first order linear homogeneous dynamic equations.

3. SOLUTIONS OF FIRST AND SECOND ORDER DYNAMIC EQUATIONS

Now we give the definitions and notation for first and second order linear dynamic equations on a time scale and then talk about the Hilger complex plane.

For the first order problem, consider a function $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation

$$y^{\Delta} = f(t, y, y^{\sigma}) \tag{3.1}$$

is called a first order dynamic equation.

Equation (3.1) is said to be a linear equation if we express $f(t, y, y^{\sigma})$ in the form

$$f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t) \text{ or } f(t, y, y^{\sigma}) = f_1(t)y + f_2(t).$$

A function $u : \mathbb{T} \to \mathbb{R}$ is called a solution of equation (3.1) if $y = u(t)$ satisfies the dynamic equation (3.1), that is, $u^{\Delta}(t) = f(t, u(t), u^{\sigma}(t))$ is satisfied for all $t \in \mathbb{T}^k$.

Assume that f, p and q are right-dense continuous functions. Then the second order linear dynamic equation takes the form:

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t).$$

If $t_0 \in \mathbb{T}^k$, then the second order initial value problem is given by

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t). \tag{3.2}$$

with the initial conditions $y(t_0) = y_0,$ and $y^{\Delta}(t_0) = y_0^{\Delta}$, where y_0 and y_0^{Δ} are given constants.

We want to obtain y, the solution of equation (3.2) in a time scale $\mathbb{T}$. For this, let us begin with an operator L_2.

Definition 3.1. (Bohner and Peterson [1]) For all $t \in \mathbb{T}^{k^2}$, $L_2 : \mathbb{C}^2_{rd} \to \mathbb{C}_{rd}$ is given by the expression

$$L_2 y(t) = y^{\Delta\Delta}(t) + p(t)y^{\Delta} + q(t)y(t).$$

We note that $\mathbb{C}_{rd}$ is the set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$. The set $\mathbb{C}^1_{rd}$ contains rd-continuous functions that are once differentiable and similarly the set $\mathbb{C}^2_{rd}$ contains functions that are twice differentiable. ▲

Now the DE (3.2) can be written in the the alternate form of $L_2 y = f$ on the time scale $\mathbb{T}$. Note that $y(t)$ is said to be a solution of $L_2 y = f$ on $\mathbb{T}$ if y is in $\mathbb{C}^2_{rd}$ and $L_2 y(t) = f(t)$ for every $t \in \mathbb{T}^{k^2}$. If $f(t) = 0$, equation (3.2) yields a homogeneous dynamic equation $L_2 y = 0$, otherwise it is said to be a nonhomogeneous equation.

The solutions of a particular family of dynamic equations $y^{\Delta} = p(t)y$ are defined in terms of the exponential function. For example: when $\mathbb{T} = \mathbb{R}$, we have $y^{\Delta} = y'$ and our solution is defined in terms of the function $f(t) = e^t$. Similarly, the solutions of the family of a particular second order linear dynamic equation (3.2) has solutions defined in terms of the sine, cosine and exponential functions. To define the generalized exponential function associated with a time scale, Hilger introduced the complex plane, so called "Hilger's Complex Plane".

3.1. HILGER'S COMPLEX PLANE

From Bohner and Peterson [1], in the Hilger complex plane, the Hilger imaginary circle is tangent to the imaginary axis and the diameter of the circle is the reciprocal of the graininess, h. For all positive h, *the Hilger complex numbers*, *the Hilger real axis*, *the Hilger alternating axis* and *the Hilger imaginary circle* are defined, respectively, as follows:

$$\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -\frac{1}{h}\};$$

$$\mathbb{R}_h := \{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h}\};$$

$$\mathbb{A}_h := \{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h}\};$$

and

$$\mathbb{I}_h := \{z \in \mathbb{C}_h : |z + \frac{1}{h}| = \frac{1}{h}\}.$$

Note that for the case $h = 0$, we have $\mathbb{C}_0 \equiv \mathbb{C}, \mathbb{R}_0 \equiv \mathbb{R}, \mathbb{I}_0 \equiv i\mathbb{R}$ and $\mathbb{A}_0 \equiv \emptyset$.

Next we present the definition of the Hilger real part and imaginary part of a complex number z.

Definition 3.2. (Bohner and Peterson [1]) Let $h > 0$. Then for all $z \in \mathbb{C}_h$, we define the *Hilger real part* of z by

$$\mathbb{R}e_h(z) = \frac{|zh + 1| - 1}{h}$$

and the *Hilger imaginary part* of z by

$$\mathbb{I}m_h(z) = \frac{Arg(zh + 1)}{h},$$

where $-\pi < Arg(z) \leq \pi$.▲

To obtain the geometrical interpretation of the real part and the imaginary part of a complex number in the Hilger complex plane, let us consider Figure 3.1 and make some notes on $\mathbb{R}e_h(z)$ and $\mathbb{I}m_h(z)$ and depict them clearly.

Using the definition of the Hilger real part and the absolute value of a complex number, we have

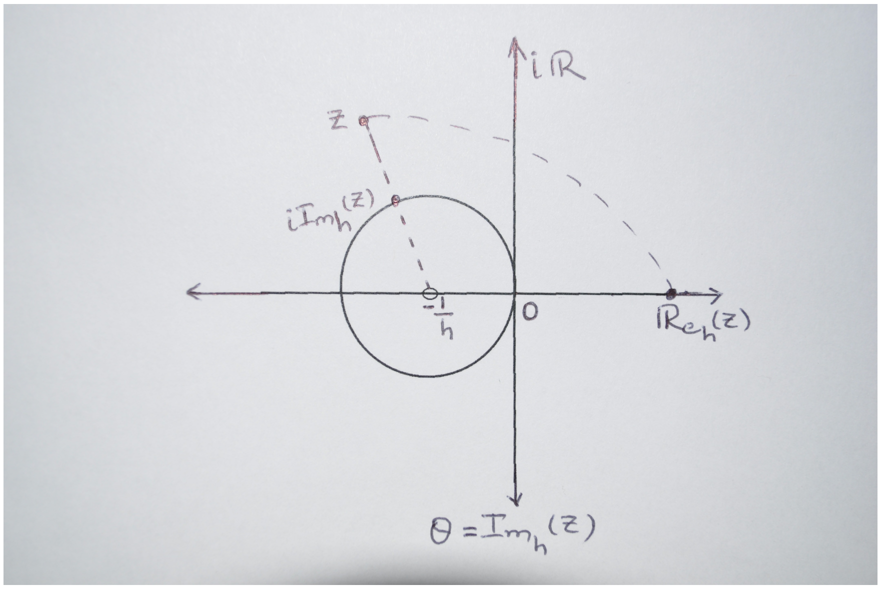

Figure 3.1. The Hilger's Complex Plane.

$$\begin{aligned}
\mathbb{R}e_h(z) &= \frac{|zh+1|-1}{h} \\
&= \frac{|xh+iyh+1|-1}{h} \\
&= \frac{\sqrt{(xh+1)^2+(yh)^2}-1}{h} \\
&= \sqrt{(x+\frac{1}{h})^2+(y)^2}-\frac{1}{h} \\
&= |z+\frac{1}{h}|-\frac{1}{h}.
\end{aligned}$$

Thus, the Hilger real part of a complex number z has value of the magnitude of $z+\frac{1}{h}$, $|z+\frac{1}{h}|$, reduced by $\frac{1}{h}$.

Similarly, from the definition of the imaginary part, we have

$$\mathbb{I}m_h(z) = \frac{Arg(zh+1)}{h}$$
$$= \frac{1}{h}Arg(z + \frac{1}{h}).$$

Therefore, the Hilger imaginary part of z is the usual principal argument of z shifted $\frac{1}{h}$ then divided by h. This is an angle measure. Hilger's real part of z is on the Hilger real axis, that is; $\mathbb{R}e_h(z) \in \mathbb{R}_h$. Thus, the Hilger real part of z lies between $-\frac{1}{h}$ and ∞. Since $-\pi \leq Arg(z) \leq \pi$, from Definition 3.1, we have $-\frac{1}{h} < \mathbb{R}e_h(z) < \infty$ and $-\frac{\pi}{h} < \mathbb{I}m_h(z) \leq \frac{\pi}{h}$.

Also important to our discussion is the Hilger purely imaginary number (see Bohner and Peterson [1])

Definition 3.3. *Hilger's purely imaginary number* is denoted by the symbol $\mathring{\iota}\omega$ and defined by the formula

$$\mathring{\iota}\omega = \frac{e^{i\omega h} - 1}{h},$$

where $-\frac{\pi}{h} < \omega \leq \frac{\pi}{h}$.▲

For a point z in the Hilger complex plane, the product of i and the Hilger imaginary part lies on the Hilger imaginary circle, that is; for each $z \in \mathbb{C}_h,\ i\mathbb{I}m_h(z) \in \mathbb{I}_h$.

We here note that

$$\mathring{\iota}\omega = \frac{e^{i\omega h} - 1}{h}$$
$$= \frac{cos\omega h + isin\omega h - 1}{h}$$
$$= \frac{cos\omega h - 1}{h} + i(\frac{sin\omega h}{h})$$
$$= (\frac{cos\omega h}{h} + i\frac{sin\omega h}{h}) - \frac{1}{h}.$$

This clarifies that the purely imaginary number $\mathring{\iota}\omega$ is on the Hilger imaginary circle, a circle of radius $\frac{1}{h}$ and positioned $\frac{1}{h}$ units to the left of the origin.

Remark 3.4. Let us state an important remark about the Hilger real and imaginary part: If h approaches zero, the Hilger real part of z tends to the real part of z and the Hilger imaginary part of z tends to imaginary part of z, that is; $\lim_{h\to 0}[\mathbb{R}e_h(z) + \mathring{\iota}\mathbb{I}m_h(z)] = \mathbb{R}e_z + i\mathbb{I}m_z$.

The following collection of definitions (Bohner and Peterson [1]) of *"circle plus"* and *"circle minus"* on the $\mathbb{C}_h$, *the cylindrical transformation* and *the inverse cylindrical transformation* will be used to define our exponential function on a time scale.

Definition 3.5. The *"circle plus"* addition $\oplus$ on $\mathbb{C}_h$ is defined by the formula

$$z \oplus w := z + w + zwh.$$

We note that $(\mathbb{C}_h, \oplus)$ forms an abelian guoup.▲

Note that for $z \in \mathbb{C}_h$,

$$\begin{aligned}
&\mathbb{R}e_h(z) \oplus \mathring{\iota}\mathbb{I}m_h(z) \\
&= \frac{|zh+1|-1}{h} \oplus \mathring{\iota}\frac{Arg(zh+1)}{h} \\
&= \frac{|zh+1|-1}{h} \oplus \mathring{\iota}\frac{exp(iArg(zh+1))-1}{h} \\
&= \frac{|zh+1|-1}{h} + \frac{exp(iArg(zh+1))-1}{h} + \frac{|zh+1|-1}{h} \times \frac{exp(iArg(zh+1))-1}{h} \times h \\
&= \frac{1}{h}|zh+1| - 1 + exp(iArg(xh+1)) - 1 + [|zh+1|-1][exp(iArt(zh+1))-1] \\
&= \frac{1}{h}|zh+1|exp(iArg(zh+1)) - 1 \\
&= \frac{(zh+1)-1}{h} \\
&= z \\
&= x + iy \\
&= \mathbb{R}e(z) + i\mathbb{I}m(z).
\end{aligned}$$

Thus, we have $\mathbb{R}e_h(z) \oplus \mathring{i}\mathbb{I}m_h(z) = z = \mathbb{R}e_(z) + i\mathbb{I}m_(z)$.

We also have the concept of the subtraction on $\mathbb{C}_h$ known as the circle minus which is defined as below:

Definition 3.6. The *"circle minus"* subtraction $\ominus$ on $\mathbb{C}_h$ is defined by the expression

$$z \ominus w := z \oplus (\ominus w),$$

where $\ominus w := \frac{-w}{1+zh}$.▲

The following definition of the cylinder transformation is very important for defining the exponential function of our interest.

Definition 3.7. For $h = 0$, we define $\xi_0(z) \equiv 0$ for all $z \in \mathbb{C}$. The *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$, for $h > 0$, is defined by the expression

$$\xi_h(z) = \frac{1}{h} \operatorname{Log}(1 + zh),$$

where Log is the principal logarithm function and $\mathbb{Z}_h$ is the strip defined by

$$\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h}\}$$

for all $h > 0$. ▲

Recall that if $z = re^{\theta i}$,

$$\begin{aligned} Logz &= lnr + i\ \theta \\ &= ln|z| + iArg\ z \end{aligned}$$

Definition 3.8. The inverse transformation of the cylinder transformation ξ_h is given by

$$\xi_h{}^{-1}(z) = \frac{1}{h}(e^{zh} - 1),$$

for $z \in \mathbb{Z}_h$.▲

The idea of cylinder transformation is that if we join (glue) the bordering line of $\mathbb{I}m(z) = -\frac{\pi}{h}$ and $\mathbb{I}m(z) = \frac{\pi}{h}$ of $\mathbb{Z}_h$, the shape will be like a cylinder. The most important remark here is that for all positive h, the cylinder transformation maps open rays coming from the point $-\frac{1}{h}$ in $\mathbb{C}$ onto horizontal lines on the cylinder ξ_h. Further, it can be seen that the circles centered at $-\frac{1}{h}$ are mapped onto the vertical lines on the strip ξ_h.

The following definition is mentioned in Bohner and Peterson [1] tells us the formula for the addition of two members on $\mathbb{Z}_h$.

Definition 3.9. We have addition on $\mathbb{Z}_h$ given by the relation

$$z + w := z + w \ (mod \frac{2\pi i}{h}) \ for \ z, w \in \mathbb{Z}_h.▲$$

Now with this foundation we are ready to define the initial value problem of a dynamic equation and discuss regressive functions before defining the generalized exponential function.

3.2. FIRST ORDER INITIAL VALUE PROBLEMS

Given $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem

$$y^{\Delta} = f(t, y, y^{\sigma}), \quad y(t_0) = y_0. \tag{3.3}$$

is called an initial value problem of the dynamic equation (3.1). A solution of (3.1) with the initial condition $y(t_0) = y_0$, is called a solution of this initial value problem (3.3).

Here, our focus is to construct solutions of the particular initial value problem

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1,$$

explicitly.

To achieve this goal, using our previous discussion about Hilger's complex plane we now offer some necessary definitions and properties. Before proving the existence and the uniqueness theorem for the initial value problem of the dynamic equation (3.3), we define a *regressive* function, p.

Definition 3.10. (Bohner and Peterson [1]) A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* if $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^k$. The collection of all *regressive and rd-continuous* functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).\blacktriangle$$

Next we present the definition of the exponential function, the solution of the first order linear homogeneous dynamic equations. For instance, if $\mathbb{T} = \mathbb{R}$, then the solution of the dynamic equation

$$y' = y, \text{ with } y(0) = 1$$

is the exponential function $y = e^t$.

Definition 3.11. (Bohner and Peterson [1]): If $p \in \mathcal{R}$, then for all $s, t \in \mathbb{T}$, the generalized exponential function is denoted by $e_p(t, s)$ and defined by the formula

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

where

$$\xi_{\mu(\tau)} = \frac{1}{\mu(\tau)} \operatorname{Log}(1 + z\mu(\tau)),$$

is the cylinder transformation with respect to $\mu(\tau)$ and Log is the principal logarithm.$\blacktriangle$

Next we establish an important lemma known as the *semigroup property* which is useful in proving nice theorems offering us solutions of the initial value problem

$$y^\Delta = p(t)y, \; y(t_0) = 1.$$

Lemma 3.12. *If $p \in \mathcal{R}$, then the semigroup property*

$$e_p(t,r)e_p(r,s) = e_p(t,s), for\ all\ r,s,t \in \mathbb{T}$$

holds.

Proof. : Let us assume that $p \in \mathcal{R}$ and r, s and $t \in \mathbb{T}$.

Using the definition of exponential function and the property of integration

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$$

we have

$$\begin{aligned} e_p(t,r)e_p(r,s) &= exp\left(\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) exp\left(\int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= exp\left(\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau + \int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= e_p(t,s). \end{aligned}$$

Hence we established that $e_p(t,r)e_p(r,s) = e_p(t,s)$, for all $r, s, t \in \mathbb{T}$. □

The following definition, as stated in Bohner and Peterson, [1], tells us about a regressive linear dynamic equation.

Definition 3.13. If $p \in \mathcal{R}$, then the first order linear dynamic equation $y^\Delta = p(t)y$ is called regressive.▲

Now we present a nice proof of the theorem which gives us $e_p(., t_0)$ as the solution of the initial value problem $y^\Delta = p(t)y$, $y(t_0) = 1$.

Theorem 3.14. *(Bohner and Peterson [1]) If the first order linear dynamic equation $y^{\Delta} = p(t)y$, $p \in \mathcal{R}$ is regressive and $t_0 \in \mathbb{T}$, then $e_p(., t_0)$ is a solution of the initial value problem*

$$y^{\Delta} = p(t)y, \; y(t_0) = 1 \quad on\ \mathbb{T}.$$

Proof. : Let $t_0 \in \mathbb{T}$ and suppose $p \in \mathcal{R}$.

If $t = t_0$, we have $e_p(t_0, t_0) = 1$. To complete the proof, we need to show that the exponential function $e_p(t, t_0)$ satisfies the dynamic equation $y^{\Delta} = p(t)y$. For this, fix $t \in \mathbb{T}^k$, then we can see two cases:

Case I:

For the case when $\sigma(t) > t$, using the definition of derivative, the Lemma 3.12 and the definition of $\xi_h^{-1}(z)$, we have

$$\begin{aligned}
e_p^{\Delta}(t, t_0) &= \frac{exp(\int\limits_{t_0}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r) - exp(\int\limits_{t_0}^{t} \xi_{\mu(r)}(p(r))\Delta r)}{\mu(t)} \\
&= \frac{exp(\int\limits_{t_0}^{t} \xi_{\mu(r)}(p(r)) exp(\int\limits_{t}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r) - exp(\int\limits_{t_0}^{t} \xi_{\mu(r)}(p(r))\Delta r)}{\mu(t)} \\
&= \frac{exp(\int\limits_{t}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r) - 1}{\mu(t)} e_p(t, t_0) \\
&= \frac{e^{\xi_{\mu(t)}(p(t))\mu(t)} - 1}{\mu(t)} e_p(t, t_0) \\
&= \xi_{\mu(t)}^{-1}(\xi_{\mu(t)}(p(t))) \times e_p(t, t_0) \\
&= p(t).e_p(t, t_0).
\end{aligned}$$

Case II: For the case when $\sigma(t) = t$, using Lemma 3.12 and the triangle inequality, we have,

$$
\begin{aligned}
&|y(t)-y(s)-p(t)y(t)(t-s)| \\
=\ & |e_p(t,t_0)-e_p(s,t_0)-p(t)e_p(t,t_0)(t-s)| \\
=\ & |e_p(t,t_0)|\times|1-e_p(s,t)-p(t)(t-s)| \\
=\ & |e_p(t,t_0)|\times|1-e_p(s,t)-p(t)(t-s)-\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau+\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau| \\
\leq\ & |e_p(t,t_0)|\times|1-\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau-e_p(s,t)|+|e_p(t,t_0)|\times|\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau-p(t)(t-s)| \\
\leq\ & |e_p(t,t_0)|\times|1-\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau-e_p(s,t)|+|e_p(t,t_0)|\times|\int_s^t [\xi_{\mu(\tau)}(p(\tau))-\xi_o(p(t))]\Delta\tau|.
\end{aligned}
$$

Let $\epsilon > 0$ be given. The function p is right-dense continuous and $\sigma(t) = t$ so that

$\lim_{r\to t} \xi_{\mu(r)}(p(r)) = \xi_0(p(t))$.

Therefore we can obtain

$$|\xi_{\mu(\tau)}(p(\tau)) - \xi_o(p(t))| < \frac{\epsilon}{3|e_p(t,t_0)|}$$

for all $\tau \in U_1$, where U_1 is a neighborhood of t.

Then, if $s \in U_1$, we have

$|e_p(t,t_0)| \times |\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_o(p(t))]\Delta\tau|$

$$
\begin{aligned}
\leq\ & |e_p(t,t_0)|\times\int_s^t |[\xi_{\mu(\tau)}(p(\tau))-\xi_o(p(t))]\Delta\tau| \\
<\ & \frac{\epsilon}{3}|t-s|.
\end{aligned}
$$

We have, by L'Hospital's rule, that,

$$\lim_{z\to 0}\frac{1-z-e^{-z}}{z}=0,$$

and hence we have

$$\left|\frac{1-\int\limits_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)}{\int\limits_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau}\right| < \epsilon^*,$$

where, $\epsilon^* = min\{1, \frac{\epsilon}{1+3|p(t)e_p(t,t_0)|}\}$ and $s \in U_2$, a neighborhood of t.

Now for $s \in U := U_1 \cap U_2$, then,

$$
\begin{aligned}
&|e_p(t,t_0)| \times |1-\int\limits_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)| \\
&< |e_p(t,t_0)| \times \epsilon^* |\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau| \\
&\le |e_p(t,t_0)| \times \epsilon^* [|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_o(p(t))]\Delta\tau| + |p(t)||t-s|] \\
&\le |e_p(t,t_0)| \times |\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_o(p(t))]\Delta\tau| + |e_p(t,t_0)| \times \epsilon^* |p(t)||t-s| \\
&\le \frac{\epsilon}{3}|t-s| + |e_p(t,t_0)| \times \epsilon^* |p(t)||t-s| \\
&\le \frac{\epsilon}{3}|t-s| + \frac{\epsilon}{3}|t-s| \\
&= \frac{2\epsilon}{3}|t-s|.
\end{aligned}
$$

Hence using previous two inequalities, we have

$$|y(t) - y(s) - p(t)y(t)(t-s)| < \quad \epsilon$$

This proves that $e_p(.,t_0)$ is a solution of the IVP $y^\Delta = p(t)y,\ y(t_0) = 1$ on $\mathbb{T}$

□

Hence we have shown the existence of a solution to the initial value problem

$$y^\Delta = p(t)y,\ y(t_0) = 1 \text{ on } \mathbb{T}.$$

Now, we want to present an important theorem which gives the uniqueness of such a solution, namely, $e_p(.,t_0)$.

Theorem 3.15. *(Bohner and Peterson [1]) If the dynamic equation $y^\Delta = p(t)y$ is regressive, then the only solution of the initial value problem $y^\Delta = p(t)y,\ y(t_0) = 1$ on $\mathbb{T}$ is given by $e_p(.,t_0)$.*

Proof. : Begin with y, a solution of the initial value problem $y^\Delta = p(t)y,\ y(t_0) = 1$. We can consider the quotient $\frac{y(t)}{e_p(.,t_0)}$, since for all $s,t \in \mathbb{T}, e_p(t,s) \neq 0$. Using the quotient rule for the delta derivative, we have,

$$\begin{aligned}
\left(\frac{y(t)}{e_p(.,t_0)}\right)^\Delta(t) &= \frac{y^\Delta(t)e_p(t,t_0) - y(t)e_p^\Delta(t,t_0)}{e_p(t,t_0)e_p(\sigma(t),t_0)} \\
&= \frac{p(t)y(t)e_p(.,t_0) - y(t)p(t)e_p(.,t_0)}{e_p(.,t_0)e_p(\sigma(t)} \\
&= 0.
\end{aligned}$$

Using the fact that f is a constant function if $f^\Delta(t) = 0$ for all t in its domain $\mathbb{T}$, we have that $\frac{y(t)}{e_p(.,t_0)}$ is constant.

Thus,

$$\begin{aligned}(\frac{y(t)}{e_p(.,t_0)}) &\equiv (\frac{y(t_0)}{e_p(t_0,t_0)}) \\ &= \frac{1}{1} \\ &= 1.\end{aligned}$$

Finally, we have $y(t) = e_p(.,t_0)$. This proves that $y(t)$ is the unique solution of the initial value problem

$$y^{\Delta} = p(t)y, \;\; y(t_0) = 1.$$

□

In the next section we will discuss second order linear equations and the behavior of the particular dynamic equation on varying time scale.

3.3. SOLVING A PARTICULAR SECOND ORDER DE ON VARYING TIME SCALES

First, we define the second order regressive dynamic equation as we have discussed for first order dynamic equation.

Definition 3.16. (Bohner and Peterson [1]) We say that the dynamic equation

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t) \tag{3.4}$$

is regressive if $1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0$, where p, q, and f are rd-continuous functions for all $t \in \mathbb{T}^k$.▲

Before discussing uniqueness and existence of solutions of the dynamic equations, we need the concept of the Wronskian.

Definition 3.17. (Bohner and Peterson [1]) If y_1 and y_2 are two differentiable functions then, we denote the Wronskian by the symbol $W = W(y_1, y_2)$ and define it by the formula

$$W(t) = det \begin{pmatrix} y_1(t) & y_2(t) \\ \\ y_1^{\Delta}(t) & y_2^{\Delta}(t) \end{pmatrix} . \blacktriangle$$

If $W(y_1(t), y_2(t)) \neq 0$, for all $t \in \mathbb{T}^k$, the solutions y_1 and y_2 form a fundamental system for the homogeneous equation $L_2 y = 0$.

The following theorem states the existence and uniqueness of solutions of dynamic equation (3.4). We omit the proof of theorems that follow because our goal is to analyze the behavior of a particular second order linear homogeneous dynamic equation and give the graphical solutions on different time scales using these theorems, definitions and properties.

Theorem 3.18. *(Bohner and Peterson [1]) Assume that the dynamic equation (3.4) is regressive. If $t_0 \in \mathbb{T}^k$, then the initial value problem*

$$L_2 y = f(t), \;\; y(t_0) = y_0, \;\; y^{\Delta}(t_0) = y_0^{\Delta},$$

where y_0 and y_0^{Δ} are given constants, has a unique solution, and this solution is defined on the whole time scale $\mathbb{T}$.

The next theorem, known as the principle of superposition, reveals that the operator L_2 is linear.

Theorem 3.19. *(Bohner and Peterson [1]) The operator $L_2 : \mathbb{C}_{rd}^2 \to \mathbb{C}_{rd}$ is a linear operator, i.e.,*

$$L_2(\alpha y_1 + \beta y_2) = \alpha L_2(y_1) + \beta L_2(y_2),$$

for all $\alpha, \beta \in \mathbb{R}$ *and* $y_1, y_2 \in \mathbb{C}^2_{rd}$.

If y_1 *and* y_2 *solve the homogeneous equation* $L_2 y = 0$, *then so does* $y = \alpha y_1 + \beta y_2$, *where* α *and* β *are real constants.*

The following theorem gives the general solution of the homogeneous linear dynamic equation $L_2 y = 0$, and the solution of the IVP

$$L_2 y = 0, \quad y(t_0) = y_0, \quad y^{\Delta}(t_0) = y_0^{\Delta}.$$

Theorem 3.20. *(Bohner and Peterson [1]). If the pair of functions* y_1, y_2 *forms a fundamental system of solutions for* $L_2 y = 0$, *then*

$$y(t) = \alpha y_1(t) + \beta y_2(t),$$

where α *and* β *are constants, is a general solution of* $L_2 y = 0$. *By general solution we mean every function of this form is a solution and every solution is in this form. In particular, the solution of the initial value problem* $L_2 y = 0,\ y(t_0) = y_0,\ y^{\Delta}(t_0) = y_0^{\Delta}$ *is given by*

$$y(t) = \frac{y_2^{\Delta}(t_0) y_0 - y_2(t_0) y_0^{\Delta}}{W(y_1, y_2)(t_0)} y_1(t) + \frac{y_1(t_0) y_0^{\Delta} - y_1^{\Delta}(t_0) y_0}{W(y_1, y_2)(t_0)} y_2(t).$$

To achieve our goal of finding the analytical and graphical solutions of the particular second order linear homogeneous DE

$$y^{\Delta\Delta} = -y, \ y^{\Delta\Delta}(0) = 0 \ \text{ and } \ y^{\Delta}(0) = 1,$$

we now present an important theorem concerning the second order linear dynamic equation with the constant coefficients, α and β in $\mathbb{R}$. This type of equation takes the form

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0, \tag{3.5}$$

with $\alpha, \beta \in \mathbb{R}$ defined on a time scale $\mathbb{T}$. The DE (3.5) is assumed to be regressive, i.e., $1-\alpha\mu(t)+\beta\mu^2(t) \neq 0$, i.e., $\beta\mu-\alpha \in \mathcal{R}$ for $t \in \mathbb{T}^k$.

Theorem 3.21. *(Bohner and Peterson [1]). Suppose $\alpha^2-4\beta \neq 0$. If $\beta\mu-\alpha \in \mathcal{R}$, then a fundamental system of (3.5) is given by*

$$e_{\lambda_1}(.,t_0) \quad \textit{and} \quad e_{\lambda_2}(.,t_0),$$

where $t_0 \in \mathbb{T}^k$ and λ_1, λ_2 are given by the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ of the DE (3.5). The solution of the initial value problem

$$y^{\Delta\Delta}+\alpha y^{\Delta}+\beta y=0, \quad y(t_0)=y_0, \quad y^{\Delta}(t_0)=y_0^{\Delta}$$

is given by

$$y(t)=y_0\frac{e_{\lambda_1}(.,t_0)+e_{\lambda_2}(.,t_0)}{2}+\frac{\alpha y_0+2y_0^{\Delta}}{\sqrt{\alpha^2-4\beta}}\times\frac{e_{\lambda_2}(.,t_0)-e_{\lambda_1}(.,t_0)}{2}.$$

The following theorem addresses the case when $\alpha^2-4\beta>0$.

Theorem 3.22. *(Bohner and Peterson [1]) Suppose $\alpha^2-4\beta>0$. Define*

$$p=-\frac{\alpha}{2} \;\; \textit{and} \;\; q=\frac{\sqrt{\alpha^2-4\beta}}{2}.$$

If p and $\mu\beta-\alpha$ are regressive, then a fundamental system of (3.5) is given by

$$cosh_{\frac{q}{1+\mu p}}(.,t_0)e_p(.,t_0) \quad \textit{and} \quad sinh_{\frac{q}{1+\mu p}}(.,t_0)e_p(.,t_0),$$

where $t_0 \in \mathbb{T}$ and the Wronskian of these two solutions is

$$q\ e_{\mu\beta-\alpha}(.,t_0).$$

The solution of the IVP

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0, y(t_0) = y_0, y^{\Delta}(t_0) = y_0^{\Delta}$$

is given by

$$[y_0 \cosh_{\frac{q}{1+\mu p}}(., t_0) + \frac{y_0^{\Delta} - p y_0}{q} \sinh_{\frac{q}{1+\mu p}}(., t_0)] e_p(., t_0).$$

Note that $\cosh_p = \frac{e_p + e_{-p}}{2}$ and $\sinh_p = \frac{e_p - e_{-p}}{2}$

Also, we include a theorem as stated (Bohner and Peterson [1]) which gives the solution for the case $\alpha^2 - 4\beta < 0$.

Theorem 3.23. *Suppose $\alpha^2 - 4\beta < 0$. Define*

$$p = -\frac{\alpha}{2} \quad \text{and} \quad q = \frac{\sqrt{4\beta - \alpha^2}}{2}.$$

If p and $\mu\beta - \alpha$ are regressive, then a fundamental system of (3.5) is given by

$$\cos_{\frac{q}{1+\mu p}}(., t_0) e_p(., t_0) \;\; \text{and} \;\; \sin_{\frac{q}{1+\mu p}}(., t_0) e_p(., t_0),$$

where $t_0 \in \mathbb{T}$ and the Wronskian of these two solutions is

$$q e_{\mu\beta - \alpha}(., t_0).$$

The solution of the IVP

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0, \;\; y(t_0) = y_0 \;\; \text{and} \;\; y^{\Delta}(t_0) = y_0^{\Delta}$$

is given by

$$[y_0 \cos_{\frac{q}{1+\mu p}}(., t_0) + \frac{y_0^{\Delta} - p y_0}{q} \sin_{\frac{q}{1+\mu p}}(., t_0)] e_p(., t_0).$$

Note that $\cos_p = \frac{e_{ip}+e_{-ip}}{2}$ and $\sin_p = \frac{e_{ip}-e_{-ip}}{2i}$ where $e_{ip}(t,t_0) = cos_p(t,t_0) + i\sin_p(t,t_0)$. The solution of the second order DE depends on whether the roots are distinct, repeated or complex.

(i) If the characteristic equation has distinct roots λ_1 and λ_2, the general solution is given by

$$y(t) = c_1 e_{\lambda_1}(t,t_0) + c_2 e_{\lambda_2}(t,t_0),$$

where constants c_1 and c_2 can be obtained using the initial conditions.

(ii) If the characteristic equation has repeated roots λ, then the general solution is given by

$$y(t) = c_1 e_\lambda(t,t_0) + c_2 e_\lambda(t,t_0) \int_{t_0}^{t} \frac{1}{\mu\tau + \lambda} \Delta\tau,$$

where constants c_1 and c_2 can be obtained using the initial conditions.

(iii) If the dynamic equation has complex roots of the form $\alpha \pm \beta$, the the general solution is represented by

$$y(t) = [c_1 \cos_{\frac{\beta}{1+\alpha\mu(t)}}(t,t_0) + c_2 \sin_{\frac{\beta}{1+\alpha\mu(t)}}(t,t_0)] e_p(t,t_0),$$

where constants c_1 and c_2 can be obtained using the initial conditions.

In the next section, we give a short discussion of the Differential Analyzer Machine.

4. THE DIFFERENTIAL ANALYZER MACHINE

4.1. INTRODUCTION

The differential analyzer was designed to solve differential equations using the method of mechanical integration. A wheel and disk system was developed to accomplish the mechanical equivalent of integration. The differential analyzer was the first analogue computer used to solve nonlinear differential equations. In this thesis, our goal is to analyze the graphical behavior of solutions of dynamic equations using the Marshall Differential Analyzer Machine. For this purpose, we now give a short description of the differential analyzer in its early period and some discussion of present progress. In addition, we will give a description of how this type of machine solves a dynamic equation on a given time scale.

4.2. HISTORY OF THE DIFFERENTIAL ANALYZER

In 1876, James Thompson and his brother, Lord Kelvin, published a paper [2] that was the description of a device (James named this an "integrating machine") that could solve differential equations of any order, theoretically. Thompson and Kelvin did not ever build such a machine to solve DE's. The first credit for constructing a differential analyzer goes to Dr. Vannevar Bush, a professor of electrical engineering at Massachusetts Institute of Technology. Using the idea from Thompson and Kelvin's paper, Dr. Bush designed and built the first differential analyzer with six integrators in the late 1920's at MIT [3]. A few years later, using the ideas of Dr. Bush, a machine with four integrators was built under the direction of Dr. Douglas Hartree by his student Arthur Porter at the Manchester University in England. The Manchester DA, the first DA in England, was constructed from

Meccano components. After Bush's and Hartree's work, additional machines were built at Cambridge University, Queen's University Belfast, and the Royal Aircraft Establishment in Farnborough. Also, the Oslo Analyzer, a Bush type machine, with 12 integrators was built in Norway in 1938.

4.3. PRESENT

Historical DA machines, built in the beginning of the early 1900s are now in museums. The Marshall University Differential Analyzer machine, the newest machine of its size was built by a team lead by Dr. Bonita Lawrence. Dr. Lawrence and her husband Dr. Clayton Brooks started researching of differential analyzers after their trip to London Science of Museum, where they saw part of Manchester Machine in 2004. In the process of their work, the team found a working DA built by an electronics engineer in California, Mr.Tim Robinson. Mr. Robinson incorporated Dr. Porter's torque amplifier design and used many of Dr. Bush's ideas and built his own. The team was happy to meet Dr. Arthur Porter in 2005 in North Carolina. With the inspiration of Dr. Porter and Mr. Tim Robinson's guidance concerning torque amplification, the team built another DA with two integrators in 2006 and named this machine "Lizzie". After a persistent attempt, the Marshall Differential Analyzer Team built a machine with four integrators in 2009 and they named this machine "Art" after Dr. Arthur Porter. The DA team recently built a small sized DA with two integrators and named it "DA Vinci". The DA Vinci is designed for effective classroom use. To obtain the results for our work, we used "Art" and "DA Vinci". These machines at Marshall University are used for demonstrating the visual concept of mathematics to the students at the university and in high schools around the country and students interested in researching about the behavior of solutions of dynamic equation.

4.4. HOW DOES THE DA WORK?

A differential analyzer machine is made of metal rods and gears. A particular machine can solve DE's of order up to the number of integrator units that it contains. We need the combination of the following main components to run and obtain the desired outcome from a DA machine. They are: input and output tables, integrators, multipliers, the system of inter-connect, adders and torque amplifiers. The integrators, input-output tables and multipliers are joined together using the section of inter-connect with a series of cross shafts and metal gears. Pre-plotted information can be fed into the machine by way of an input table. We draw a desired function first on a paper using the output table. The output table plots a graphical solution of the differential equation drawn by a pen on paper in the form of curve. Figure 4.1 shows the mechanical components of integration on differential analyzer machine. The integrator consists of a horizontally placed disk rotating about a vertical rod through its center. The wheel is positioned on the top of the disk. Three main shafts are associated with each integrator: two input shafts (integrand and independent variable), and one output shaft.

As shown in the Figure 4.1, let the radius of the wheel be a inches and y' be the distance in inches the wheel sits from the center of the disk. If the wheel is at this position, we have $y' = a$. We note that the number of turns made by the wheel is equal to the number of turns made by the disk. The circumference of the wheel is $2\pi a$ inches and the length of the path of the wheel on the disk is $2\pi y'x'$ inches, where x' denote the number of the turns of the disk. The number of the turns of the wheel can be described by

$$\frac{1}{a}y'(x)\Delta(x').$$

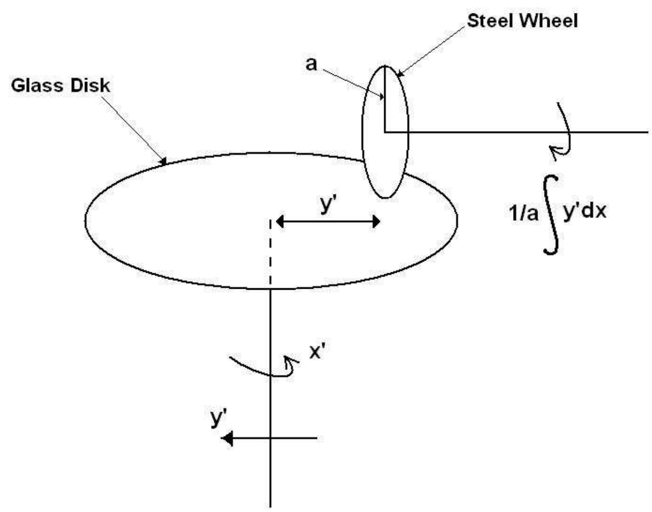

Figure 4.1. Principle of Integrator

For a discrete set of positions of the wheel on the disk, the number of turns of the wheel is given by the expression

$$\frac{1}{a}\sum_{i=1}^{n} y'(x_i')\Delta(x_i').$$

If $\Delta(x_i') \to 0$ then the number of turns of the wheel is

$$\frac{1}{a}\int_{x_0}^{x_n} y'(x')dx'.$$

Now, we would like to rewrite the expression $\frac{1}{a}\int_{x_0}^{x_n} y'(x')dx'$ in terms of the shaft rotations of the three integrator shafts. Let y be the number of turns of the integrand shaft required to produce a linear displacement y' of the disk along the center of the disk, then we have the relation $y = \frac{y'}{P}$ where P is the pitch of the displacement lead screw. If K is the reduction gear between the shaft representing the variable of integration and the disk axle, the number of turns x of the disk is given by $x = \frac{x'}{K}$. Now, substituting the value of x' and y' in the expression

$$\frac{1}{a}\sum_{i=1}^{n} y'(x_i')\Delta(x_i'),$$

we get the following expression for rotation of the output shaft from an integrator

$$\frac{KP}{a}\int y dx.$$

The term $\frac{a}{KP}$ is called the *integrator constant.*

Another set of components that we need within the structure of the machine is adders. Sometimes we need to change the sign for a term within the differential equation that we would like to solve. For example, to set up the differential equation $y^{\Delta} = -y$ on the machine, we need to change the sign of the motion $y(t)$ to get $-y(t)$. For this we use two gears of same size and we mesh them on two adjoining rods. They turn the opposite direction. This will give us the negative sign and we can send this motion as needed. Similarly, torque amplifiers are very important on the differential analyzer machine. We need to send the motion of the output shaft of the wheel through the system of inter-connect into the next integrator. This process can be done using the torque amplifier. The motion of the shaft of the integrator wheel is fed into a torque amplifier which amplifies the torque and sends to the next integrator to use as needed. Marshall DA Art Lab uses torque amplifiers designed by Mr. Tim Robinson.

4.5. THE BUSH SCHEMATIC

In this section, our goal will be to consider solutions of a particular initial value problem on different time scales. For this, we select an IVP

$$y^{\Delta} = \frac{1}{2}y$$

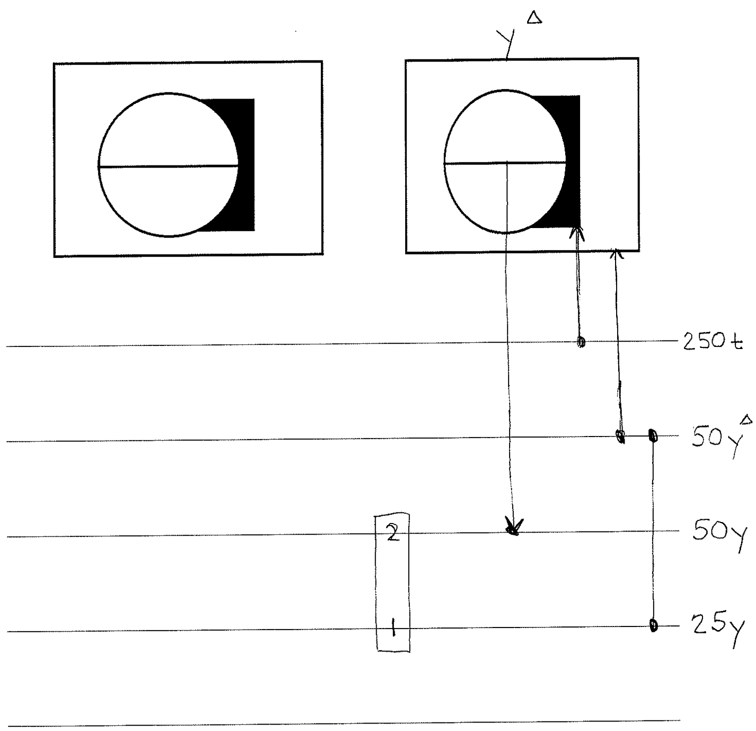

Figure 4.2. Schematic Diagram

with an initial condition $y^{\Delta}(0) = \frac{1}{2}$.

To solve this type of problem, it is essential to give some details about how we set up in the Differential Analyzer machine. For simplicity, we use a diagram called a schematic diagram introduced by Dr. Bush at MIT. According to Bush's schematic, we start with the highest order of the derivative and we work down to the dependent variable, along the way getting the lower order derivatives. Figure (4.2) is the Bush's schematic for the chosen dynamic equation

$$y^{\Delta} = \frac{1}{2}y, \quad y^{\Delta}(0) = \frac{1}{2}.$$

In the Figure (4.2), one can see that the larger rectangular part represents the integrator, the circular part represent the disk, the part which seems like a straight line at the center of the disk represents the wheel (a view from above) and the smaller rectangular shaped piece is the disc drive. The horizontal lines represent the connecting rods. The first arrow sent to the integrator represents the independent variable, (scaled)$250t$, which is fed into the disk drive. This means 250 rotations of the disk is equivalent to one unit of time in the analytical sense on the counter. Similarly, the first derivative term we scale as $50y^{\Delta}$ is the motion which is sent through the second horizontal rod and directs the carriage movement. The output now can be seen in the third horizontal rod as $50y$ which came from the output of the integrator.

For the integrators on our machine, incorporating the gears connected to the counters, $K = \frac{2}{5}\frac{3}{10}, P = \frac{1}{32}$ and $a = \frac{15}{16}$, so using $\frac{a}{KP} = 250$, one can easily see that

$$\begin{aligned}
\frac{KP}{a} \int 50y^{\Delta}(t)d(250t) & \\
&= \frac{1}{250} \int 50y^{\Delta}(t)d(250t) \\
&= 50y(t).
\end{aligned}$$

Then we gear down the motion by one-half to obtain $25y(t)$, as presented on the fourth horizontal rod. Finally, we send this motion to the second horizontal rod to equalize the motions and obtain our first order dynamic equation $50y^{\Delta} = 25y$ which is equivalent to

$$y^{\Delta}(t) = \frac{1}{2}y(t).$$

A schematic for our second order problem is presented in Section 6.3. Now, we analyze the chosen first and second order dynamic equations in different time scales.

5. STATEMENT OF THE PROBLEM WITH ANALYTICAL SOLUTIONS

In this Section, we discuss the solutions of a particular first and second order linear dynamic equations and give a brief analytical discussion of solutions on some given time scales.

5.1. SOLVING A PARTICULAR FIRST ORDER DE ON VARYING TIME SCALES

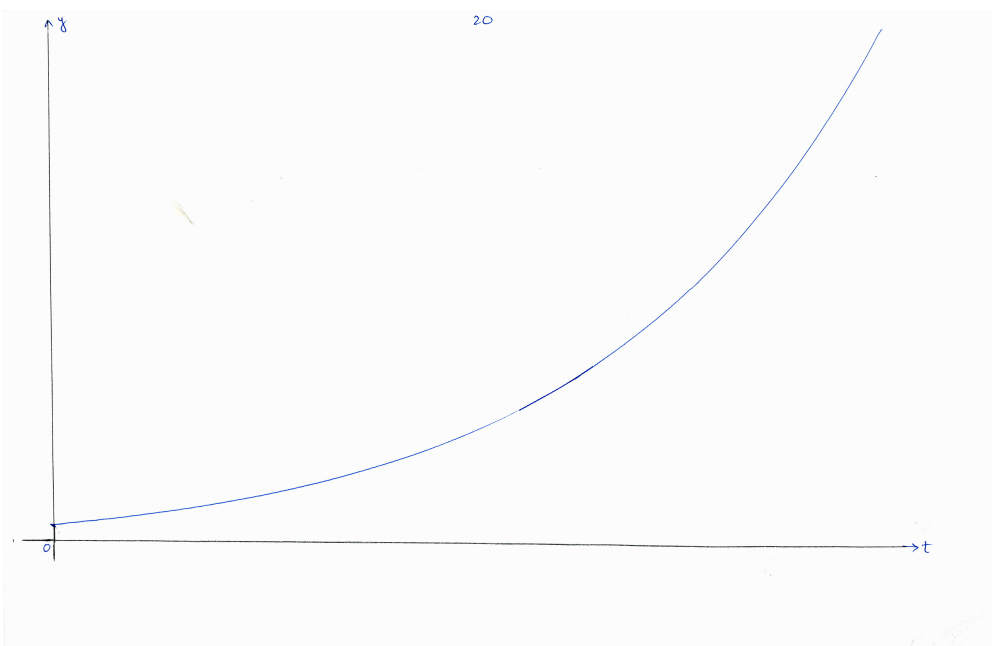

Figure 5.1. Solution of $y^{\Delta} = \frac{1}{2}y,\ y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T} = [0, 6]$

We know that our initial value problem, IVP,

$$y^{\Delta} = \frac{1}{2}y, \ y^{\Delta}(0) = \frac{1}{2} \tag{5.1}$$

has a unique solution. We chose a particular time scale $\mathbb{T} = [0, 6]$ and set up this problem on the differential analyzer machine. We will give the details of the process in Section 6.

Figure 5.1 is the graph of the solution of our IVP (5.1) on $\mathbb{T} = [0, 6]$. Now consider a sequence of time scales, each the union of two closed subsets of $\mathbb{T}_0 = [0, 6]$ where $\mathbb{T}_i \subset \mathbb{T}_{i+1}$ for all $i = 1, 2, 3, 4, ...19$. For each $\mathbb{T}_i$, we plotted the solution curve using the Differential Analyzer. For simplicity, let us define $\mathbb{T}_i = [0, 0.15i] \cup [6 - 0.15i, 6]$, $i = 1, 2, 3, ...20$. Then time scales according to our plan are:

$$\mathbb{T}_0 = [0, 6],$$

$$\mathbb{T}_1 = [0, 0.15] \cup [5.85, 6],$$

$$\mathbb{T}_2 = [0, 0.30] \cup [5.70, 6],$$

$$\mathbb{T}_3 = [0, 0.45] \cup [5.55, 6],$$

$$\mathbb{T}_4 = [0, 0.60] \cup [5.40, 6],$$

.

.

.

$$\mathbb{T}_{18} = [0, 2.70] \cup [3.30, 6],$$

$$\mathbb{T}_{19} = [0, 2.85] \cup [3.15, 6]$$

and

$$\mathbb{T}_{20} = [0, 3.00] \cup [3.00, 6].$$

the solution converges towards solution on the original time scale T. For these domains for our solutions, we can see that the gap between the closed intervals are decreasing gradually. We can see an interesting pattern using these $\mathbb{T}_i's$. Note that the solutions of the sequence of time scales, $\mathbb{T}_i$, created above converge towards the solution on the original time scale $\mathbb{T}_0 = [0, 6]$. Our goal is to analyze solutions of our dynamic equation on these time scales analytically and graphically. Each time scale in the sequence is a union of two closed intervals. An important issue is the behavior of the solutions after the jump in the domain from one closed interval to the other. Solutions are obtained with the same initial condition $y^{\Delta}(0) = \frac{1}{2}$, for all $\mathbb{T}_i$. By gradually decreasing the gap between the two disconnected pieces, the solutions tend toward the solution of our first order dynamic equation on $\mathbb{T}_0 = [0, 6]$.

For the analytical discussion of the initial value problem (5.1), let us consider the time scale $\mathbb{T}_3 = [0, 0.45] \cup [5.55, 6]$ from our sequence and compare the solution obtained with the results we will obtained on the Differential Analyzer in the next section.

We know that the IVP $y^{\Delta} = p(t)y, \; y(t_0) = 1$ has unique solution given by the formula

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right)$$

with

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h} & \text{if } h \neq 0; \\ z & \text{if } h = 0. \end{cases}$$

For the IVP $y^{\Delta} = \frac{1}{2}y, \; y^{\Delta}(0) = \frac{1}{2}$, $p = \frac{1}{2}$. On the time scale $\mathbb{T}_3 = [0, 0.45] \cup [5.55, 6]$, we want to discuss the following three cases:

(i) For $t \in [0, 0.45]$, since $\sigma(t) = t$, and using $t_0 = 0$, we obtain

$$\begin{aligned}
y(t) &= e_{\frac{1}{2}}(t, 0) \\
&= e^{\left(\int\limits_0^t \xi_{\mu(\tau)}(\frac{1}{2})\Delta\tau\right)} \\
&= e^{\left(\int\limits_0^t \xi_0(\frac{1}{2})\Delta\tau\right)} \\
&= e^{\left(\int\limits_0^t (\frac{1}{2}\Delta\tau)\right)} \\
&= e^{\frac{1}{2}t}.
\end{aligned}$$

(ii) For $t = 5.55$, since $\sigma(0.45) = 5.55$, and using Theorem (2.24),

$$\begin{aligned}
y(5.55) &= e_{\frac{1}{2}}(5.55, 0) \\
&= \exp\left(\int\limits_0^{5.55} \xi_{\mu(\tau)}(\frac{1}{2})\Delta\tau\right) \\
&= \exp\left(\int\limits_0^{0.45} \xi_0(\frac{1}{2})\Delta\tau + \int\limits_{0.45}^{5.55} \xi_{5.10}(\frac{1}{2})\Delta\tau\right) \\
&= \exp\left(\int\limits_0^{0.45} (\frac{1}{2}\Delta\tau) + \int\limits_{0.45}^{5.55} \frac{\operatorname{Log}(1 + \frac{5.10}{2})}{5.10}\Delta\tau\right) \\
&= \exp\left(\frac{0.45}{2} + ln(1 + \frac{5.10}{2})\right) \\
&= \left(1 + \frac{5.10}{2}\right)\exp\frac{0.45}{2}.
\end{aligned}$$

Also, since $\sigma(t) > t$, using the simple useful formula for any differentiable function y, $y(\sigma(t)) = y(t) + \mu(t)y^{\Delta}(t)$, for our example we obtain

$$\begin{aligned}
y(\sigma(0.45)) &= y(0.45) + \mu(0.45)y^{\Delta}(0.45) \\
y(5.55) &= y(0.45) + (5.55 - 0.45) \times y^{\Delta}(0.45) \\
&= e^{\frac{1}{2}(0.45)} + 5.10 \times \frac{1}{2}e^{\frac{1}{2}(0.45)} \\
&= \left(1 + \frac{5.10}{2}\right) e^{\frac{0.45}{2}}.
\end{aligned}$$

(iii) For $t \in (5.55, 6]$, since $\sigma(t) = t$,

$$\begin{aligned}
y(t) &= e_{\frac{1}{2}}(t, 0) \\
&= \exp(\int_0^t \xi_{\mu(\tau)}(\frac{1}{2})\Delta\tau) \\
&= \exp\left(\int_0^{0.45} (\frac{1}{2})\Delta\tau + \int_{0.45}^{5.55} ln(1 + \frac{5.10}{2})\Delta\tau + \int_{5.55}^{t} (\frac{1}{2})\Delta\tau\right) \\
&= \exp\left(\frac{0.45}{2} + ln(1 + \frac{5.10}{2}) + \frac{1}{2}t - \frac{5.55}{2}\right) \\
&= \exp(\frac{0.45}{2}) \left(1 + \frac{5.10}{2}\right) \times \exp(\frac{1}{2}t - \frac{5.55}{2})
\end{aligned}$$

Note that

$$\begin{aligned}
y^{\Delta}(5.55) &= y'(5.55) \\
&= e^{\frac{0.45}{2}}(1 + \frac{5.10}{2}) \times \frac{1}{2}e^{\left(\frac{5.55}{2} - \frac{5.55}{2}\right)} \\
&= \frac{1}{2}e^{\frac{0.45}{2}}(1 + \frac{5.10}{2}) \\
&= \frac{1}{2}y(5.55).
\end{aligned}$$

Also, note that for $t \in (5.55, 6]$, $y(t)$ is not equivalent to

$$\begin{aligned} e_{\frac{1}{2}}(t, 5.55) &= \exp\left(\int_{5.55}^{t} \xi_0(\frac{1}{2})\Delta t\right) \\ &= \exp\left(\int_{5.55}^{t} \frac{1}{2}\Delta t\right) \\ &= \exp\left(\frac{1}{2}t - \frac{5.55}{2}\right). \end{aligned}$$

In the next section, we analyze the chosen second order linear homogeneous dynamic equation $y^{\Delta\Delta} = -y$, with the initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta}(0) = 1$.

5.2. SOLVING A PARTICULAR SECOND ORDER DE ON VARYING TIME SCALES

Now, from the discussion in Section 5.1, we are ready to discuss the solution of the second order linear dynamic equations analytically. For this, let us take a time scale $\mathbb{T}_0 = [0, 3\pi]$ and create a sequence of time scales as below:

$$\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$$

$$\mathbb{T}_2 = [0, \frac{7\pi}{10}] \cup [\frac{13\pi}{10}, 3\pi]$$

and

$$\mathbb{T}_3 = [0, \frac{9\pi}{10}] \cup [\frac{11\pi}{10}, 3\pi].$$

From this sequence, we choose $\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$ and particular second order linear homogeneous IVP

$$y^{\Delta\Delta} = -y, \quad y^{\Delta\Delta}(0) = 0 \text{ and } y^{\Delta}(0) = 1. \tag{5.2}$$

Comparing this IVP with the DE $y^{\Delta\Delta}+\alpha y^{\Delta}+\beta y=0,\ \ y(t_0)=y_0,\ \ y^{\Delta}(t_0)=y_0^{\Delta}$, we have $\alpha^2-4\beta=0^2-4*(1)=-4<0$. Thus using Theorem 3.23, we analyze three cases as given below:

We have $\alpha=0$, $\beta=1$ so that $p=-\frac{\alpha}{2}=0$ and $q=\frac{\sqrt{4\beta-\alpha^2}}{2}=1$. The characteristic equation of the DE $y^{\Delta\Delta}=-y$ is $\lambda^2+1=0$ whose roots are $\lambda_1=i$ and $\lambda_1=-i$.

(i) For $t\in[0,\frac{\pi}{2}]$, since $y_0=0$, and $y_0^{\Delta}=1$, we have

$$\begin{aligned}
y(t) &= [y_0\cos_{\frac{q}{1+\mu p}}(t,t_0)+\frac{y_0^{\Delta}-py_0}{q}\sin_{\frac{q}{1+\mu p}}(.,t_0)]e_p(t,t_0)\\
&= [0\times\cos_{\frac{1}{1+0}}(t,0)+\frac{1-0}{q}\sin_{\frac{1}{1+0}}(t,0)]e_0(t,0)\\
&= 0+\sin_1(t,0)\\
&= \frac{e_i-e_{-i}}{2i}\\
&= \frac{\exp\left(\int\limits_0^t(i)\Delta\tau\right)-\exp\left(\int\limits_0^t(-i)\Delta\tau\right)}{2i}\\
&= \frac{e^{it}-e^{-it}}{2i}\\
&= \frac{\cos t+i\sin t-(\cos t-i\sin t)}{2i}\\
&= \frac{2i\sin t}{2i}\\
&= \sin t.
\end{aligned}$$

(ii) For $t = \frac{\pi}{2}$, since $\sigma(\frac{\pi}{2}) = \frac{3\pi}{2}$, we have by using the definition of $\sin_1(t, t_0)$,

$$\begin{aligned}
y(t) &= [y_0 \cos_{\frac{q}{1+\mu p}}(t, t_0) + \frac{y_0^{\Delta} - p y_0}{q} \sin_{\frac{q}{1+\mu p}}(., t_0)] e_p(t, t_0) \\
y((\frac{3\pi}{2})) &= [0 * \cos_{\frac{1}{1+0}}(\frac{3\pi}{2}, 0) + \frac{1-0}{1} \sin_{\frac{1}{1+0}}(\frac{3\pi}{2}, 0)] e_0(\frac{3\pi}{2}, 0) \\
&= \sin_1(\frac{3\pi}{2}, 0) \exp\left(\int_0^{\frac{3\pi}{2}} \xi_\pi(0) \Delta\tau \right) \\
&= \sin_1(\frac{3\pi}{2}, 0) \exp\left(\int_0^{\frac{\pi}{2}} \xi_0(0) \Delta\tau \ + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \xi_\pi(0) \Delta\tau \right) \\
&= \sin_1(\frac{3\pi}{2}, 0) \exp\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{Log[1 + 0 * \pi]}{\pi} \Delta\tau \right) \\
&= \exp\left(\frac{ln[1]}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \Delta\tau \right) \\
&= \exp\left(\frac{\pi}{\pi} \times \ln[1] \right) \\
&= 1
\end{aligned}$$

Also, since $\sigma(t) > t$, using the simple useful formula for any differentiable function y, we have, $y(\sigma(t)) = y(t) + \mu(t) y^{\Delta}(t)$. Then;

$$\begin{aligned}
y(\sigma(\frac{\pi}{2})) &= y(\frac{\pi}{2}) + \mu(\frac{\pi}{2}) y^{\Delta}(\frac{\pi}{2}) \\
y(\frac{3\pi}{2}) &= y(\frac{\pi}{2}) + (\frac{3\pi}{2} - \frac{\pi}{2}) * y^{\Delta}(\frac{\pi}{2}) \\
&= \sin(\frac{\pi}{2}) + \pi \times \cos(\frac{\pi}{2}) \\
&= 1 + \pi \times 0 \\
&= 1
\end{aligned}$$

(iii) For $t \in (\frac{3\pi}{2}, 3\pi]$, since $\sigma(t) = t$ and using the definition of $\sin_1(t, t_0)$,

$$\begin{aligned} y(t) &= [y_0 \cos_{\frac{q}{1+\mu p}}(t, t_0) + \frac{y_0^\Delta - py_0}{q} \sin_{\frac{q}{1+\mu p}}(., t_0)]e_p(t, t_0) \\ &= [0 + \frac{1-0}{1} sin_{\frac{1}{1+0}}(t, 0)]e_0(t, 0) \\ &= -sint. \end{aligned}$$

Therefore our solution on $\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$ has the form

$$y(t) = \begin{cases} \sin t & \text{for } t \in [0, \frac{\pi}{2}]; \\ 1 & \text{if } t = \frac{3\pi}{2}; \\ -\sin t & \text{if } t \in (\frac{3\pi}{2}, 3\pi]. \end{cases}$$

Now, we discuss plotting graphical solution of the IVP $y^\Delta = \frac{1}{2}y,\ \ y^\Delta(0) = \frac{1}{2}$ and $y^{\Delta\Delta} = -y,\ y^{\Delta\Delta}(0) = 0,\ y^\Delta(0) = 1$ on our chosen sequences of time scales using the Differential Analyzer.

6. SOLVING DE'S ON VARYING TIME SCALESUSING THE DIFFERENTIAL ANALYZER

There are many methods of obtaining the numerical solutions and graphical solutions of differential equations of any order. In this study, we will examine the behavior of the solution of the first and second order DE's on varying time scales. We start the problem with a given time scale and gradually increase the size of time scales and obtain solutions on the sequence of time scales. That means we decrease the gap between the two disconnected closed intervals of real line. We gave an introduction to the differential analyzer machine in Section 4. Now we want to discuss some details about how we set up our problem in the Differential Analyzer to achieve our results. We use the method of mechanical integration to lower the order as required (we also discussed this in the Bush's Schematic). The Differential Analyzer's disk and wheel mechanism integrates derivatives of the functions y. This type of machine mainly consists of integrator units, torque amplifiers (one per integrator unit), counters (counting units), adding units (adders), input and output tables and gearing and shafting system for sign changes and to gear down within the differential equation and link the motions together. For this process, for a linear equation, we send the independent variable motion, denoted by t, to each of the integrator units through the independent variable motor. We also send the motion describing the derivative of a function to the integrator unit through a lead screw to obtain the desired output. The integrator consists of horizontally positioned disc and a vertically placed wheel at the top of the disc. We note here that the disk rotates about a vertical rod through its center. When the machine is in motion, we measure the motion made by the turns of the disk and rotation of the wheel in terms of shaft rotations. The output of the integrator is sent to the other integrator, if required, through the system of interconnect and finally it is sent to the output table to get the curve that we targeted.

6.1. SOLVING A PARTICULAR FIRST ORDER DE ON THE DA

To find the solution of the differential equation $y^{\Delta} = \frac{1}{2}y$, we use only one integrator, because it is of first order. We set the initial condition using the counters for the derivatives and the output table so it is ready to draw the solution curve. We send the motion of the independent variable and the motion of $y^{\Delta}(t)$ to the integrator. After the process of integration, we obtain the motion $y(t)$ and make a careful connection to the output table. We also gear down $y(t)$ by 2 in this problem to obtain $\frac{1}{2}y(t)$ and then connect it to the rod that turns the lead screw on the integrator.

The interesting point made here is how we run the problem across the gap. In particular, if our time scale is $\mathbb{T}_3 = [0, 0.45] \cup [5.55, 6]$, how do we run the problem? For this, on the output table, we will let the pen plot the solution up to the point $t = 0.45$. At this point we lift the pen up from the paper and we disengaged the lead screw, representing the first derivative and moving the first integrator, via the clutch. Then, we run the machine up to the point $t = 5.55$. Now, we disengage the independent and dependent variable from the output table via the clutches. We then reengage the integrator clutches and run the independent variable in the positive direction until the corresponding derivative, as we read on the counter, reads a value consistent with the value described by the differential equation with respect to the corresponding y value at the point $t = 5.55$. At this point, we stopped the machine and reengage the derivative clutch and proceed on to the next y values up to $t = 6$.

The main reason for disengaging the clutch is that the function is not defined for values between $t = 0.45$ to $t = 5.55$, but from 0 to 0.45 and 5.55 to 6 we have an exponential curve as the solution. We use the simple useful formula for the differential function and compute

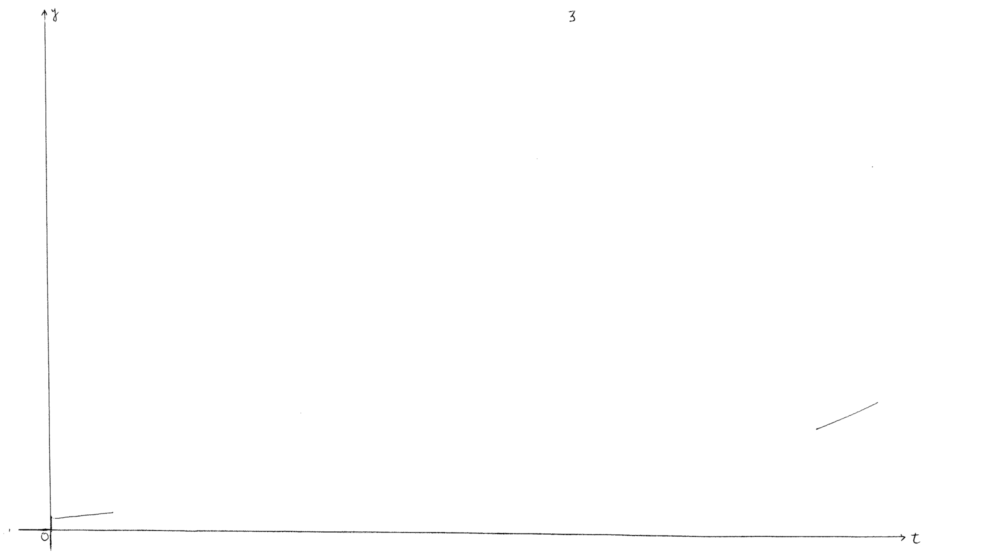

Figure 6.1. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_3 = [0, 0.45] \cup [5.55, 6]$

$y(5.55)$, where $y(5.55) = y(0.45) + (5.55 - 0.45) \times y^{\Delta}(0.45)$. At this point, we are at $y(0.45)$, the graininess is $(5.55 - 0.45)$ and we want the direction $y^{\Delta}(0.45)$. Most importantly, we also note that only one initial condition is expected for this type of problem. Another important thing we need to note is that the machine automatically gives the value for y where we engaged the clutch at $t = 5.55$.

Figure 6.1 is the graphical solutions of the initial value problem

$$y^{\Delta} = \frac{1}{2}y, \; y^{\Delta}(0) = \frac{1}{2}$$

on the time scale $\mathbb{T}_3 = [0, 6] \cup [5.55, 6]$. Figures 6.2 shows the solutions of the dynamic equation

$$y^{\Delta} = \frac{1}{2}y, \; y^{\Delta}(0) = \frac{1}{2}$$

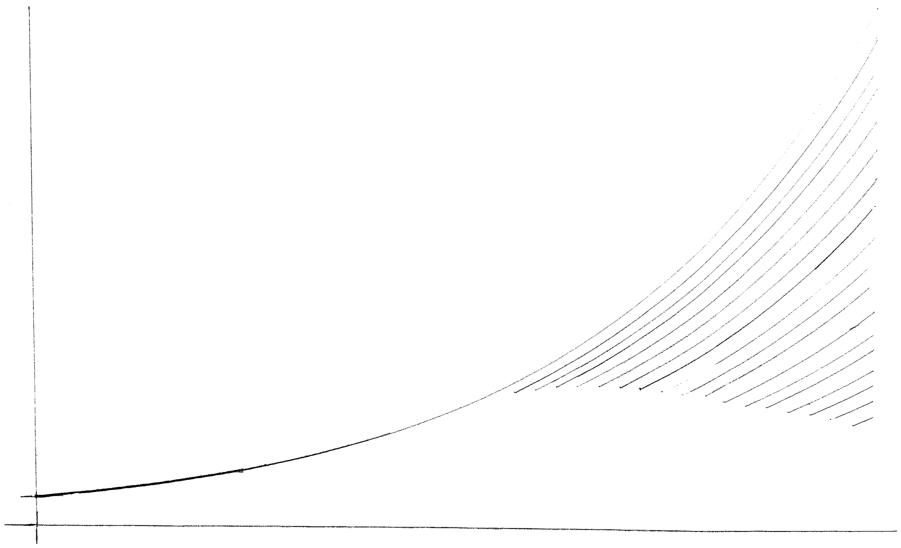

Figure 6.2. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on Varying Time Scales

on varying time scales $\mathbb{T}_i's$ that we have discussed in the Section 5.1. We can clearly see that the gap between two disconnected pieces are decreasing and the solutions of $\mathbb{T}_i's$, $i = 1, 2, 3, ...20$ converges towards the solution of the time scale $\mathbb{T}_0$.

6.2. SOLVING A PARTICULAR SECOND ORDER DE ON THE DA

We take time scales $\mathbb{T}_0 = [0, 3\pi]$, $\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$, $\mathbb{T}_2 = [0, \frac{7\pi}{10}] \cup [\frac{13\pi}{10}, 3\pi]$ and $\mathbb{T}_3 = [0, \frac{9\pi}{10}] \cup [\frac{11\pi}{10}, 3\pi]$ as we have discussed for the analytical solution for our second order homogeneous linear dynamic equation

$$y^{\Delta\Delta} = -y, \; y^{\Delta\Delta}(0) = 0 \text{ and } y^{\Delta} = 1.$$

Let us discuss how we set up this problem on the Differential Analyzer. For this we first describe the Bush Schematic as shown in Figure 6.3. We clearly can see in this figure that we need two integrators for this second order problem. From the first horizontal line (which represents time), the independent variable $250t$ is sent to both of the integrators representing the first and second derivatives. The motion of $250y^{\Delta\Delta}$ moves the carriage of the first integrator. After a process of integration, we can see the output motion as $250y^{\Delta}$ which is represented by the second horizontal rod. Similarly, we now send the motion of $250y^{\Delta}$ to the second integrator and the resulting motion can be seen on the third horizontal rod which is $250y$. Now now connect two gears of same size to allow running the motion in the opposite direction so that we obtain the motion $-250y$. Finally, we sent the motion $-250y$ to the first integrator to equalize and obtain $y^{\Delta\Delta} = -y$.

Particularly, if our time scale is $\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$, let us explain how we run this second order problem on the DA. We need two integrators because the equation is of second order. We set up the initial conditions using the counters and we run the solution curve to the point $t = \frac{\pi}{2}$ with the first integrator, the second integrator and the output table engaged. We let the pen draw the solution curve on the paper set on the output table. We are now at the corresponding y value of $\frac{\pi}{2}$. Since we have the common useful formula of the differential function $y(\frac{3\pi}{2}) = y(\frac{\pi}{2}) + \mu(\frac{\pi}{2})y^{\Delta}(\frac{\pi}{2})$, we need to run from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ with the slope of $y^{\Delta}(\frac{\pi}{2})$. For this we run the output table with the first derivative held constant at $y^{\Delta}(\frac{\pi}{2})$. Then we run the machine up to the point $t = \frac{3\pi}{2}$. We have the corresponding y value for $\frac{3\pi}{2}$ at this point. Then, we disconnect the output table. We require the opposite y value on the counter of the integrator which represents the second derivative, $y^{\Delta\Delta}$, and the related y^{Δ} value. We reconnect the lead screws to both of the integrators. Then we lift the pen disconnect the output table and run the machine until we have the appropriate $y(\frac{3\pi}{2})$ on the counter for the first integrator but of opposite sign. Now we reconnect all the lead screws. Finally we run up to the point $t = 3\pi$.

Figure 6.4 was obtained on the Differential Analyzer for our second order DE $y^{\Delta\Delta} = -y$, with the initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta} = 1$ on time scale $\mathbb{T}_2 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$.

Next, we describe our results and present the curves that we obtained on different time scales for the chosen first and second order dynamic equations.

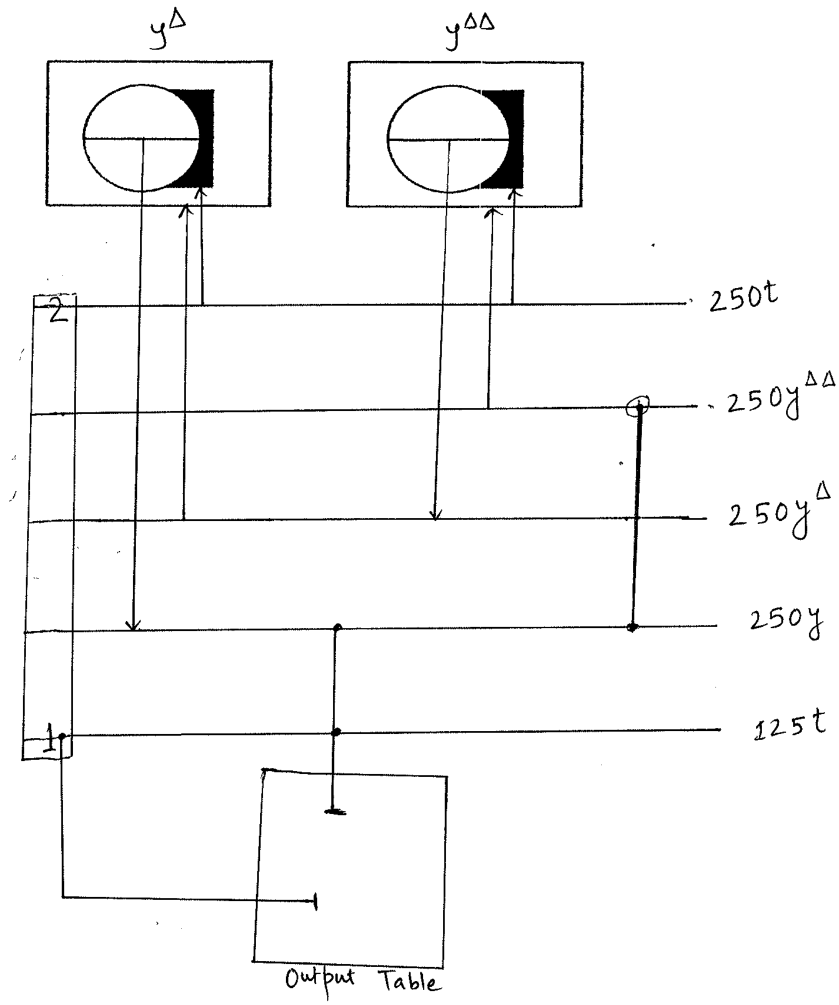

Figure 6.3. The Bush Schematic

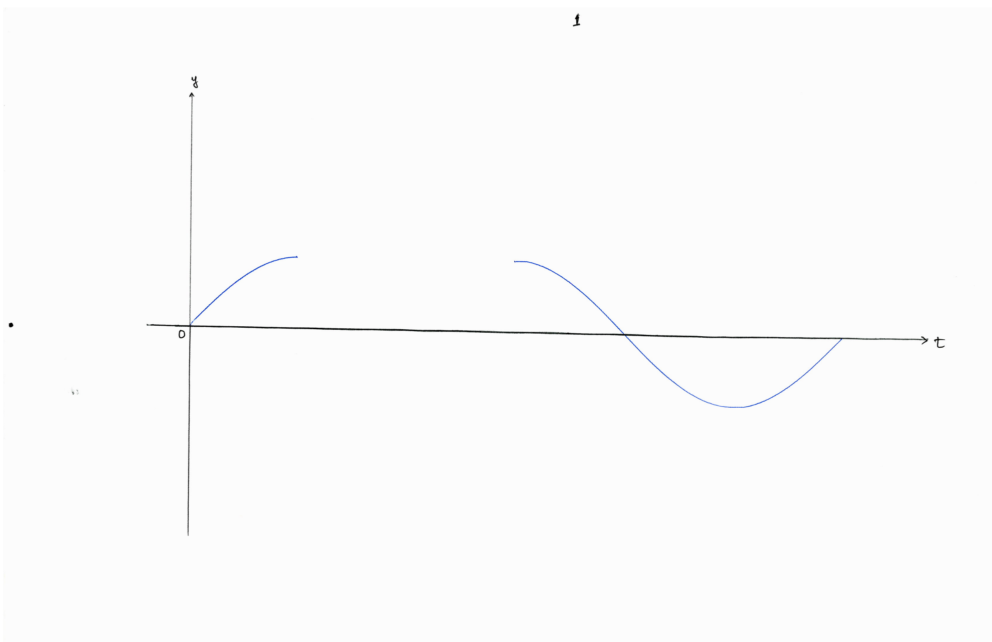

Figure 6.4. Solution of $y^{\Delta\Delta} = -y$, $y^{\Delta\Delta}(0) = 0$, $y^{\Delta}(0) = 1$ on The Time Scale $\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$

7. CONCLUSIONS AND DA RESULTS

In this work, we have sufficiently discussed the basic terms and useful properties, with some examples, on time scale calculus. Further, we presented the concepts of the delta derivative, delta antiderivative, first and second order initial value problem starting with the Hilger Complex Plane and the generalized exponential function (the solution of a first order dynamic equations). We demonstrated analytical and graphical solutions of some initial value problems. Particularly, we have chosen

$$y^{\Delta} = \frac{1}{2}y,\ y^{\Delta}(0) = \frac{1}{2}$$

as our first order dynamic equation and the second order linear homogeneous dynamic equation

$$y^{\Delta\Delta} = -y,\ y^{\Delta\Delta}(0) = 0 \text{ and } y^{\Delta} = 1.$$

Additionally, we demonstrated the analytical and graphical solution to these equations on varying time scales. A major focus of this work is the use of the Differential Analyzer Machine at Marshall University.

There are many methods for studying the graphical behavior of first and second order dynamic equations on time scales calculus. To find the solution on the union of two disconnected closed intervals of real line, we realized that the best way is to analyze the behavior of the solutions using the Differential Analyzer. This type of machine gives the qualitative behavior of the dynamic equations of any order. The Marshall DA Team exerted painstaking effort to make the successful models on the machine and particularly demonstrating the visual concept to the mathematics and engineering students of the university as well as high school students. Including the author of this thesis, we have had a very exciting time working with the machine. In the process of its study, we realized that this kind of machine is very much useful for achieving the results for a variety of differential equations. The beauty behind this machine is that one can use this machine as a working tool for his research and create a mechanical model to get the output curve as desired for the DE problems. There is not a unique way to solve a DE in the machine. We can get results by setting up the problem in various ways. After understanding the visual perspective of the machine offers, everybody can realize the beauty of the Differential Analyzer.

For our work, particularly, we used the differential analyzer machine to find graphical solutions. For this we started with a time scale $\mathbb{T}$ and further we created a sequence of time scales $\mathbb{T}_i's$ and showed that the solution on the $\mathbb{T}_i's$ (decreasing the gap between two closed intervals) converges to the solution on time scale $\mathbb{T}_0$. We gave the detailed steps to set up the first and second order problem on the differential analyzer.

Thus, we concluded that the solutions of these two linear dynamic equations of first and second order in different time scales converges to the solution after gradually increasing the time scales where $\mathbb{T}_{i+1}'s \subset \mathbb{T}_i, i = 0, 1, 2, 3, 4, ...$

The figures that follow were obtained after setting up our first order DE $y^\Delta = \frac{1}{2}y,\ y^\Delta(0) = \frac{1}{2}$ on the Differential Analyzer in 21 different time scales and for the second order linear homogeneous dynamic equation $y^{\Delta\Delta} = -y$ with the initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^\Delta = 1$ on 4 different time scales. Note the behavior of the solutions as the time scales converges to the associated solution on $\mathbb{T}_0$.

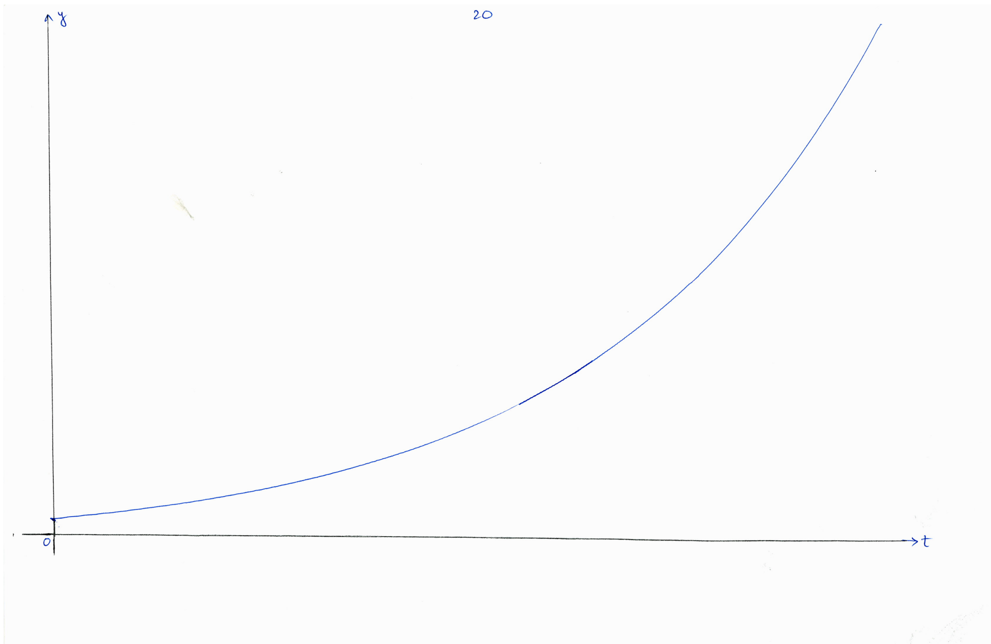

Figure 7.1. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_0 = [0, 6]$

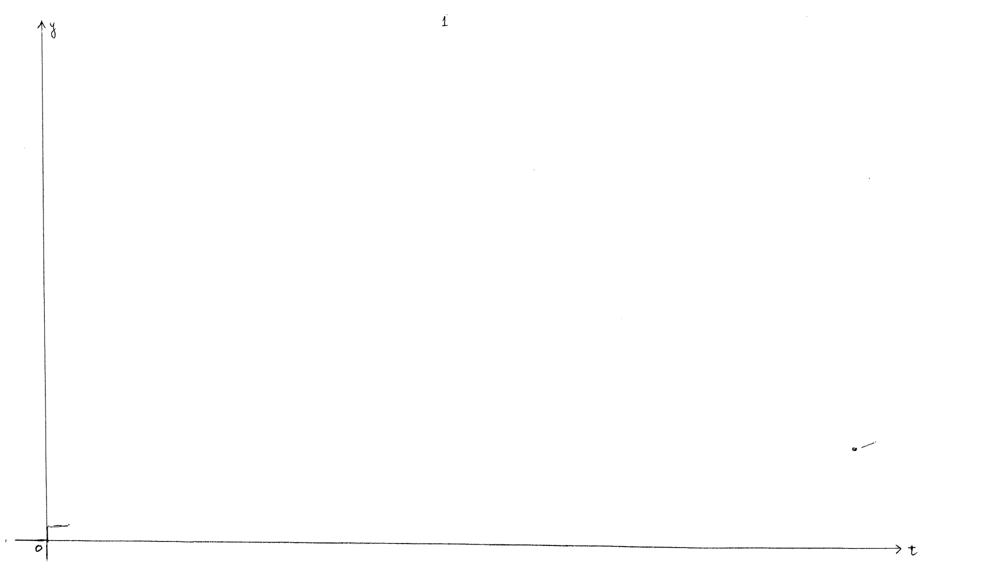

Figure 7.2. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_1 = [0, 0.15] \cup [5.85, 6]$

Figure 7.3. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_2 = [0, 0.30] \cup [5.70, 6]$

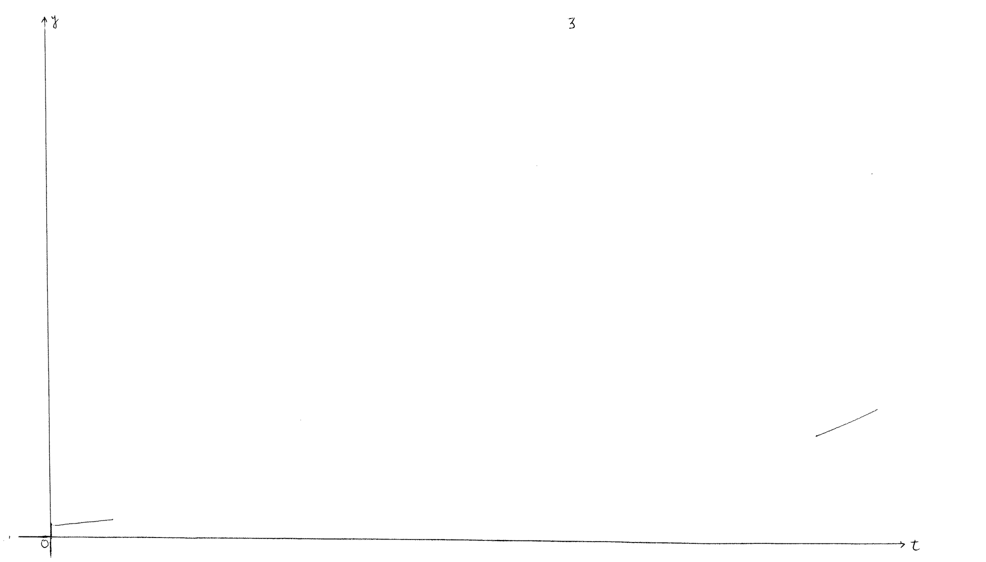

Figure 7.4. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_3 = [0, 0.45] \cup [5.55, 6]$

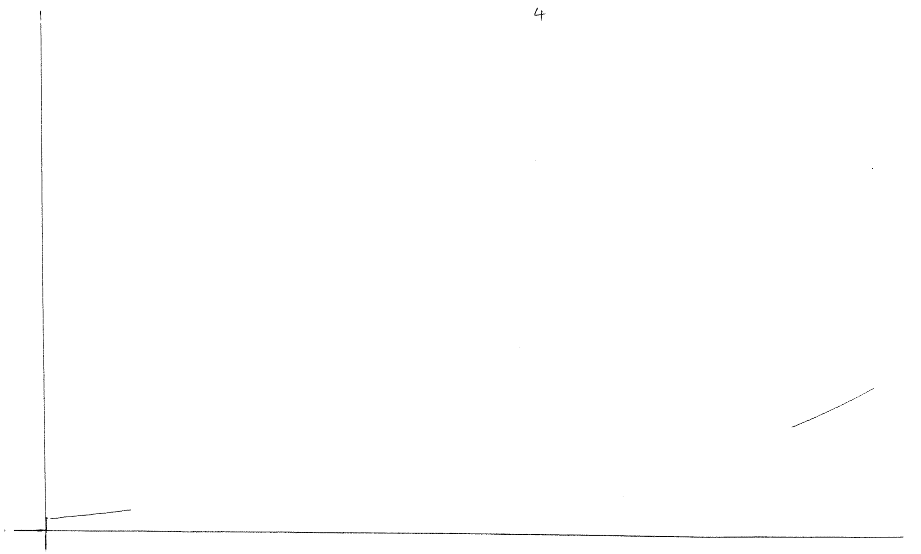

Figure 7.5. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_4 = [0, 0.60] \cup [5.40, 6]$

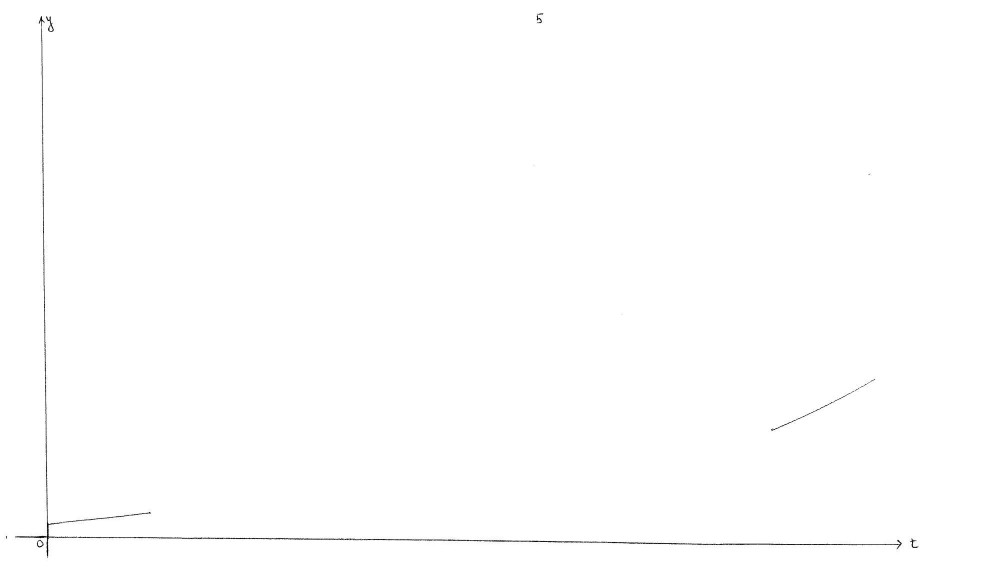

Figure 7.6. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_5 = [0, 0.75] \cup [5.25, 6]$

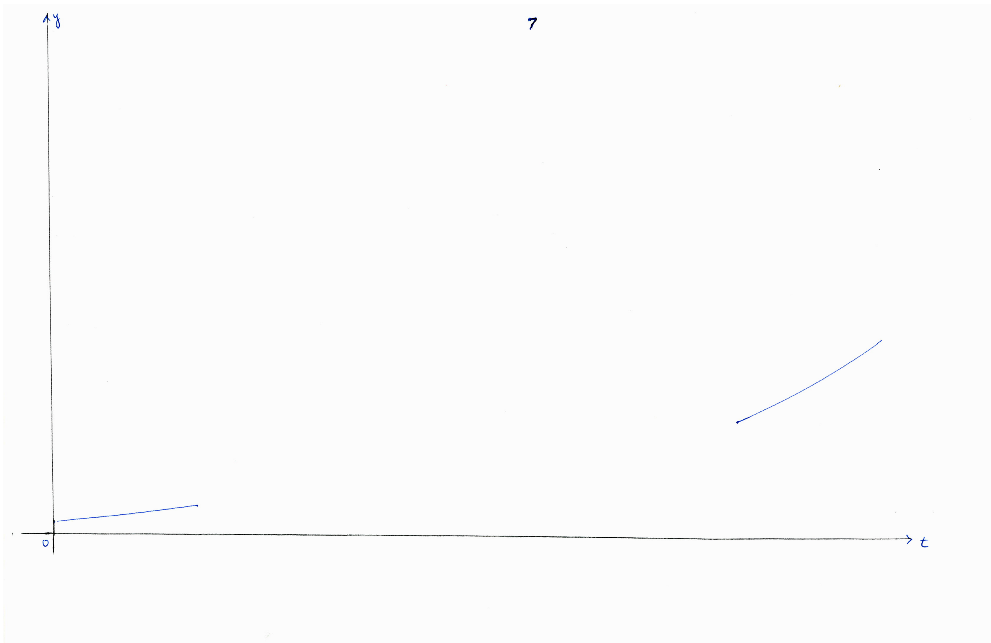

Figure 7.7. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_7 = [0, 1.05] \cup [4.95, 6]$

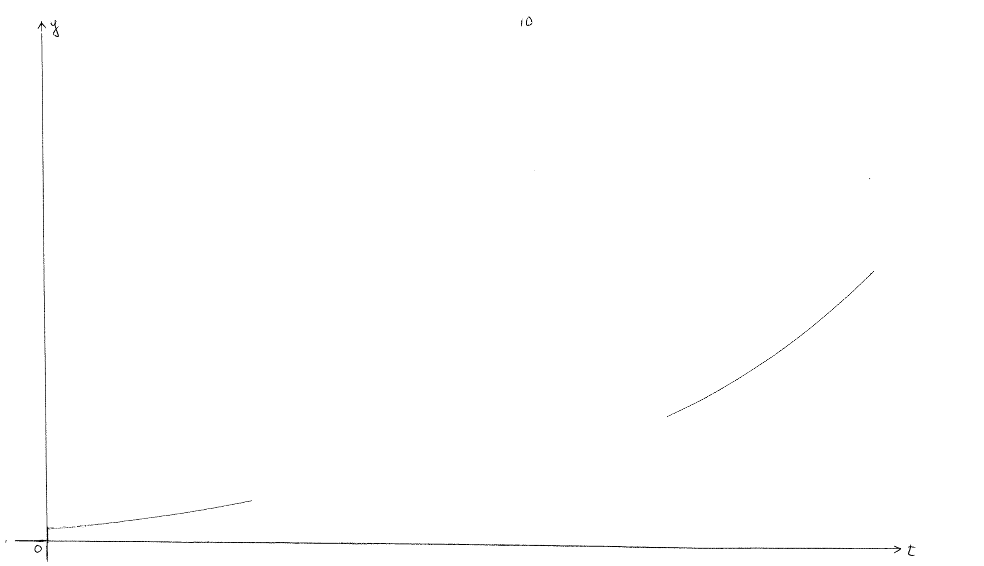

Figure 7.8. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_{10} = [0, 1.50] \cup [4.50, 6]$

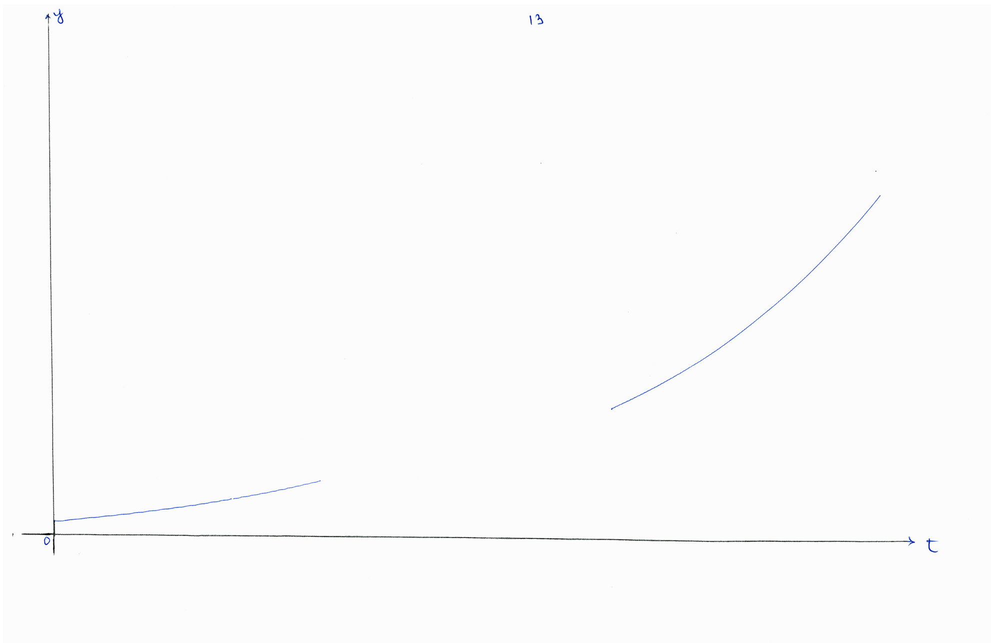

Figure 7.9. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_{13} = [0, 1.95] \cup [4.05, 6]$

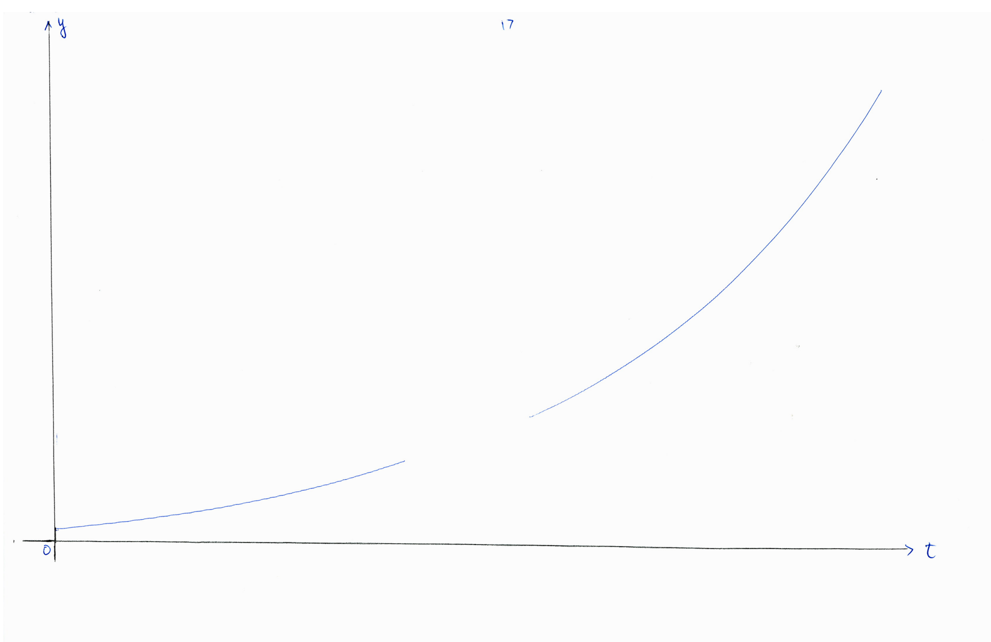

Figure 7.10. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_{17} = [0, 2.55] \cup [3.45, 6]$

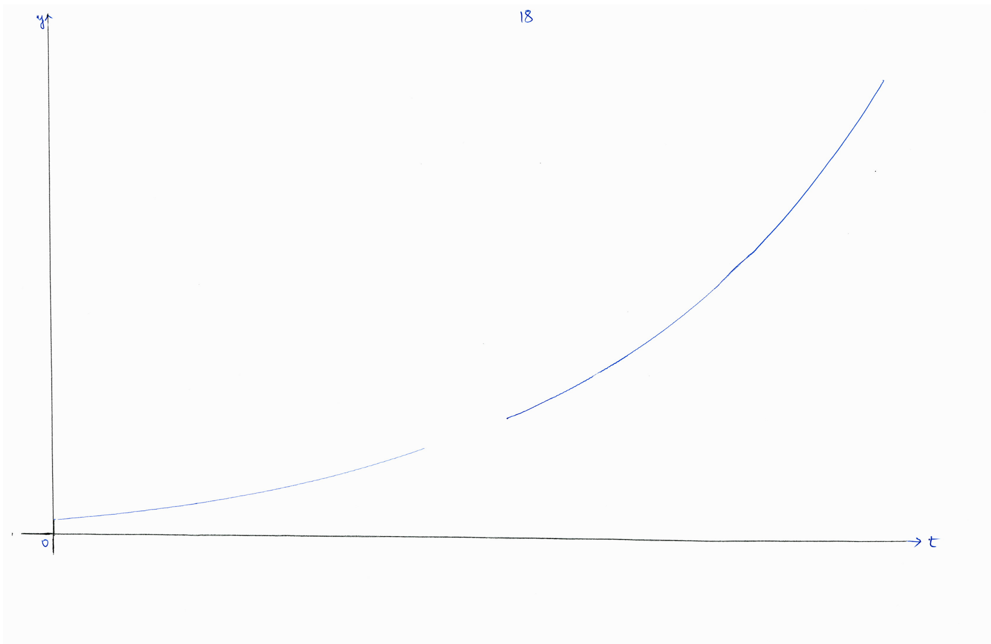

Figure 7.11. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_{18} = [0, 2.70] \cup [3.30, 6]$

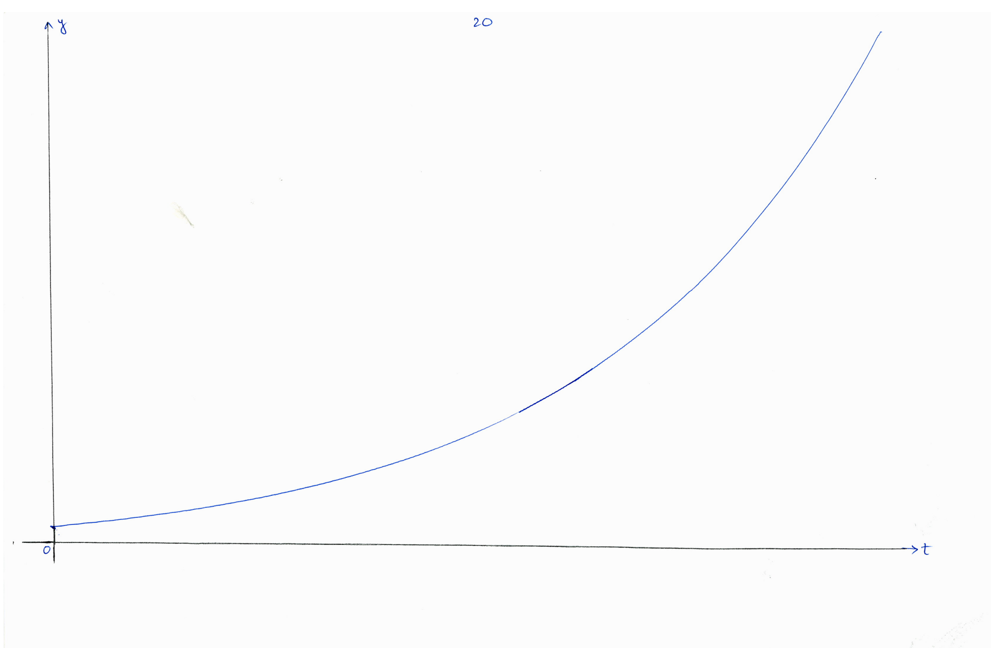

Figure 7.12. Solution of $y^{\Delta} = \frac{1}{2}y$, $y^{\Delta}(0) = \frac{1}{2}$ on $\mathbb{T}_{20} = [0, 3.00] \cup [3.00, 6]$

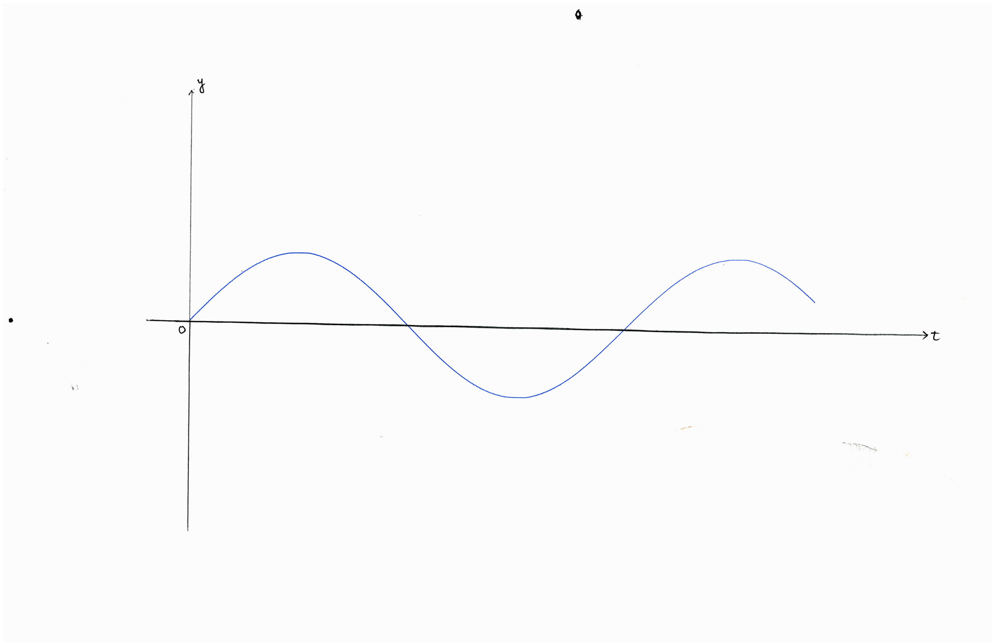

Figure 7.13. Solutions of $y^{\Delta\Delta} = -y$, $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta} = 1$ on $\mathbb{T}_0 = [0, 3\pi]$

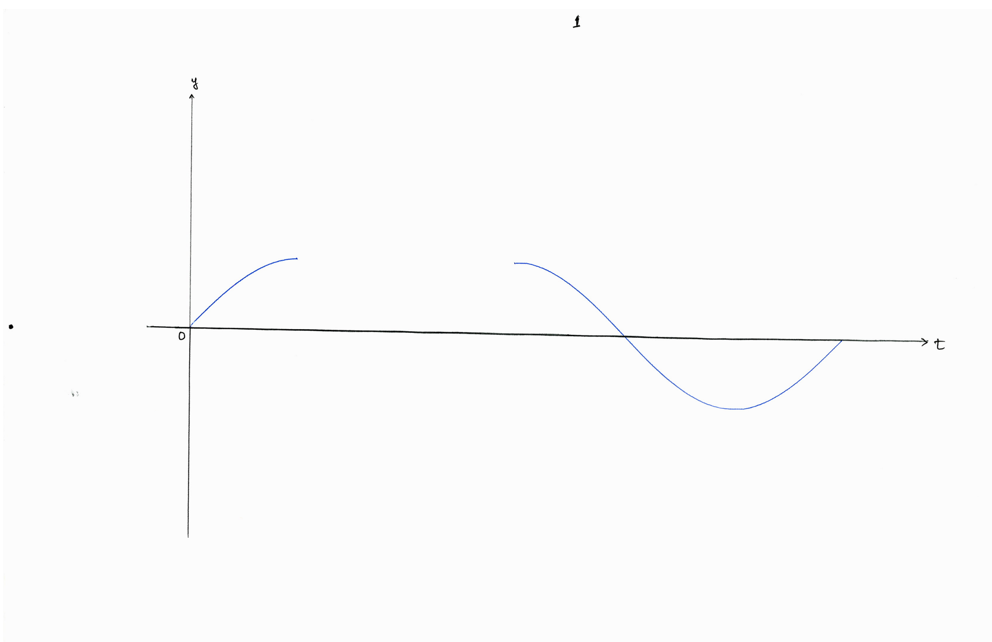

Figure 7.14. Solutions of $y^{\Delta\Delta} = -y$, $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta} = 1$ on $\mathbb{T}_1 = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 3\pi]$

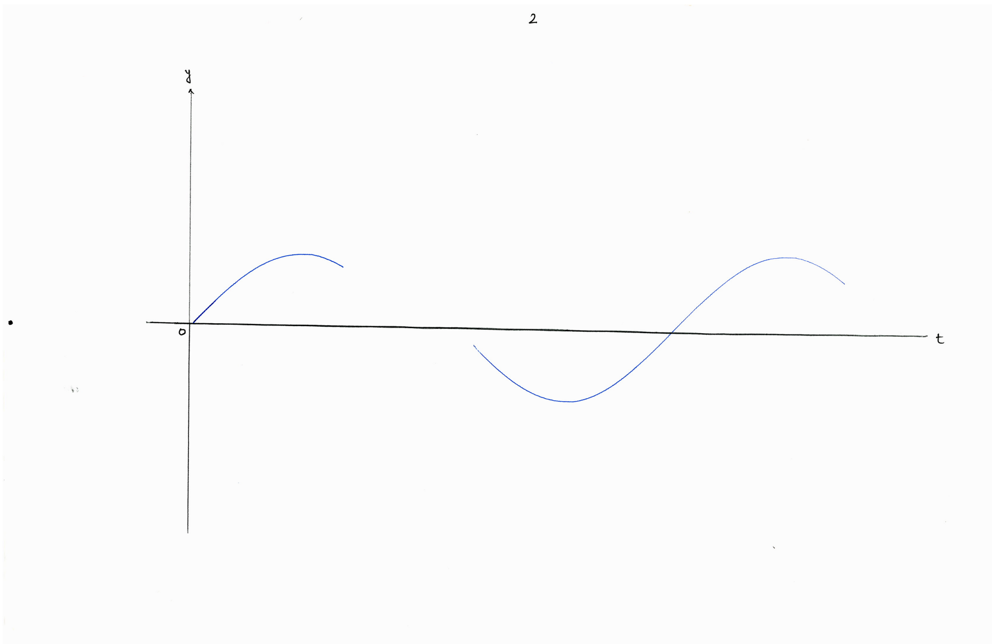

Figure 7.15. $y^{\Delta\Delta} = -y$, $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta} = 1$ on $\mathbb{T}_2 = [0, \frac{7\pi}{10}] \cup [\frac{13\pi}{10}, 3\pi]$

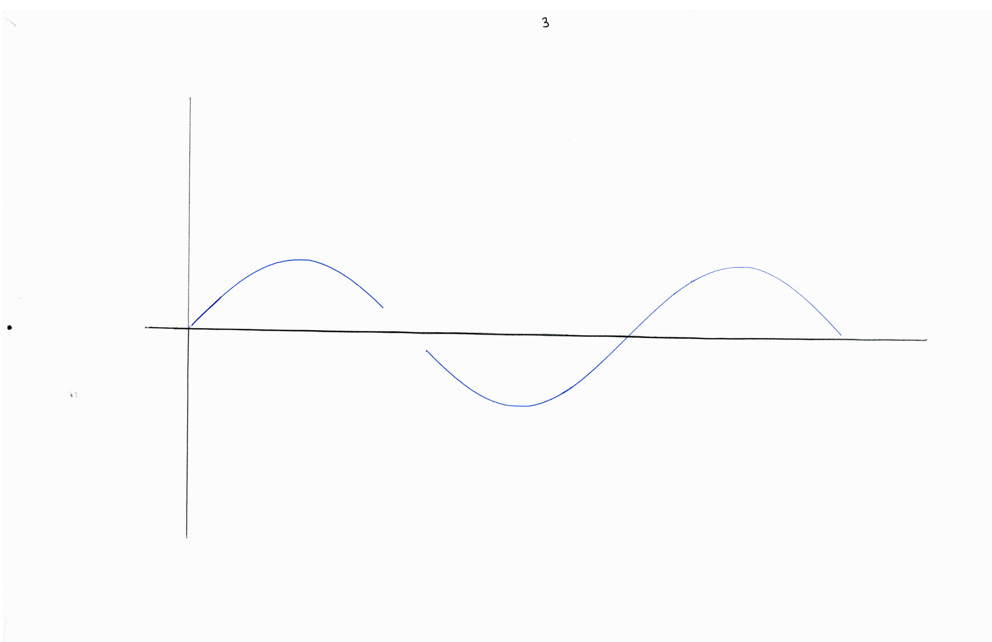

Figure 7.16. $y^{\Delta\Delta} = -y$, $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta} = 1$ on $\mathbb{T}_3 = [0, \frac{9\pi}{10}] \cup [\frac{11\pi}{10}, 3\pi]$

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