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Using Radial Basis Functions to Interpolate Along Single-Null Characteristics

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- **Gravitational radiation is properly defined only at future null infinity, but mathematically it is estimated at a finite radius.**

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- **The correct modeling of gravitational radiation is tough!**
- **Gravitational radiation is properly defined only at future null infinity, but mathematically it is estimated at a finite radius.**
- **Cauchy-Characteristic Extraction is the most precise method.**
- **Satisfies the detection criteria required by Advanced LIGO.**

HIGHLIGHTS OF CCE TOOL

- **Flexibility and control in prescribing initial data**
- **Rigid coordinates implies little gauge freedom**
- **Constraint violations fall off asymptotically as *1/r***
- **Grid domain is the region in which waves propagate**
- **No outgoing radiation or other artificial boundary**
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N	1.59	1.56
Ψ	1.16	1.14

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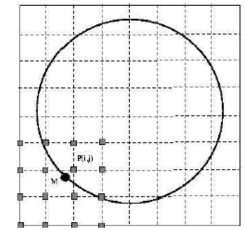
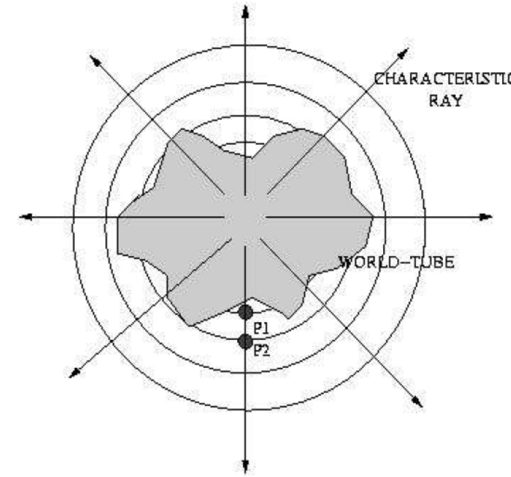
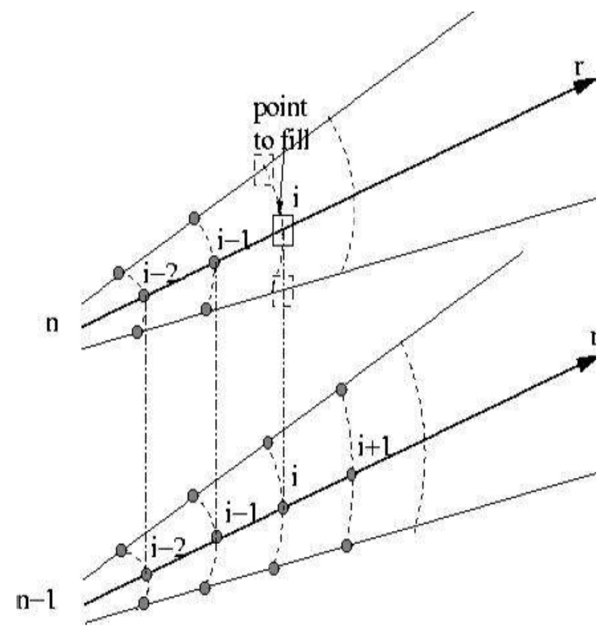
- **Overall 1st order of accuracy ought to be improved.**
- **Goal: extend the overall accuracy of the CCE tool.**

SOURCES OF ERROR

1. **Inner boundary: Characteristic data is extracted from the Cauchy data**
2. **Start-up: the extraction worldtube moves with respect to the null grid**
3. **Evolution: radial and time integration that propagate the field equations in Bondi coordinates.**
4. **Outer boundary: asymptotic limits of the approximations used in the computation of the waveform.**







NUMERICAL METHODS

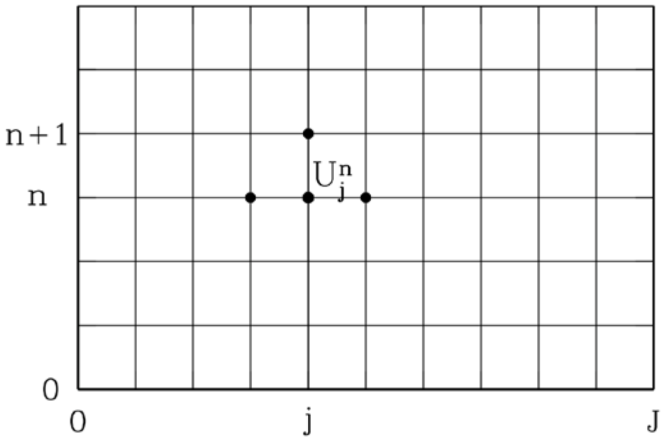
Classically based on polynomial interpolation.

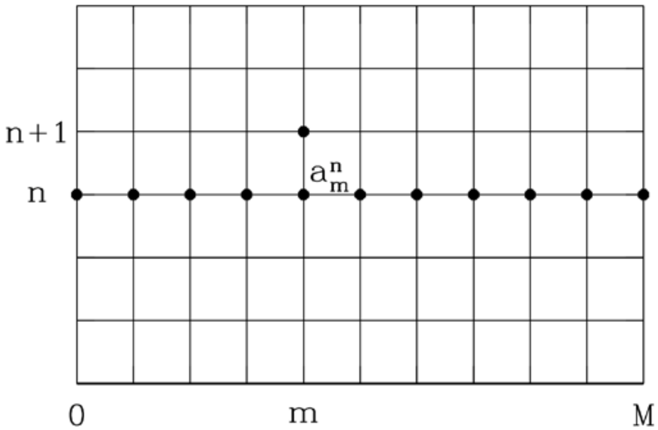
Finite-Difference (FD)

- **Taylor series expansion**
- **Local Grid flexibility**
- **Ease of implementation**
- **Low convergence rate**

Pseudospectral Methods

- **Orthogonal (basis) functions**
- **Tied to a global grid**
- **Hard to implement**
- **Exponential convergence rate**





RADIAL BASIS FUNCTIONS

Exact function interpolators: passes through all points.

$$f(r) = \sum_{i=0}^{N} a_i h_i(r)$$

Function of the Euclidian distance of the point to origin

$$h(r) = \phi(\|r - r_i\|)$$

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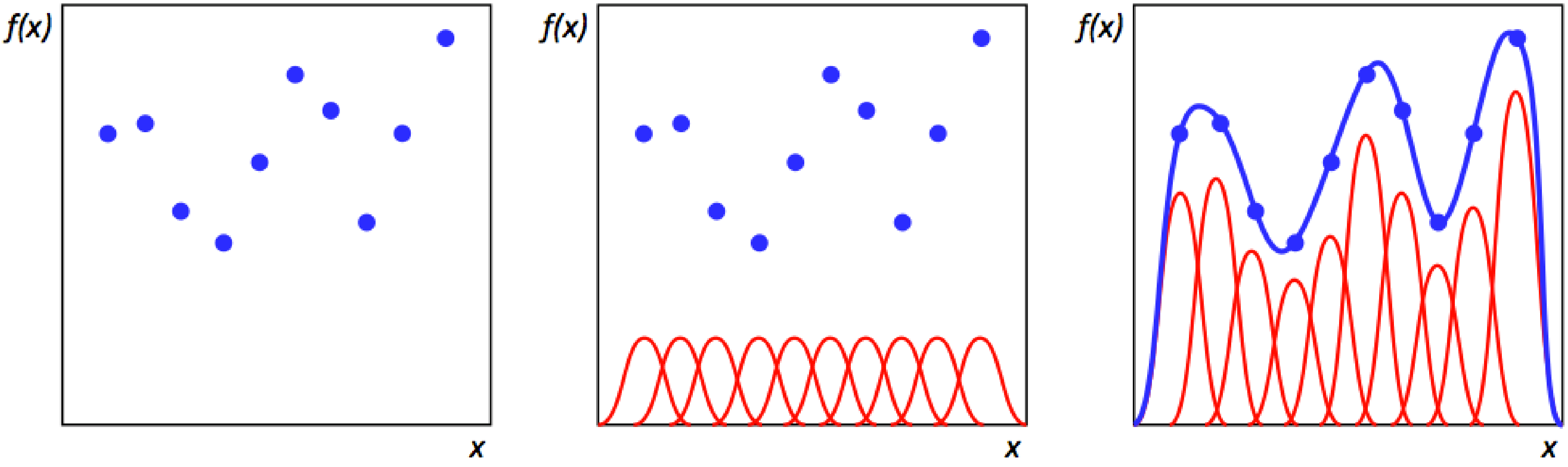

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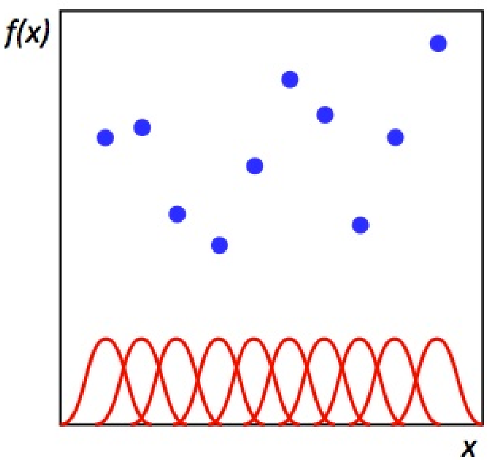
Bottom line regarding RBFs:

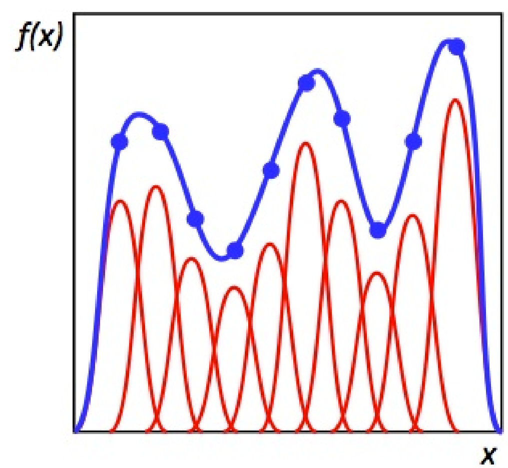


Simple to implement

Spectral accuracy

Not tied to a grid (meshless)

Independent of dimensions.



THE 1D PROBLEM

- **Given a set of interpolation data $Y = [y_i]_N$, at centers $X=[x_i]_N$, solve for the unknown expansion coefficients $A=[a_{ij}]_N$:**

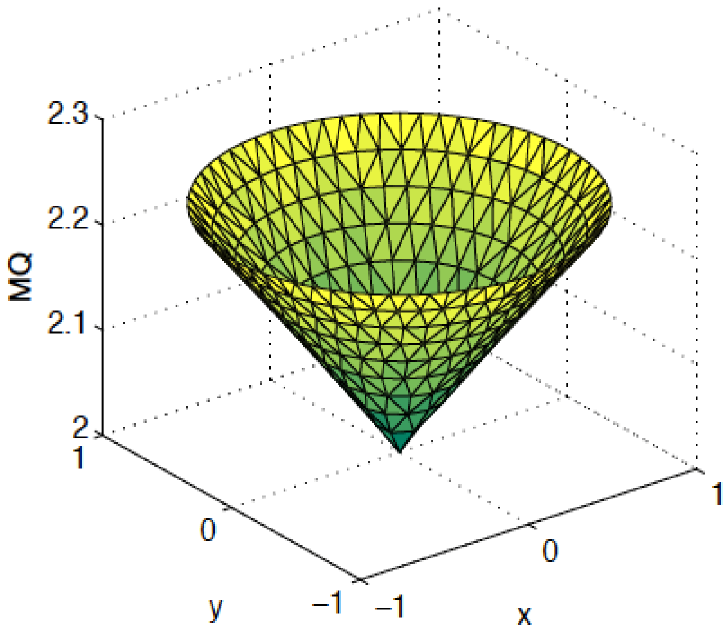
$$\begin{bmatrix} \phi(x_1-x_1) & \phi(x_1-x_2) & \dots & \phi(x_1-x_N) \\ \phi(x_2-x_1) & \phi(x_2-x_2) & & \phi(x_2-x_N) \\ \vdots & & & \vdots \\ \phi(x_N-x_1) & \phi(x_N-x_1) & \dots & \phi(x_N-x_N) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

- **$\Phi=[\varphi]_{NXN}$ Toeplitz bisymmetric, (2N+1) degrees of freedom.**
- **If uniform grid, only N degrees of freedom, Fourier coeffs.**
- **Easier to solve, numerically stable only if well-conditioned**

$$A = Y\Phi^{-1}$$

MULTIQUADRIC RBFS

$$\phi(r,\varepsilon)=\sqrt{1+(\varepsilon r)^2}$$



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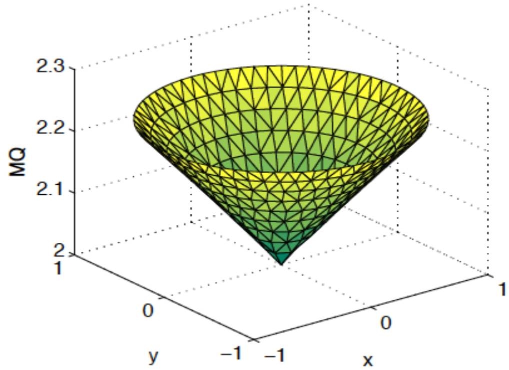

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- **Non-polynomial generalization of all global polynomial interpolation methods on structured grids.**

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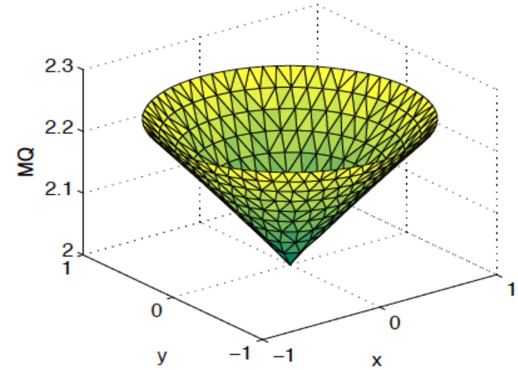

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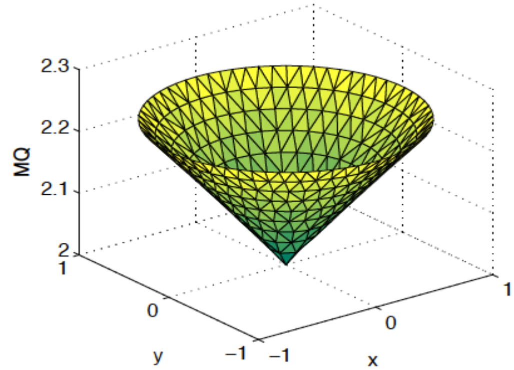
- **Non-polynomial generalization of all global polynomial interpolation methods on structured grids.**
- **In the limit $\varepsilon \rightarrow 0$, MQ-RBF reproduces all the classical pseudospectral methods, on pseudospectral grids.**
- **Continuous, infinitely differentiable.**

MULTIQUADRIC RBFS



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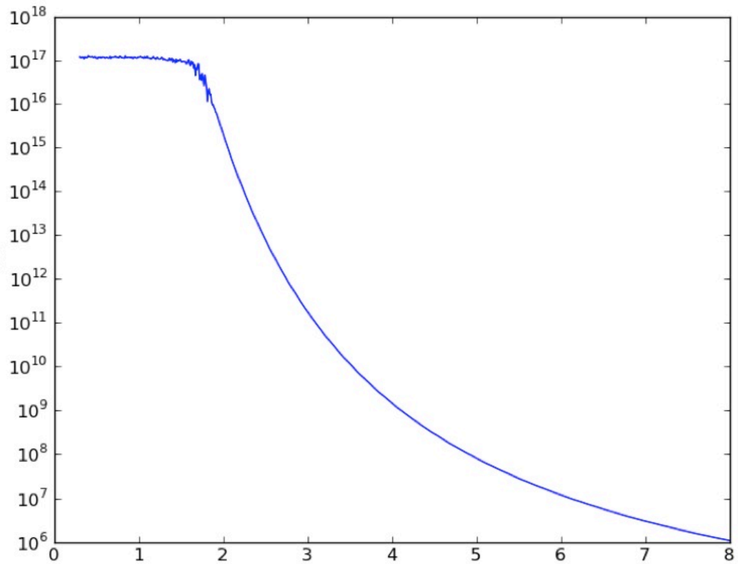
- **Non-polynomial generalization of all global polynomial interpolation methods on structured grids.**
- **In the limit $\varepsilon \to 0$, MQ-RBF reproduces all the classical pseudospectral methods, on pseudospectral grids.**
- **Continuous, infinitely differentiable.**
- ***It is not tame: ill-conditioned, nearly singular!***
- **But it is good: the best interpolant yet.**

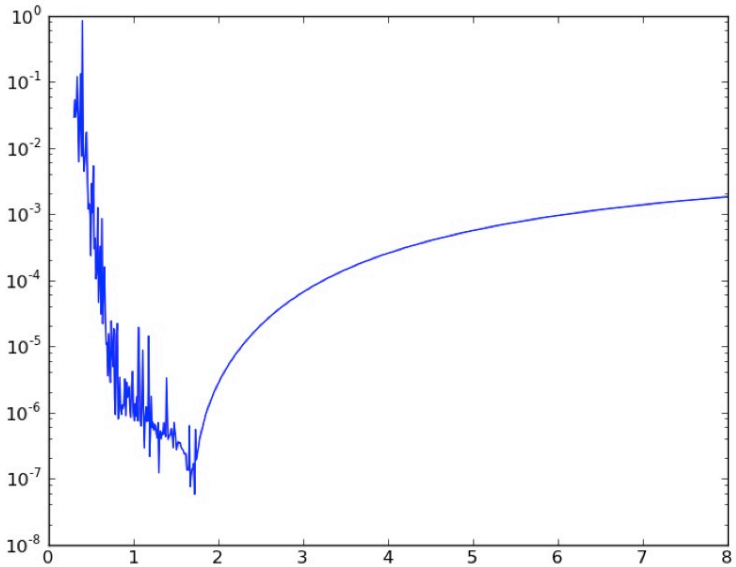
SHAPE PARAMETER TESTS

$$Cond(\Phi) = \left\|\Phi\right\| \left\|\Phi^{-1}\right\| = \left|\lambda_{max}\right| / \left|\lambda_{min}\right| = 10^{k}$$

Conditioning vs shape parameter



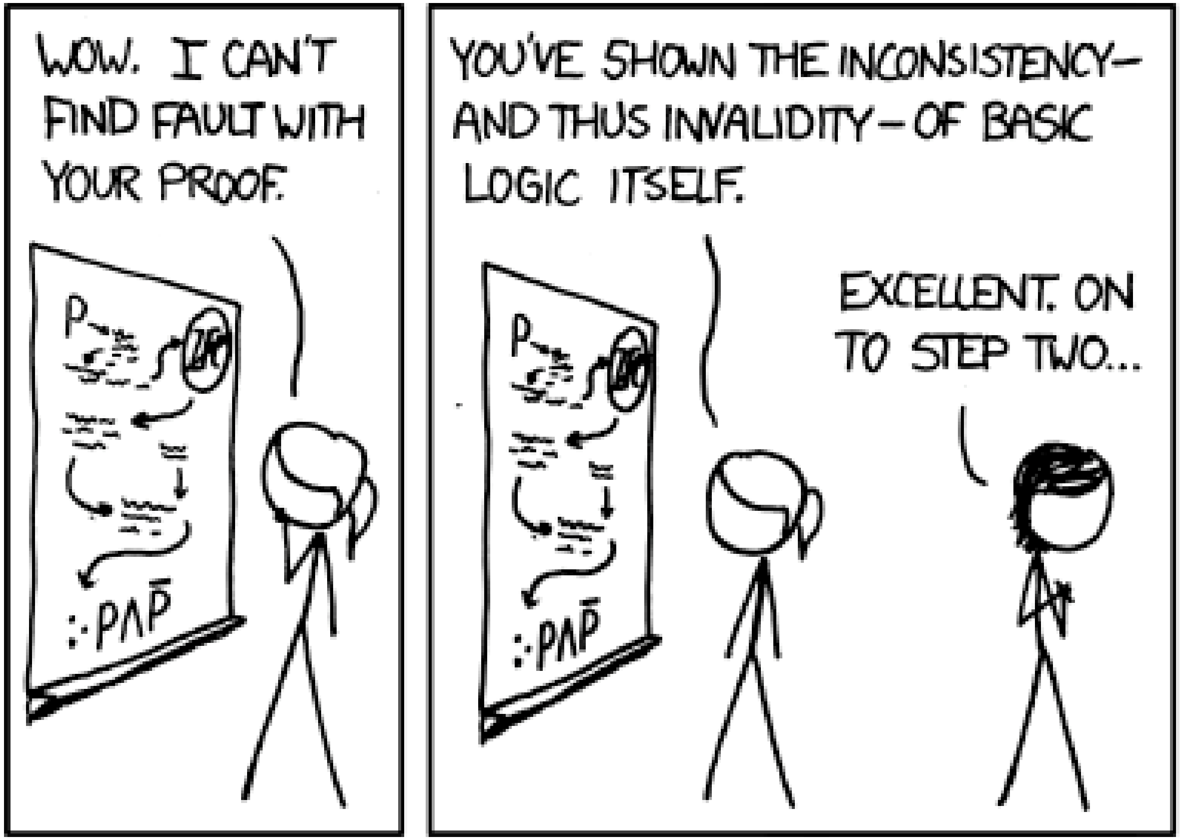
Error vs shape parameter



- **‘RBF Uncertainty principle’: accuracy vs conditioning.**
- **The higher condition number, the most accurate results!**

CRITICALLY CONDITIONING

Limited by the machine precision: $10^{15} \leq Cond(\Phi) \leq 10^{17}$



WAVE EQUATION ON 1+1 SINGLE-NULL CHARACTERISTIC GRID

- **Characteristic wave equation:**

$$\left(\partial_t^2 - \partial_r^2\right) f = 0 => 4\partial_u \partial_v f = 0;$$

- **Single-null characteristic SWE:**

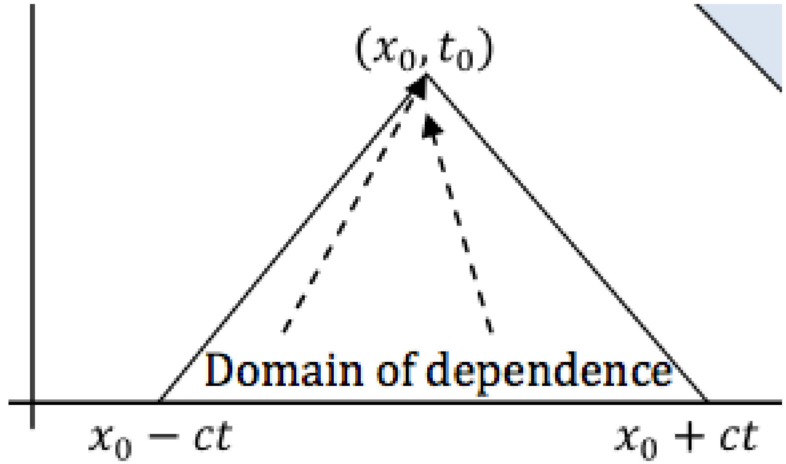
$$\partial_\xi \left(2\partial_\tau + \partial_\xi\right) f = 0; \tau = t + r, \xi = r$$

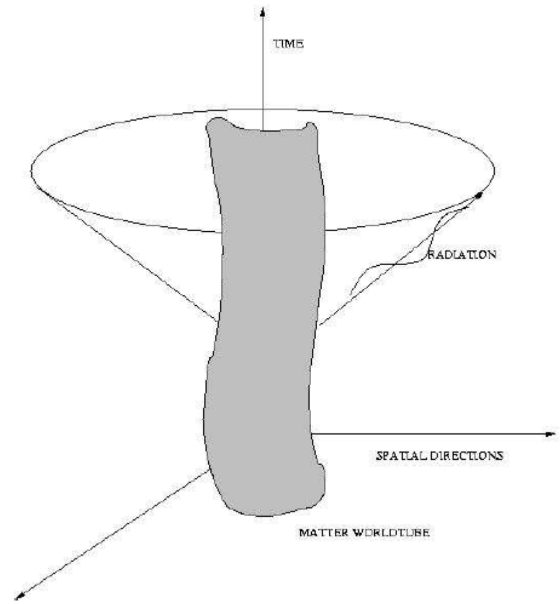
- **With compactification**

$$x = r/(1+r);$$

- **The wave equation is:**

$$\partial_x \left[2\partial_t + (1-x^2)\partial_x\right] = 0$$





THE INTEGRATION ALGORITHM

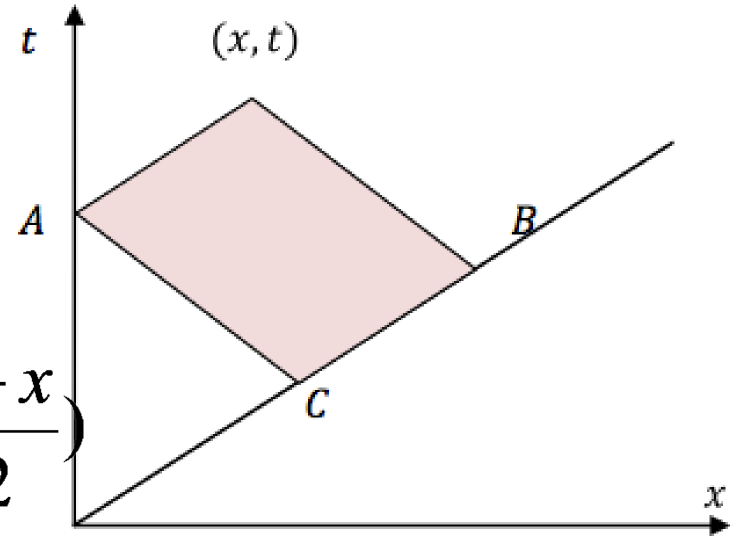
- **The parallelogram rule (d'Alembert)**

$$u(A)+u(C)=u(B)+u(D)$$

- **Winicour marching algorithm:**

$$f(x,t)=f(A)+f(B)-f(D)+IC;$$

$$A(0,t-x),\ B(\frac{t+x}{2},\frac{t+x}{2}),\ C(\frac{t-x}{2},\frac{t-x}{2})$$



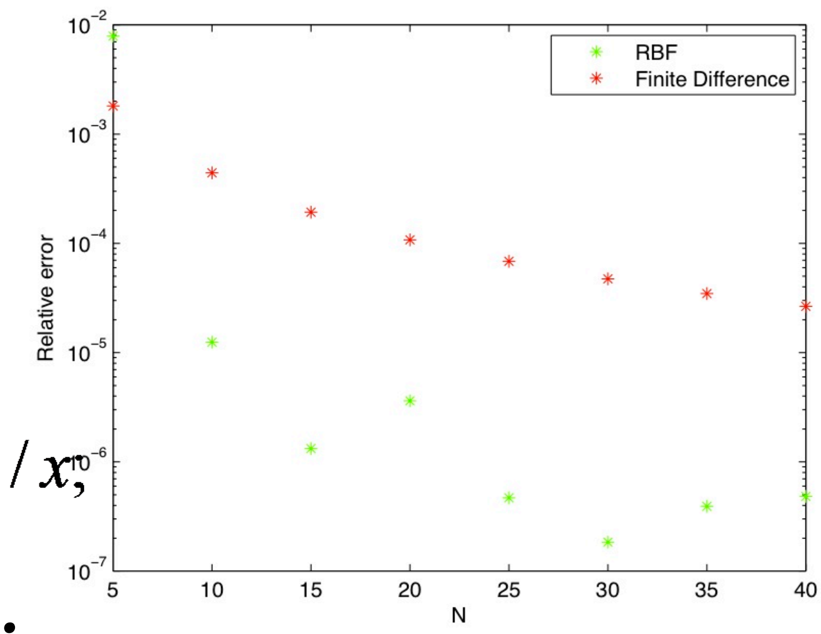
- **Interpolation points:** $x_i \pm \Delta x/2$

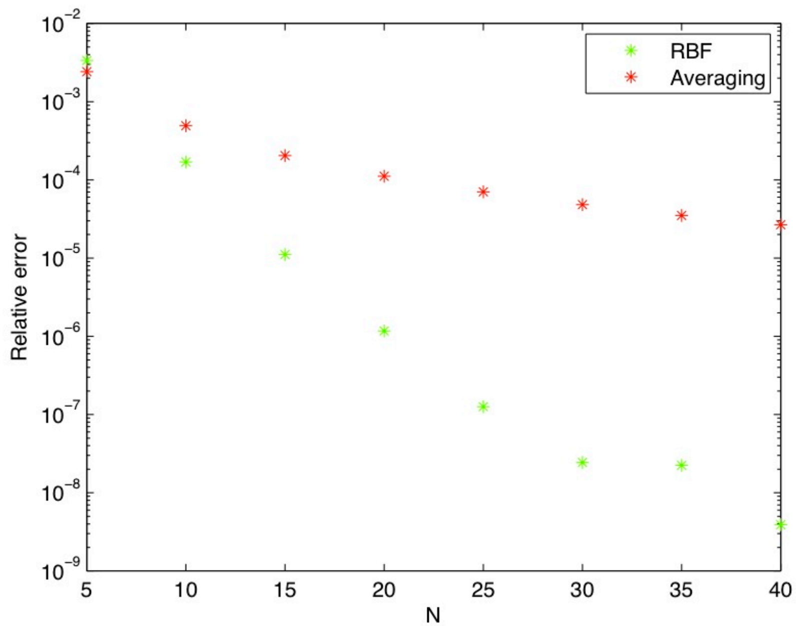
TEST RESULTS

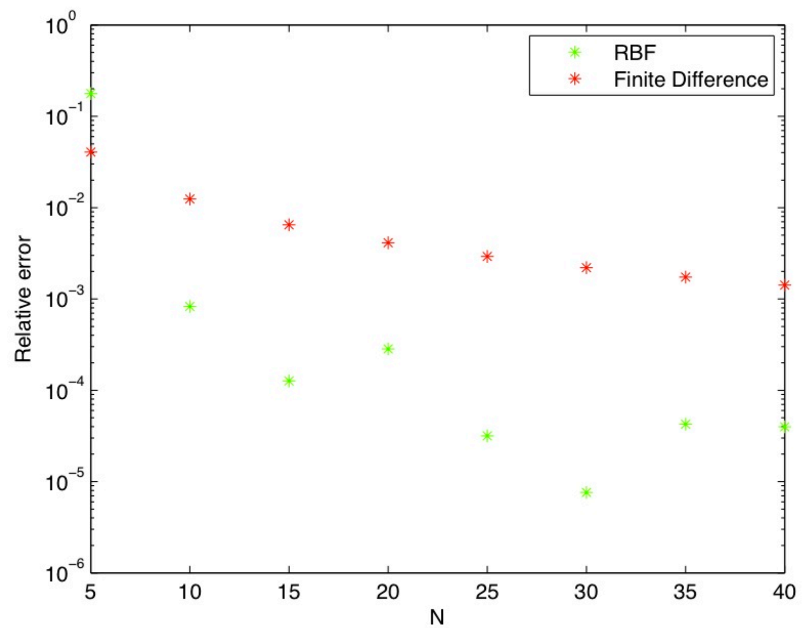
- **Initial data:**

$$f(t) = A\cos(\omega t);\ f(0,x) = f\big|_{x_E}(1-x)/x;$$

- **Shape parameter: 2.5,**
- **Number of centers: N= gridsize.**

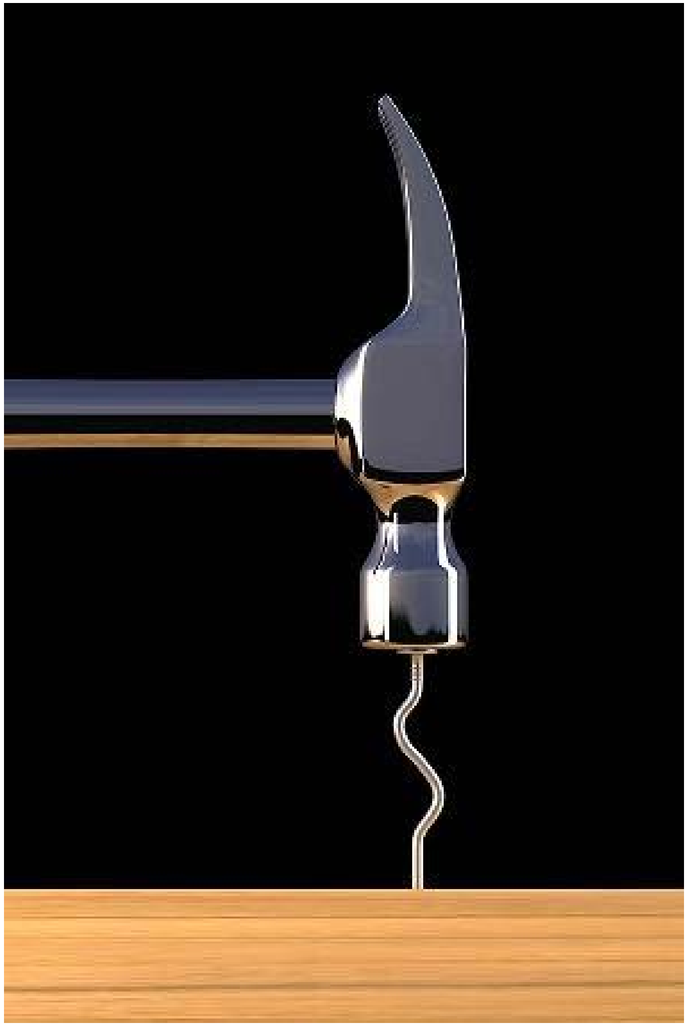






DISCUSSION

- **RBF interpolation is better than any other interpolation schemes.**
- **At the limits of floating point precision, spectral convergence lost.**
- ***Is about precision, not power!***



THE PUZZLING FUTURE

- **Shall we adopt the MQ-RBF?**
- **Explore training methods:**
  - Preconditioning and Regression (Greedy).
- **To do: Cauchy boundary and Evolution (RBF on sphere).**