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Towards a Fully Nonlinear Cauchy Characteristic Extraction

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Present Errors

- The artificial finite outer boundary present in Cauchy codes introduce two sources of error:
- The outer boundary condition,
- Waveform extraction at an inner worldtube.
- The problem of proper boundary condition for a radiating system can be solved only by extension to I^{+} (conformal compactification).
- Cauchy Characteristic Extraction (CCE) offers a means to avoid these errors.

Potential Advantages

- The potential advantages of characteristic over traditional boundary conditions are:
  - accurate waveform and polarization state at infinity,
  - computational efficiency for radiation problems in terms of both the grid domain and the computational algorithm,
  - elimination of an artificial outer boundary condition on the Cauchy problem, which eliminates contamination from back-reflection and clarifies the global initial value problem,
  - a global picture of the space-time exterior to the horizon.

The Characteristic Method

- Extends the solution to infinity using Cauchy-Characteristic Extraction (CCE):
  - Change the coordinates between the Cauchy and characteristic metric at the Cauchy-characteristic boundary,
  - Extract the characteristic data at the inner worldtube, and transforms it into boundary data for the metric describing the light cones,
  - Propagate the field and the coordinates to infinity by evolving along the outgoing light cones,
  - Cauchy evolve to the next level,
- Finally, extracts the gravitational wave at infinity, on an asymptotically inertial frame.

Characteristic Evolution

- The fundamental ingredient is a foliation by null hypersurfaces u = const. generated by a two-dimensional set of null rays, labeled x^A, with a coordinate λ varying along the rays.
- In (u, λ, x^A) null coordinates, the main set of Einstein equations, written in the Bondi formalism, decompose into a set of hierarchical hypersurface equations, which can be integrated in terms of the characteristic data for the evolution variables and prior members of the hierarchy.
- In addition, there are 4 Einstein equations, which have the physical interpretation of conservation laws.

Computational Advantages

- The initial data is free - no elliptic constraints on the initial data.
- The coordinates are very rigid - very little remaining gauge freedom.
- The constraints satisfy ordinary differential equations along the characteristics - any constraint violation falls off asymptotically as $1/r^2$.
- The main Einstein equations form a system of coupled ordinary differential equations along the characteristics - march along the characteristics.
- The grid domain is exactly the region in which waves propagate - the radiation is calculated immediately (in retarded time).

The PITT Null Code

- Implements the characteristic method of computing gravitational waves at infinity, in terms of "compactified" light cones.

$$x = \frac{r}{\rho + r}, r \to \infty \quad \textit{is located at} \quad x = 1$$

- Interior
  - The "Cauchy surface"
  - Near the source BBH
- Exterior
  - The "Characteristic"
  - Far from source BBH
- Match the two at the boundary

Characteristic Formulation

- Based on a family of outgoing null hypersurfaces, from the worldtube to infinity, in Bondi-Sachs metric:

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB} U^A U^B\right) du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B$$

$$J = \frac{1}{2} h_{AB} q^A q^B, \quad q_{AB} = \frac{1}{2}\left(q_A \bar{q}_B + \bar{q}_A q_B\right)$$

- The Einstein equations ***$G_{\mu\nu}=0$*** decompose into hypersurface, evolution and conservation equations. The evolution equation takes the form:

$$2(rJ)_{,ur} - \left(r^{-1}V(rJ)_{,r}\right)_{,r} = -r^{-1}\left(r^2 \eth U\right)_{,r} + 2r^{-1}e^{\beta}\eth^2 e^{\beta} - \left(r^{-1}V\right)_{,r} J + N_J$$

- The code implements this as a second order finite difference scheme, all angular derivatives first order.

Implementation

- Uses a standard Bondi–Sachs null coordinate system. The hypersurface equations derive from the $G_{\mu}{}^{\nu}\nabla_{\nu} u$ components of the Einstein tensor.
- Given the null data on an outgoing null hypersurface, this hierarchy of equations can be integrated radially in order to determine the Bondi metric variables on the hypersurface in terms of integration constants on an inner boundary.
- The evolution equations for the u-derivative of the null data derive from the trace-free part of the angular components of the Einstein tensor.

Code Agorithm

- Explicit second order finite difference evolution algorithm based upon retarded time steps on a uniform three-dimensional null coordinate grid,
- Handles tensor fields and their derivatives on the sphere, by incorporating a computational version of the Newman–Penrose eth-formalism.
- Data is posed on an initial null hypersurface and on a worldtube boundary, and evolve the exterior spacetime out to a compactified version of null infinity, where the waveform is computed.

Angular dissipation

- Numerical dissipation is necessary to:
  - stabilize the intergrid interpolation error,
  - suppress the circular boundary high frequency error
- The evolution equation takes the form:

$$\partial_u\left(\left(1-x\right)\Phi_{,x}+\Phi\right)=S,\quad x=r/\left(R+r\right),\quad \Phi=xJ,$$

- We introduce angular dissipation in the retarded time *u* and radial *r* evolutions:

$$\partial_u\left(\left(1-x\right)\Phi_{,x}+\Phi\right)\rightarrow\partial_u\left(\left(1-x\right)\Phi_{,x}+\Phi\right)+\varepsilon_u h^3\eth^2 W\bar{\eth}^2\partial_u\left(\left(1-x\right)\Phi_{,x}+\Phi\right)$$

$$\partial_u\left(\left(1-x\right)\Phi_{,x}+\Phi\right)\rightarrow\partial_u\left(\left(1-x\right)\Phi_{,x}+\Phi\right)+\varepsilon_x h^3\eth^2 W\bar{\eth}^2\Phi_{,u}$$

- We dissipate also the hypersurface equations.

Waveforms at null infinity

- Conformal Penrose compactification of Bondi metric:

$$l = 1/r, \quad \hat{g}_{\mu\nu} = l^2 g_{\mu\nu}$$

$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = -\left(e^{2\beta} V l^3 - h_{AB} U^A U^B\right) du^2 + 2e^{2\beta} du dl - 2h_{AB} U^B du dx^A + h_{AB} dx^A dx^B$$

- Future null infinity ***I*⁺** is at *l=0*. The Bondi mass (total energy), news *N* and Ψ_4^0 (radiation power), are constructing from expansion of metric in powers of *l*.

$$2H_{C(A} D_{B)} L^C + \partial_u H_{AB} - H_{AB} D_C L^C = O(l)$$

- H, H_{AB}, c_{AB} and L^A are expansion coefficients.
- the waveform characteristic extraction is done in null coordinates.

Calculation of the News

- In an inertial conformal Bondi frame the News are :

$$N = \lim_{\Omega \to 0} \frac{1}{2\Omega} Q^\alpha Q^\beta \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Omega$$

- where:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = \omega^2 \hat{g}_{\mu\nu}, \quad \Omega = \omega l, \quad Q_{AB} := \tilde{g}_{ab}\big|_{I^+} = \omega^2 H_{AB}$$

$$H^{AB} = \left(F^A \overline{F}^B + \overline{F}^A F^B\right)/2 \; F^A = q^A \sqrt{\frac{K+1}{2}} - \overline{q}^A J \sqrt{\frac{1}{2(K+1)}}, Q^\beta = e^{-i\delta} \omega^{-1} F^\beta + \lambda \tilde{n}^\beta$$

- An explicit calculation leads to:

$$N = \frac{1}{4} e^{-2i\delta} \omega^{-2} e^{-2H} F^\alpha F^\beta \left\{ \left(\partial_u + Ł_L\right) c_{AB} - \frac{1}{2} c_{AB} D_C L^C + 2\omega D_A \left[\omega^{-1} D_B \left(\omega e^{2H}\right)\right] \right\}$$

- In inertial Bondi coordinates: $N = \frac{1}{4} Q^A Q^B \partial_u c_{AB}$
- The general form is used, which is challenging because of second order angular derivatives of ω.

Calculation of Weyl tensor

- Weyl tensor vanishes at ***I*⁺** (asymptotic flatness)

$$\hat{\Psi} := -\frac{1}{2}\lim_{l\to 0}\frac{1}{l}\hat{n}^{\mu}\hat{\bar{m}}^{\nu}\hat{n}^{\rho}\hat{\bar{m}}^{\sigma}\hat{C}_{\mu\nu\rho\sigma} = -\frac{1}{2}\overline{\Psi}_4^{\ 0}, \quad \hat{n}^{\mu} = \hat{\nabla}^{\mu} l, \quad \hat{l}^{\mu}\partial_{\mu} = \partial_l$$

- The inertial radiation field in terms of code variables:

$$\Psi = \frac{1}{2}\omega^{-3}e^{-2i\delta}\hat{n}^{\mu}F^{A}F^{B}\left(\partial_{\mu}\hat{\Sigma}_{AB} - \partial_{A}\hat{\Sigma}_{\mu B} - \hat{\Gamma}^{\alpha}_{\mu B}\hat{\Sigma}_{A\alpha} + \hat{\Gamma}^{\alpha}_{AB}\hat{\Sigma}_{\mu\alpha}\right)\Big|_{I^+}$$

- involves lengthy algebra. In inertial Bondi coordinates

$$\Psi = \frac{1}{4}Q^{A}Q^{B}\partial_{\mu}^{2}c_{AB} = \partial_{\mu}^{2}\partial_{l}J\Big|_{I^+} = \partial_{u}N$$

- However, general form is used, which is challenging because of third order angular derivatives of ω.

Linearized Expressions

- One can require the Bondi coordinate to be inertial (Minkowsky) at ***I*****+** but it is not assumed.
- The general nonlinear representation of Ψ in terms of the computational variables reduces to a simpler form in first order perturbations off Minkowski background.

$$\Psi = \frac{1}{2}\partial_u^2 \partial_l J - \frac{1}{2}\partial_u J - \frac{1}{2}\eth L - \frac{1}{8}\eth^2\left(\eth L + \bar{\eth} L\right) + \partial_u \eth^2 H$$

$$N = \frac{1}{2}\partial_u \partial_l J + \frac{1}{2}\eth^2\left(\omega + 2H\right)$$

- ω propagates across patches

$$2\hat{n}^\alpha \partial_\alpha \log\omega = -e^{-2H} D_A L^A .$$

Circular patches

- Complex stereographic coordinates cover the sphere

$$\xi_N = q_N + ip_N = \tan(\theta/2)e^{i\varphi}, \xi_S = 1/\xi_N$$

$$F_S(\xi_S = 1/\xi_N) = F_N(\xi_N)(-1)^s e^{-2is\varphi}$$

- Unit sphere metric in each patch:

$$q_{AB}dx^A dx^B = \frac{4}{P^2}(dq^2 + dp^2), P = 1 + q^2 + p^2, q^A = \frac{P}{2}(1,i), \sqrt{q^2 + p^2} = 1$$

- All boundary points of one patch are interior points of another patch. The overlapping of the patches is key to the stability of method. The discretization is:

$$q_i = -1 + (i - O - 1)\Delta, p_j = -1 + (j - O - 1)\Delta, 1 \le i, j \le M + 1 + 2O$$

- The active finite difference grid: $\sqrt{q_i^2 + p_j^2} \le 1 + (O - R_E)\Delta$
- Stability requires that the interpolation stencil for one patch ghosts points lies below equator in other patch.

Sources of Error

- Perturbative regime tests compares favorably CCE with Zerilli extraction, and show CCE advantage at small radii.
- Nonlinear tests show CCE stable, but plagued by numerical error in the numerical postprocesing at null infinity.
- New work towards the numeric and geometric improvement of the accuracy of the waveform.

Ways to improve accuracy

- Geometrical: computation of the asymptotic of part of Ψ_4 and comparison with the news *N*.
- Numerical: improvement of intergrid interpolations between the patches smoothly covering the sphere. Comparison between:
- The circular stereographic patching,
- The cubed-sphere patching.
- Alternatives: higher order finite difference approximations, adaptive mesh refinement.

New Tests Conclusions

- All errors are second order convergent: higher order finite difference approximations might supply the accuracy needed for realistic astrophysical applications.
- Intrinsic difficulty in extracting waveforms due to the delicate cancellation of leading order terms in the metric and connections.
- The excellent accuracy for the metric suggests that perturbative waveform extraction must suffer the same difficulty: waveforms are not easy to extract accurately.

Possible Applications

- Whether the advantages of CCE prove to be significant will depend upon the results of future application in the nonlinear regime.
- Clarify the difference between EOB and NR - is it a ystematic error in the numerical amplitude due to the extraction radius, or higher order PN corrections are necessary?
- Clarify the deviation observed for the $l \neq m$ modes in the waveforms from the analytic fit model- might be caused by extracting the waveform too cose to the source, not yet in the wave zone.

Future Steps

- Revive the full extraction module, testing it against previous results with Abigel harmonic Cauchy code and the Teukolsky wave.
- Match the CCE code with the Hahndol BSSN code under Cactus (preferably implementing the Teukolsky wave?) - make it work
- Move forward, to matching CCE with a fully nonlinear BBH evolution with Hahndol.
- Work on enabling CCE to start extract at specified time, and accept mesh refinement.