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The Quotient of the beta-Weibull Distribution

Nonhle Channon Mdziniso
nonhlemdz@gmail.com

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Nonhle Channon Mdziniso

Approved by

Dr. Alfred Akinsete, Committee Chairperson
Dr. Laura Adkins
Dr. Ari Aluthge

Marshall University
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CONTENTS

ABSTRACT

The Quotient of the beta-Weibull Distribution

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A new class of distributions recently developed involves the logit of the beta distribution. Among this class of distributions are, the *beta-Normal* (Eugene *et al.* [15]); *beta-Gumbel* (Nadarajah and Kotz [18]); *beta-Exponential* (Nadarajah and Kotz [19]); *beta-Weibull* (Famoye *et al.* [6]); *beta-Rayleigh* (Akinsete and Lowe [3]); *beta-Laplace* (Kozubowshi and Nadarajah [20]); and *beta-Pareto* (Akinsete *et al.* [4]), among a few others. Many useful statistical properties arising from these distributions and their applications to real life data have been discussed in literature. One approach by which a new statistical distribution is generated is by the transformation of random variables having known distribution function(s). The focus of this work is to investigate the statistical properties of the quotient of the beta-Weibull distribution. The latter was defined and extensively studied by Famoye et al. [6]. That is, if X and Y are random variables having a beta-Weibull distribution with parameters α_1, β_1, c_1 and γ_1, and α_2, β_2, c_2 and γ_2, respectively, i.e. $X \sim \mathrm{BW}(\alpha_1,\beta_1,c_1,\gamma_1)$, and $Y \sim \mathrm{BW}(\alpha_2,\beta_2,c_2,\gamma_2)$, what then is the distribution of the quotient of X and Y? That is, the distribution of the random variable $V = \frac{X}{Y}$. We obtain the probability density function (pdf) and the cumulative distribution (cdf) of this distribution. Various statistics of the distribution are obtained, including, for example, moments, moment and characteristic generating functions, hazard function, and the entropy. We propose the method of Maximum Likelihood Estimator (MLE) for estimating the parameters of the distribution. The open source software R and Python are used extensively in implementing our results.

Chapter 1

Introduction

Distribution functions, their properties and interrelationships play a significant role in modeling naturally occuring phenomena. For this reason, a large number of distribution functions which are found applicable to many events in real life have been proposed and defined in literature. Various methods exist in defining statistical distributions. Many of these arose from the need to model naturally occuring events. For example, the Normal distribution addresses real-valued variables that tend to cluster at a single mean value, while the Poisson distribution models discrete rare events. Yet few other distributions are functions of one or more distributions. For example, a random variable T is said to have a t - distribution if T = $\frac{Z}{\sqrt{W/n}}$, where Z has the standard normal distribution, and W has the Chi-square distribution with n degrees of freedom.

A new class of distributions recently developed involves the logit of the beta distribution, where the logit for a probability p is defined as

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right) = \log(p) - \log(1-p).$$

Among this class of distributions are the *Beta-Normal* (Eugene *et al.* [15]); *Beta-Gumbel* (Nadarajah and Kotz [18]); *Beta- Exponential* (Nadarajah and Kotz [19]); *Beta-Weibull* (Famoye *et al.* [6]); *Beta-Rayleigh* (Akinsete and Lowe [3]); *Beta-Laplace* (Kozubowshi and Nadarajah [20]); and *Beta-Pareto* (Akinsete *et al.* [4]), among a few others. Many useful statistical properties arising from these distributions and their applications to real life data have been discussed in the literature. One approach by which a new statistical distribution is generated is by the transformation of random variables having known distribution function(s). Many useful properties of statistical distributions are revealed by transformations of random variable. For example, if X and Y are independent and identically distributed random variables having the gamma distribution with parameters (α, s) , and (β, s), respectively, then a random variable Z defined by Z = $\frac{X}{X+Y}$ is known to have a beta distribution with parameter α and β.

The focus of this work is to investigate the statistical properties of the quotient of the beta-Weibull distribution. The beta-Weibull distribution was defined and extensively studied by Famoye et al. [6]. If X is a random variable having a beta-Weibull distribution with parameters α_1, β_1, c_1 and γ_1, i.e. $X \sim$ BW(α_1,β_1,c_1,γ_1), and $Y \sim$ BW(α_2,β_2,c_2,γ_2), we then seek to find the distribution of the quotient of X and Y. That is, the distribution of the random variable $V = \frac{X}{Y}$. In this study, we obtain the probability density function (pdf) and the cumulative distribution (cdf) of the quotient convoluted distribution. Various statistics of this distribution are obtained, including, for example, moments, moment and characteristic generating functions, hazard function, and the entropy. We propose the method of Maximum Likelihood Estimator (MLE) for estimating the parameters of the distribution. The open source software R and Python are used extensively in implementing our results.

Chapter 2

Literature Review

The beta distribution has been widely applied as a statistical distribution to address various kinds of problems in reliability. According to Nadarajah [17], a generalized class of beta distribution has been introduced in recent years. Under this scheme, the cumulative distribution function (cdf) for the generalized class of distributions for the random variable X is generated by applying the inverse of the cdf of X to a beta distributed random variable to obtain,

$$\text{F(x)} = \frac{1}{B(\alpha,\beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1}dt; \quad 0 < \alpha, 0 < \beta.$$

The corresponding probability density function (pdf) from G(x) is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} [F(x)]^{\alpha-1}[1-F(x)]^{\beta-1}F'(x),$$

where $F'(x) = f(x)$ is the pdf of X.

We discuss, in what follows, summaries of some of the beta compounded distributions that have been defined and studied in literature.

2.1 The beta-Exponential distribution (BED)

The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability. The beta exponential distribution, defined and studied by Nadarajah and Kotz [19] , is generated from the logit of a beta random variable. In the paper, the author provides a comprehensive treatment of statistical properties of the beta exponential distribution. The paper also discusses and derives expressions for the moment generating function, characteristic function, the first four moments, variance, skewness, kurtosis, mean deviation about the mean, mean deviation about the median, R$\acute{e}$nyi Entropy, and the Shannon Entropy.

The paper proposes a generalization of the exponential distribution with the hope that it would attract wider applications in reliability. The generalization is motivated by the following general class:

If G denotes the cumulative distribution function of a random variable, then the cdf of a generalized class of distribution can be defined by

$$F(x) = I_{G(x)}(a, b);\ a > 0, b > 0,$$

where, $I_y(a, b) = \frac{B_y(a,b)}{B(a,b)}$, denotes the incomplete beta function ratio, and

$$B_y(a, b) = \int_0^y w^{a-1}(1 - w)^{b-1}dw,$$

denotes the incomplete beta function.

The author defined the beta exponential distribution by taking G to be the cdf of an exponential distribution with parameter λ. The cdf of the beta exponential distribution then becomes,

$$F(x) = I_{1-e^{(-\lambda x)}}(a, b), \qquad x > 0, a > 0, b > 0, \lambda > 0,$$

and the corresponding probability density function as obtained by Nadarajah and Kotz [19] is,

$$f(x) = \frac{\lambda}{B(a,b)} e^{(-b\lambda x)}(1 - e^{(-\lambda x)})^{a-1} \qquad a > 0, b > 0, \lambda > 0.$$

This distribution is the generalization of the exponentiated exponential distribution defined by Gupta and Kundu [16] . When $b = 1$ and $a = 1$, the beta exponential distribution reduces to the exponential distribution with parameter λ .

Besides its mathematical simplicity, when compared with other beta compounded distributions, the beta exponential distribution can be used as an improved model for the failure time data. The distribution exhibits both increasing and decreasing failure rates, and the shape of the failure rate function depends on the parameter a.

2.2 The beta-Gumbel distribution (BGD)

The Gumbel distribution is perhaps the most widely applied statistical distribution for problems in engineering. Nadarajah and Kotz [18] introduced and defined the beta-Gumbel distribution from the logit of a beta random variable. Their paper provides a comprehensive treatment of the mathematical properties of the beta-Gumbel distribution, and discusses the analytical shapes of the corresponding probability density function and the hazard rate function. Expressions for the moment generating function, variation of the skewness and kurtosis, asymptotic distribution of the extreme order statistics and estimation are also discussed in the paper.

In the essence of the logit of the beta distribution, the cumulative distribution function G(x) of the Gumbel distribution is defined by

$$G(x) = e^{-\exp\left(\frac{-(x-\mu)}{\sigma}\right)},$$

where $-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$.

Thus, the CDF of the BGD is given by

$$F(x) = I_{e^{(-u)}}(a,b) \qquad \text{where,} \qquad u = e^{\frac{-(x-\mu)}{\sigma}}.$$

The corresponding probability density function (pdf) is

$$f(x) = \frac{u}{\sigma B(a,b)} e^{-au}(1-e^{-u})^{b-1}$$

The above has the equivalent form

$$f(x) = \frac{\Gamma(a+b)}{\sigma\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k u e^{-(a+k)u}}{k!\Gamma(b-k)}$$

The beta-Gumbel distribution allows for greater flexibility of its tail, which enables some real life problems with tail features to be analyzed more accurately, leading to a better estimation and prediction of parameters.

2.3 The beta-Rayleigh distribution (BRD)

According to Akinsete and Lowe [3], the problem of estimating the reliability of components is of utmost importance in many areas of research, for example, in medicine, engineering and control systems. If X represents a random strength capable of withstanding a random amount of stress Y in a component, the quantity $R = P(Y < X)$ measures the reliability of the component. In the paper, the authors defined and studied the beta-Rayleigh distribution (BRD), and obtained a measure of reliability when both X and Y have the beta-Rayleigh distribution. Some properties of the BRD are discussed in the paper, including for example, special cases of the distribution, moments, and parameter estimation.

By taking $F(x)$ as the cdf of the beta-Rayleigh distribution, the pdf for the beta Rayleigh distribution can be written as

$$g(x) = \frac{x}{\sigma^2 B(\alpha,\beta)} e^{\frac{-x^2\beta}{2\sigma^2}} (1 - e^{\frac{-x^2}{2\sigma^2}})^{\alpha-1}; \quad x \geq 0.$$

Using the relationship between the incomplete beta function and the Gauss Hypergeometric function, the cdf for the BRD can be expressed as

$$G(x) = 1 - \frac{e^{\frac{-\alpha x^2}{2\sigma^2}}}{\alpha B(\alpha,\beta)} \; {}_2\mathrm{F}_1(\alpha, 1-\beta; 1+\alpha; e^{\frac{-\alpha x^2}{2\sigma^2}})$$

where ${}_2\mathrm{F}_1(a, b; c; z)$ is the second order hypergeometric cdf function.

The distribution is used in calculating the measure of reliability, which is vital in many fields requiring safety. The reliability measure obtained from the BRD is seen to generalize the known Rayleigh reliability measure, and addresses more cases of reliability measures.

2.4 The beta-Laplace distribution (BLD)

Motivated by the work of Eugene *et al.* [15], Kozubowski and Nadarajah [20] introduced the beta Laplace distribution generated from the logit of a beta random variable. The basic theoretical properties of the distribution are diiscussed, including, for example, modality and concavity of the density, moments and related parameters, and stochastic representations that aid in random variate generation from the model.

By the usual method of the logit of the beta distribution, and using the cumulative distribution function of the Laplace distribution, the pdf of the beta-Laplace distribution is given by

$$f_{a,b}(x) = \left(\tfrac{1}{2}\right)^{a+b+1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \begin{cases} e^{ax}(2 - e^{x})^{b-1} & \text{if } x \leq 0, \\ e^{-bx}(2 - e^{-x})^{a-1} & \text{if } x > 0. \end{cases}$$

2.5 The beta-Pareto distribution (BPD)

According Akinsete et al. [4], the family of the Pareto distribution is well known in literature for its capability in modeling the heavy-tail distribution, such as the data on income distribution, city population size, and size of companies. Some other quantities measured in the physical, biological, technological and social systems of various kinds have been found to follow the Pareto distribution.

Different types of Pareto distributions and their generalizations exist in literature. In the paper by Akinsete *et al.* [4], a four-parameter beta-Pareto distribution is generated and studied. Some properties are discussed in the paper, including the unimodality of the distribution, the unimodal or decreasing hazard rate, the expressions for the mean, mean deviation, variance, skewness, kurtosis, Renyi and Shannon entropy, maximum likelihood estimates of the parameters and applications to real life data.

A random variable Y is said to have the Pareto distribution with parameters k and θ if its probability density function is given as

$$g(y) = \frac{k\theta^k}{y^{k+1}}\ ; \quad k > 0, \theta > 0, y \geq \theta.$$

The Pareto distribution is skewed to the right and characterized by a shape parameter k and a scale parameter θ. The cdf $f(y)$ is a decreasing function and achieves its maximum when y is smallest.
The probability density function of the beta-Pareto distribution is given in Akinsete *et al.* [4] as

$$f(x) = \frac{k}{\theta B(\alpha,\beta)}(1-(x/\theta)^{-k})^{\alpha-1}(x/\theta)^{-k\beta-1}; \quad x \geq 0, \alpha, \beta, \theta, k > 0.$$

2.6 The beta-Weibull distribution (BWD)

The Weibull distribution has a wide range of applications in many fields of studies. One generalization of the Weibull distribution is the beta-Weibull distribution, defined by Famoye *et al.* [6].

The authors discussed some properties of the four-parameter beta-Weibull distribution. The distribution is shown to have a bathtub, unimodal, increasing, and decreasing hazard functions. The distribution is applied to censored data sets on bus-motor failures, a censored data set on head-and-neck-cancer clinical trial, and also to survival data.
By taking $G(x)$ to be the cumulative distribution function of a Weibull random variable X, the corresponding probability density function for the beta-Weibull random variable is expressed as:

$$f(x) = \frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{c}{\gamma} (x/\gamma)^{c-1} (1 - e^{-(x/\gamma)^c})^{\alpha-1} e^{-\beta(x/\gamma)^c},$$

where, $x > 0, \alpha > 0, \beta > 0, c > 0, \gamma > 0.$

We discuss in the following section, the quotient of two beta-Weibull distributed random variables. Various properties of this distribution are obtained.

Chapter 3

The quotient of the beta-Weibull distribution

3.1 Definition, Density and Distribution Functions

Let $f(x)$ be the pdf of a beta-Weibull random variable X. According to Famoye *et al.* [6], the pdf of the distribution is expressed as,

$$f(x) = \frac{1}{B(\alpha_1, \beta_1)} \frac{c_1}{\gamma_1} \left(\frac{x}{\gamma_1}\right)^{c_1 - 1} \left(1 - e^{-(x/\gamma_1)^{c_1}}\right)^{\alpha_1 - 1} e^{-\beta_1 (x/\gamma_1)^{c_1}}, \tag{3.1}$$

$$\alpha_1 > 0, \beta_1 > 0, c_1 > 0, \gamma_1 > 0, x > 0.$$

A random variable X with pdf expressed in Equation(3.1) is said to have the beta-Weibull distribution with parameters $\alpha_1, \beta_1, c_1, \gamma_1$. By notation, we write $X \sim BW(\alpha_1, \beta_1, c_1, \gamma_1)$. Similarly, we write $Y \sim BW(\alpha_2, \beta_2, c_2, \gamma_2)$ for the random variable Y having the beta-Weibull distribution with parameters $\alpha_2, \beta_2, c_2, \gamma_2$. Let $V = \frac{X}{Y}$ be a random variable, the quotient or ratio of the random variables X and Y. Using the transformation method, we compute the pdf of V as follows:

Let $V = \frac{X}{Y}$ and $U = Y$. Then $Y = U$ and $X = UV$.

The set $A = \{(x, y) | x > 0, y > 0\}$ with $V = \frac{X}{Y}$ and $U = Y$ maps onto

$B = \{(v,u)|v > 0, u > 0\}$. In this way, the Jacobian of the tranformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} u & v \\ 0 & 1 \end{vmatrix} = u \neq 0.$$

The joint pdf of the independent random variables X and Y is given by

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) \cdot f_Y(y) \\ &= \frac{1}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)} \frac{c_1 c_2}{\gamma_1 \gamma_2} \left(\frac{x}{\gamma_1}\right)^{c_1-1} \left(\frac{y}{\gamma_2}\right)^{c_2-1} \left(1 - e^{-\left(\frac{x}{\gamma_1}\right)^{c_1}}\right)^{\alpha_1-1} \\ &\times \left(1 - e^{-\left(\frac{y}{\gamma_2}\right)^{c_2}}\right)^{\alpha_2-1} e^{-\beta_1\left(\frac{x}{\gamma_1}\right)^{c_1}} e^{-\beta_2\left(\frac{y}{\gamma_2}\right)^{c_2}} \end{aligned}$$

With $Y = U$ and $X = UV$, the joint pdf of U and V is

$$\begin{aligned} f(uv,u) &= \frac{1}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)} \frac{c_1 c_2}{\gamma_1 \gamma_2} \left(\frac{uv}{\gamma_1}\right)^{c_1-1} \left(\frac{u}{\gamma_2}\right)^{c_2-1} \left(1 - e^{-\left(\frac{uv}{\gamma_1}\right)^{c_1}}\right)^{\alpha_1-1} \\ &\times \left(1 - e^{-\left(\frac{u}{\gamma_2}\right)^{c_2}}\right)^{\alpha_2-1} e^{-\beta_1\left(\frac{uv}{\gamma_1}\right)^{c_1}} e^{-\beta_2\left(\frac{u}{\gamma_2}\right)^{c_2}} \end{aligned}$$

The marginal pdf of V is a procedure in statistical literature. By this procedure, the pdf of the random variable V is expressed by:

$$f_V(v) = \int_{-\infty}^{\infty} f(uv,u)\cdot|J|du = \int_0^{\infty} f(uv,u)\cdot|J|du$$

$$\Rightarrow f_V(v) = \int_0^{\infty} \frac{1}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\frac{c_1c_2}{\gamma_1\gamma_2}\left(\frac{uv}{\gamma_1}\right)^{c_1-1}\left(\frac{u}{\gamma_2}\right)^{c_2-1}\left(1-e^{-\left(\frac{uv}{\gamma_1}\right)^{c_1}}\right)^{\alpha_1-1}$$
$$\times\left(1-e^{-\left(\frac{u}{\gamma_2}\right)^{c_2}}\right)^{\alpha_2-1} e^{-\beta_1\left(\frac{uv}{\gamma_1}\right)^{c_1}} e^{-\beta_2\left(\frac{u}{\gamma_2}\right)^{c_2}}\cdot u\,du \tag{3.2}$$

For mathematical simplicity, let $\alpha_1 = \alpha_2 = 1$ in Equation(3.2). Then,

$$f_V(v) = \int_0^{\infty} \frac{\beta_1\beta_2c_1c_2}{\gamma_1\gamma_2}\left(\frac{uv}{\gamma_1}\right)^{c_1-1}\left(\frac{u}{\gamma_2}\right)^{c_2-1} e^{-\beta_1\left(\frac{uv}{\gamma_1}\right)^{c_1}} e^{-\beta_2\left(\frac{u}{\gamma_2}\right)^{c_2}}\cdot u\,du \tag{3.3}$$

Again, we set $c_1 = c_2 = 1$ in Equation(3.3) for simplicity. Then we have,

$$f_V(v) = \int_0^{\infty} \frac{\beta_1\beta_2}{\gamma_1\gamma_2} e^{-\beta_1\left(\frac{uv}{\gamma_1}\right)} e^{-\beta_2\left(\frac{u}{\gamma_2}\right)}\cdot u\,du = \frac{\beta_1\beta_2}{\gamma_1\gamma_2}\int_0^{\infty} ue^{-\left(\frac{\beta_1\gamma_2 v+\beta_2\gamma_1}{\gamma_1\gamma_2}\right)u}\,du \tag{3.4}$$

Let $w = \frac{\beta_1\gamma_2 v+\beta_2\gamma_1}{\gamma_1\gamma_2}$ in Equation(3.4). Then, Equation(3.4) becomes

$$f_V(v) = \frac{\beta_1\beta_2}{\gamma_1\gamma_2}\int_0^{\infty} ue^{-uw}\,du$$

$\Rightarrow$

$$f_V(v) = \frac{\beta_1\beta_2}{\gamma_1\gamma_2 w}\int_0^{\infty} uwe^{-uw}\,du \tag{3.5}$$

By definition,

$$\int_0^{\infty} uwe^{-uw}\,du = E(U),$$

for a random variable U having the exponential distribution with parameter w. This implies that

$$E(U) = \int_0^\infty uwe^{-uw}\, du = \frac{1}{w}$$

Substituting this result back into Equation(3.5) we obtain,

$$f_V(v) = \frac{1}{w} \cdot \frac{\beta_1\beta_2}{\gamma_1\gamma_2} \cdot \frac{1}{w}$$

Therefore

$$f_V(v) = \frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2}, \tag{3.6}$$

where $\beta_1 > 0, \beta_2 > 0, \gamma_1 > 0, \gamma_2 > 0$, and $v > 0$

Equation (3.6) is the pdf of the random variable V, the quotient of two independent, beta-Weibull distributed random variables. We say that V has a four parameter quotient beta-Weibull distribution (QBWD), with parameters $\beta_1, \beta_2, \gamma_1, \gamma_2$, and write, for notational purposes; $V \sim$QBWD$(\beta_1, \beta_2, \gamma_1, \gamma_2)$.

This is a special case of QBWD$(\alpha_1, \alpha_2, \beta_1, \beta_2, c_1, c_2, \gamma_1, \gamma_2)$ with $c_1 = c_2 = \alpha_1 = \alpha_2 = 1$. In that case, QBWD$(1, 1, \beta_1, \beta_2, 1, 1, \gamma_1, \gamma_2)$ = QBWD$(\beta_1, \beta_2, \gamma_1, \gamma_2)$.

Theorem 1. *Let V be a non-negative random variable having the quotient beta-Weibull distribution with parameters $\beta_1, \beta_2, \gamma_1$ and γ_2. Then*

$$f_V(v) = \frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2}, \qquad \beta_1 > 0, \beta_2 > 0, \gamma_1 > 0, \gamma_2 > 0, v > 0,$$

is a legitimate probability density function.

Proof. The above requires us to show that

$$f_V(v) \geq 0 \quad \text{and} \quad \int_0^\infty f_V(v)\, dv = 1.$$

Now,

$$\int_0^\infty f_V(v)dv = \int_0^\infty \frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2} dv$$

Set $w = \beta_1\gamma_2 v + \beta_2\gamma_1$. Then $dw = \beta_1\gamma_2 dv$, $v = 0 \Rightarrow w = \beta_2\gamma_1$, and $v \to \infty \Rightarrow w \to \infty$.

This implies that

$$\begin{aligned}
\int_0^\infty f_V(v)dv &= \int_{\beta_2\gamma_1}^\infty \frac{\beta_1\beta_2\gamma_1\gamma_2}{w^2} \cdot \frac{dw}{\beta_1\gamma_2} \\
&= \beta_2\gamma_1 \int_{\beta_2\gamma_1}^\infty \frac{1}{w^2} dw \\
&= \beta_2\gamma_1 \lim_{c\to\infty} \int_{\beta_2\gamma_1}^c \frac{1}{w^2} dw
\end{aligned}$$

$$\begin{aligned}
&= \beta_2\gamma_1 \lim_{c\to\infty} \left[\frac{-1}{w}\right]_{\beta_2\gamma_1}^c \\
&= \beta_2\gamma_1 \lim_{c\to\infty} \left[\frac{-1}{c} + \frac{1}{\beta_2\gamma_1}\right] \\
&= \beta_2\gamma_1 \left[0 + \frac{1}{\beta_2\gamma_1}\right] = 1.
\end{aligned}$$

This concludes the proof, showing that $f_V(v)$ is a pdf. □

By definition, the cumulative distribution function of a random variable V is defined as $F(v) = P(V \leq v)$. Using Equation (3.6), the cdf of the quotient beta-Weibull distribution becomes,

$$F(v) = \int_0^v f(t)dt$$

$$= \int_0^v \frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 t + \beta_2\gamma_1)^2} dv$$

$$= \beta_2\gamma_1 \int_0^v \frac{\beta_1\gamma_2}{(\beta_1\gamma_2 t + \beta_2\gamma_1)^2} dt$$

$$= \beta_2\gamma_1 \left[\frac{-1}{\beta_1\gamma_2 t + \beta_2\gamma_1}\right]_0^v$$

$$= \beta_2\gamma_1 \left[\frac{-1}{\beta_1\gamma_2 v + \beta_2\gamma_1} + \frac{1}{\beta_2\gamma_1}\right]$$

$$F(v) = 1 - \left[\frac{\beta_2\gamma_1}{\beta_1\gamma_2 v + \beta_2\gamma_1}\right] = \frac{\beta_1\gamma_2 v}{\beta_1\gamma_2 v + \beta_2\gamma_1} \tag{3.7}$$

We immediately see from Equation(3.7) that $\lim_{v\to\infty} F(v) = 1$.

The graph of the cdf of the quotient beta-Weibull distribution is shown in Figure (3.1).

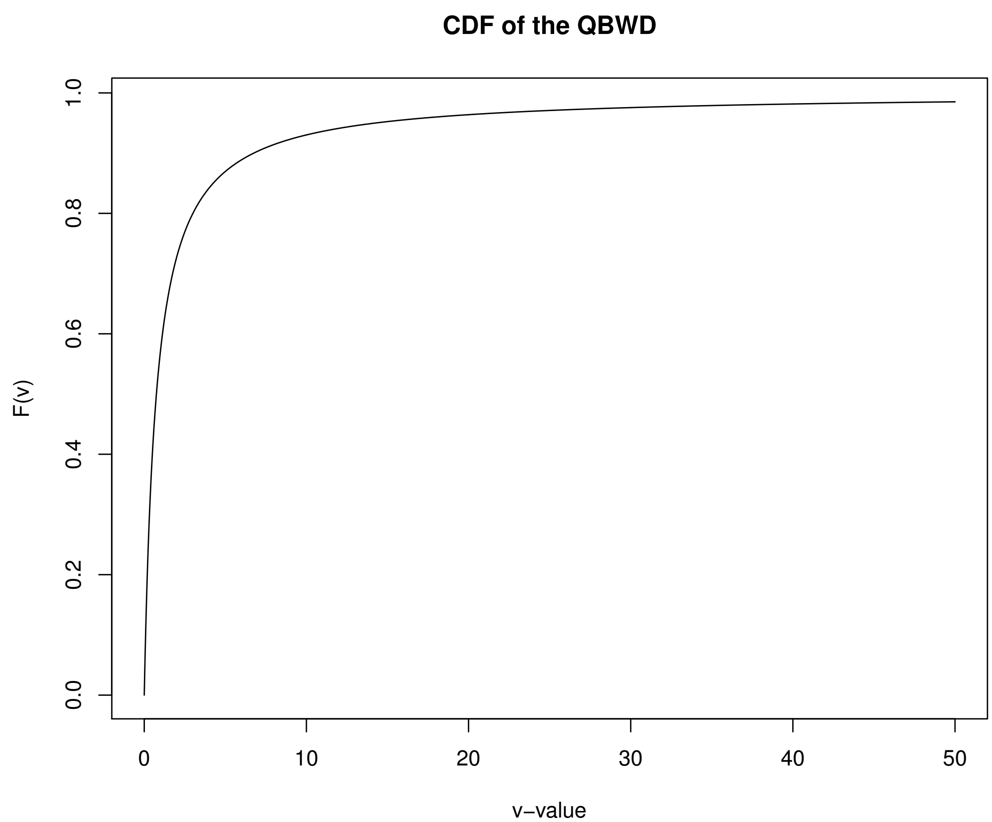

Figure 3.1: The CDF of the QBWD with parameters ($\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2\gamma_2 = 4$)

3.2 Shape

The shape of the QBWD is investigated in this section under various parameter values. Again, recall that the pdf of a QBWD with parameters $\beta_1 > 0, \beta_2 > 0, \gamma_1 > 0, \gamma_2 > 0$ is given as

$$f_V(v) = \frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2}. \tag{3.8}$$

The behavior of $f(v)$ given $\beta_1, \beta_2, \gamma_1$, and γ_2 with $v \to 0$ or $v \to \infty$ is as follows;

$f(v)$ has a vertical asymptote given by $v = -\frac{\beta_2\gamma_1}{\beta_1\gamma_2}$.

We see that

$$\lim_{v \to -\left(\frac{\beta_2\gamma_1}{\beta_1\gamma_2}\right)} f(v) = \infty, \quad \forall \beta_1 > 0, \beta_2 > 0, \gamma_1 > 0, \gamma_2 > 0, \quad \text{and}$$

$$\lim_{v \to \infty} f(v) = 0, \quad \forall \beta_1 > 0, \beta_2 > 0, \gamma_1 > 0, \gamma_2 > 0.$$

The first and second derivatives of Equation(3.8) with respect to v are

$$f'(v) = \frac{-2\beta_1^2\beta_2\gamma_1\gamma_2^2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^3}, \quad \text{and} \tag{3.9}$$

$$f''(v) = \frac{6\beta_1^3\beta_2\gamma_1\gamma_2^3}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^4}. \tag{3.10}$$

The possible extreme values of $f(v)$ are given by the roots of $f'(v) = 0$:

$$f'(v) = 0 \Rightarrow \frac{-2\beta_1^2\beta_2\gamma_1\gamma_2^2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^3} = 0 \Rightarrow -2\beta_1^2\beta_2\gamma_1\gamma_2^2 = 0, \tag{3.11}$$

which shows that the function $f(v)$ has no extreme values.

The possible points of inflection result from $f''(v) = 0$:

$$f''(v) = 0 \Rightarrow \frac{6\beta_1^3\beta_2\gamma_1\gamma_2^3}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^4} = 0 \Rightarrow 6\beta_1^3\beta_2\gamma_1\gamma_2^3 = 0 \tag{3.12}$$

which also shows that the function $f(v)$ has no points of inflection.

By choosing different values for the parameters $\beta_1, \beta_2, \gamma_1$, and γ_2 in the distribution, corresponding shapes of the distribution are shown in Figure 3.2.

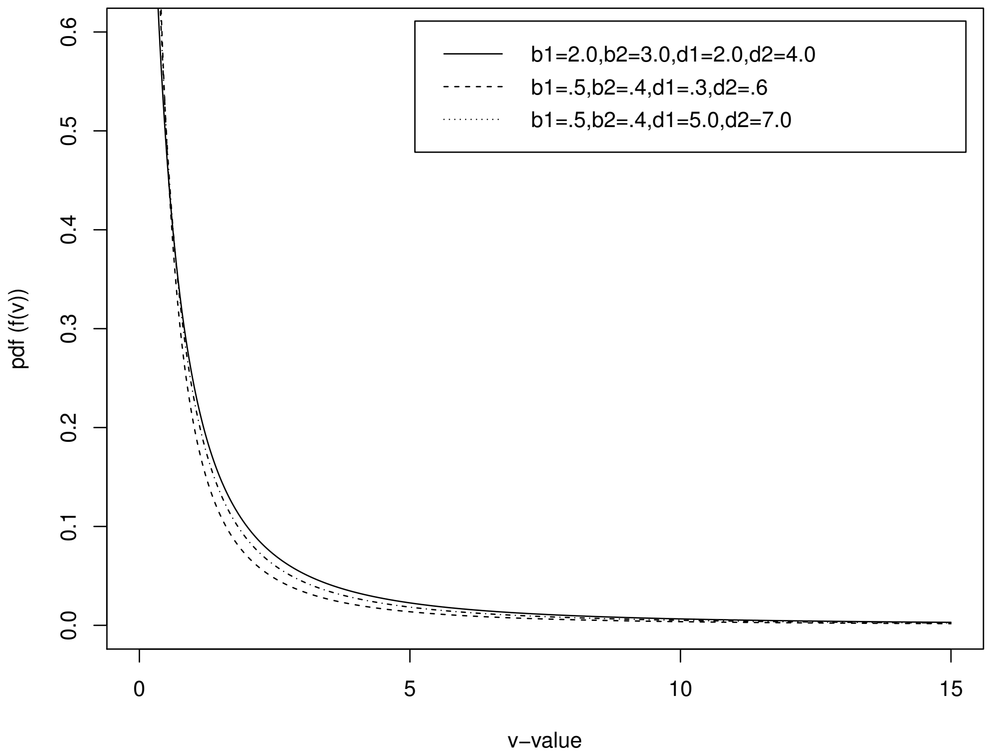

Figure 3.2: The graph of the pdf of the QBWD with various values of the parameters

From the graphs of the distribution, we note that all the parameters of the QBWD are location parameters. Changing the values of the parameters changes the heaviness of the tail of the function. We also noticed the following about the variation of the location parameters $\beta_1, \beta_2, \gamma_1$ and γ_2. The vertical asymptote is $v = -1$, if $\beta_2\gamma_1 = \beta_1\gamma_2$. The vertical asymptote is $v < -1$, if $\beta_2\gamma_1 > \beta_1\gamma_2$. This will result in the graph of the pdf shifting towards the left horizontally. The vertical asymptote is $v > -1$, if $\beta_2\gamma_1 < \beta_1\gamma_2$. This will result in the graph of the pdf shifting towards the right horizontally. However, the vertical asymptote is always $v = k$, k being a negetive constant, and $v \neq 0$ since $\beta_1 > 0, \beta_2 > 0, \gamma_1 > 0, \gamma_2 > 0$.

Moreover, from the analysis of the graph of the pdf of the QBWD we make the following conclusions;

- The QBWD has a heavy right tail and therefore is positively skewed.
- It is a heavy tailed distribution, meaning that a random variable following the QBWD can have extreme values.
- The modal value of the QBWD is also the minimum value of the random variable V, where $V \sim$QBWD$(\beta_1, \beta_2, \gamma_1, \gamma_2)$. From the graph of the pdf, we see that the modal value occurs at $v = 0$, since $V = \{v | v > 0\}$.

From the shape of the pdf of the QBWD, we can see that this distribution can be applied to many situations in which an equilibrium is found in the distribution of the "small" to the "large". An example would be in describing the allocation of wealth which is defined by the Pareto principle or the "80-20 rule" which says that 20% of the population controls 80% of the wealth. According to John Reh [7], it can be shown that from a pdf graph of the population $f(v)$, the probability, or fraction, of $f(v)$ that own a small amount of wealth per person is high. The probabitlity then decreases steadily as wealth increases.
Another example would be in modelling the number of students taking upper-level math courses. Suppose that we monitor a large group of students in a certain university over an eight-year period starting from their first year in college. Let X represent the level of the math course, and $f(x)$ be its probability distribution, or fraction of students who are taking an X-level math course. We would notice that probability for students taking the 100-level math courses is high, mainly because of the requirements in the different majors, which then decreases as the level of the Math courses increases.

Another example is the distribution of the length of the programs for the number of jobs assigned to supercomputers, consisting of few large ones and many small ones.

3.3 Asymptotic behavior

The asymptotic properties of the quotient beta-Weibull distribution are investigated by considering the behaviors of $\lim_{v\to 0} f(v)$ and $\lim_{v\to\infty} f(v)$ as follows.
Considering the situation when $v \to 0$ and $v \to \infty$ in Equation(3.6), we have,

$$\lim_{v\to 0} f(v) = \lim_{v\to 0}\left(\frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2}\right) = \frac{\beta_1\gamma_2}{\beta_2\gamma_1},$$

and

$$\lim_{v\to\infty} f(v) = \lim_{v\to\infty}\left(\frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2}\right) = 0,$$

and these analytical results agree with the numerical results from the shape of the graph of the pdf $f(v)$.

3.4 Relationship with other distributions

3.4.1 The beta-Weibull distribution(BWD)

Let $f(x)$ be the pdf of a beta-Weibull random variable X. According to Famoye *et al.* [6], the pdf of the distribution is given by,

$$f(x) = \frac{1}{B(\alpha,\beta)}\frac{c}{\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\left(1 - e^{-(x/\gamma)^c}\right)^{\alpha-1} e^{-\beta(x/\gamma)^c}, \tag{3.13}$$

$$\alpha > 0, \beta > 0, c > 0, \gamma > 0, x > 0.$$

We compare the graph of the pdf of the BWD with parameters $\alpha = \beta = c = \gamma = 1$, and the graph of the QBWD with parameters $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$. The graphs are shown in Figure 3.3.

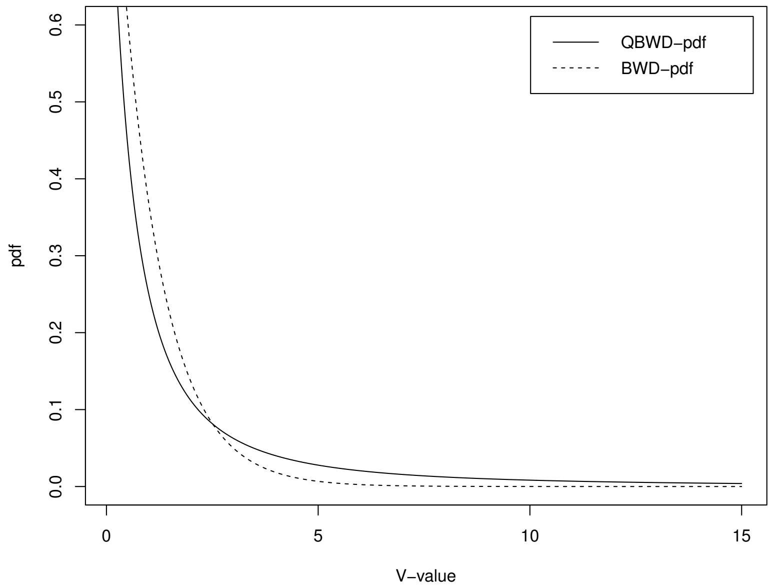

Figure 3.3: The pdf of the QBWD with parameters ($\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$) and the BWD with parameters ($\alpha = \beta = c = \gamma = 1$)

From the graph, we see that the BWD has the same shape as the QBWD with the given parameters.

3.4.2 The Pareto Distribution

According to Balakrishnan *et al.* [14], the pdf of a random variable X distributed as the Pareto distribution is given by

$$f(x) = \begin{cases} \frac{\alpha h^{\alpha}}{(x-a)^{\alpha+1}} & \text{if } x \geq a+h, \\ 0 & \text{if } x < a+h, \end{cases}$$

where $-\infty < a < \infty$, $h > 0$.

The QBWD has the same pdf as the Pareto distribution when $\alpha = 1$, $a = -\beta_2\gamma_1$ and $h = \beta_1\beta_2\gamma_1\gamma_2$. We compare the graph of the pdf of the Pareto distribution with parameters $\alpha = 1, a = -6, h = 6$, and the graph of the QBWD with parameters $\beta_1 = 2, \beta_2 = 3, \gamma_1 =$

$2, \gamma_2 = 0.5$. The graphs are shown in figure 3.4.

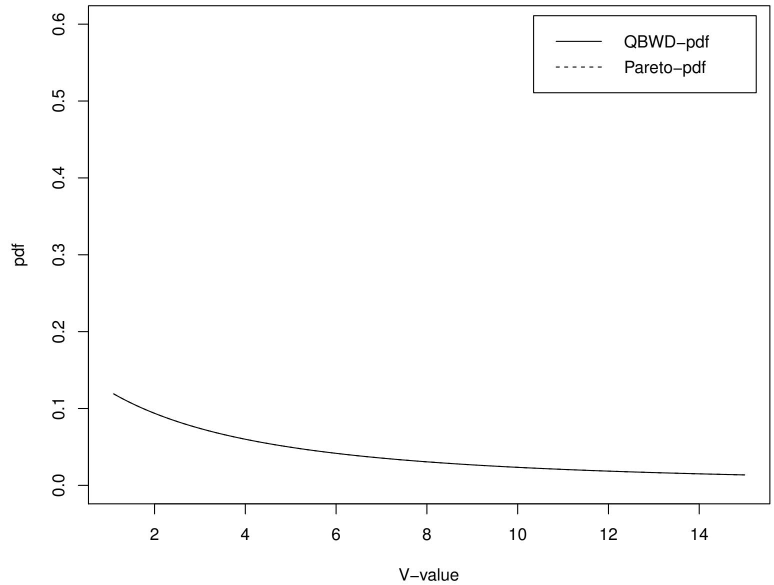

Figure 3.4: The pdf of the QBWD and the pdf of the Pareto distribution

3.4.3 The Webull-Pareto distribution(WPD)

According to Alzaatreh [5], the pdf of a random variable $X \sim WPD(c, \beta, \theta)$ distributed as the Weibull-Pareto distribution is given by

$$f(x) = \frac{\beta c}{x}\left(\beta \log\left(\frac{x}{\theta}\right)\right)^{c-1} \exp\left(-\left(\beta \log\left(\frac{x}{\theta}\right)\right)^{c}\right)$$

$$x > \theta, c > 0, \beta > 0, \theta > 0.$$

The shape of the pdf of the QBWD is the same as the shape of the pdf of the WPD with parameters $c = 0.5, \beta = 1, \theta = 1$. In fact, the QBWD has the same pdf as the WPD with parameters $c = 0.5, \beta = 1, \theta = 1$ when the QBWD has parameters $\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2\gamma_2 = 4$ on the interval $(3, \infty)$.The graphs are shown in Figure 3.5.

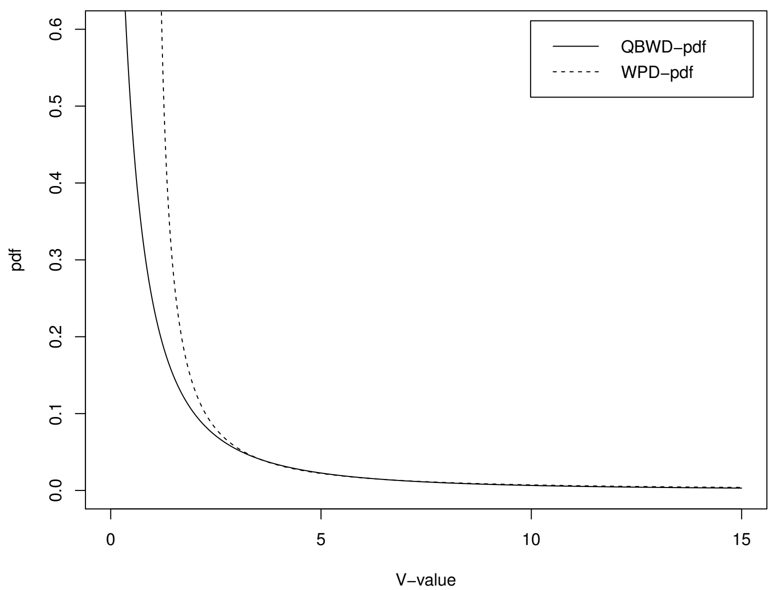

Figure 3.5: The pdf of the QBWD with parameters ($\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2, \gamma_2 = 4$) and the WPD with parameters ($c = 0.5, \beta = 1, \theta = 1$)

Chapter 4

The Hazard Function

The hazard function is defined as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. Specifically, it is computed as the number of failures per unit time in the respective interval, divided by the average number of surviving cases at the mid-point of the interval.

Mathematically, the hazard function for a random variable X is defined as

$$h(x) = \frac{f(x)}{1 - F(x)},$$

with $h(x) \geq 0$ and $\int_0^\infty h(x)dx = \infty$.

Hence the hazard function associated with the QBWD from Equation (3.6) and Equation (3.7) is

$$\begin{aligned} h(v) &= \frac{f(v)}{1 - F(v)} \\ &= \frac{\frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2}}{1 - \left[1 - \frac{\beta_2\gamma_1}{(\beta_1\gamma_2 v + \beta_2\gamma_1)}\right]} \end{aligned}$$

Therefore,

$$h(v) = \frac{\beta_1\gamma_2}{\beta_1\gamma_2 v + \beta_2\gamma_1} \tag{4.1}$$

The behavior of the hazard function as v approaches zero and as v approaches ∞ is as follows:

Taking the limit of Equation(4.1) as $v \to 0$, we have

$$\lim_{v\to 0} h(v) = \lim_{v\to 0} \frac{\beta_1\gamma_2}{\beta_1\gamma_2 v + \beta_2\gamma_1} = \frac{\beta_1\gamma_2}{\beta_2\gamma_1}.$$

Taking the limit of Equation (4.1) as $v \to \infty$, we have

$$\lim_{v\to\infty} h(v) = \lim_{v\to\infty} \frac{\beta_1\gamma_2}{\beta_1\gamma_2 v + \beta_2\gamma_1} = 0.$$

We note that $h(v) \geq 0$ as desired.

From the graph, we can see that the value of the hazard function decreases as v increases. It approaches zero as v increases. The implication of this behavior explains that the quotient beta-Weibull distribution may be appropriate in modelling events where infant mortality failures are occurring.

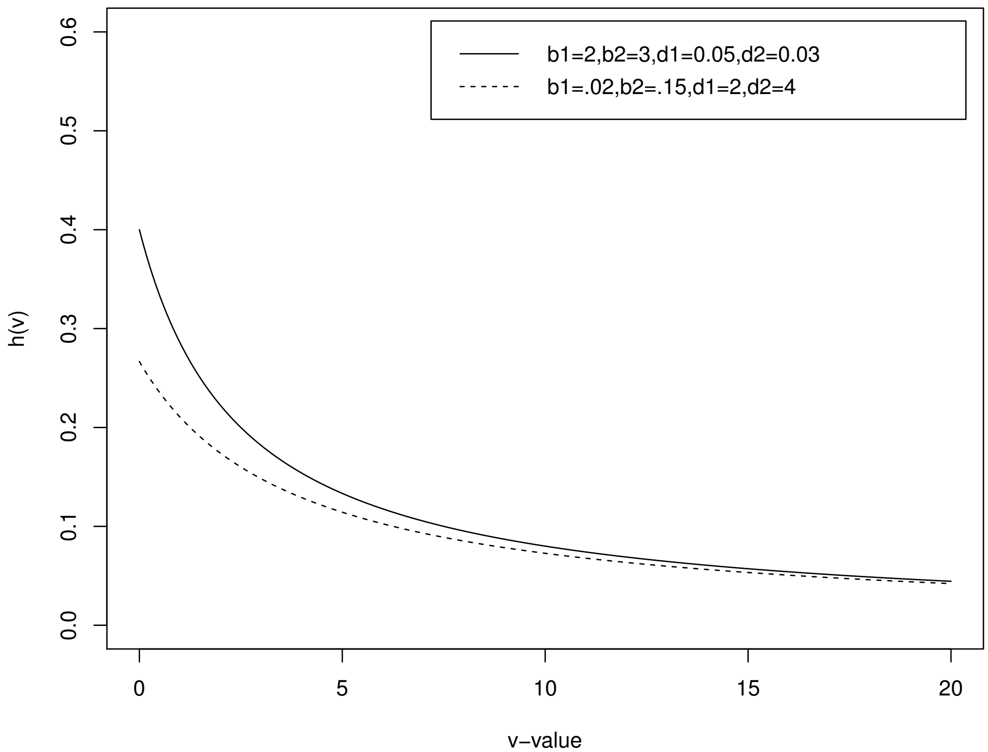

Figure 4.1: The hazard rate function of the QBWD

Chapter 5

Moments and Characteristics Functions

Moments are the expected values of certain functions of a random variable. They serve to numerically describe the variable with respect to given charateristics such as location, variation, skewness and kurtosis. Section 5.1 gives the moment generating function for a random variable with the QBWD density function. Section 5.2 gives the characteristic function and the moments, and section 5.3 gives the shape characteristics of a random variable with a QBWD density function.

5.1 Moment generating function

We derive the moment generating function for a random variable V having the QBWD density function given by Equation(3.6) as follows:
Let V be a non-negetive random variable, by definition, the moment generating function of V is given by

$$M_V(t) = E(e^{tv}) = \int_0^\infty e^{tv} f(v) dv,$$

where $|t| < 1$.

Using Equation(3.6), we have,

$$M_V(t) = \int_0^{\infty} e^{tv} f(v)dv = \beta_2\gamma_1 \int_0^{\infty} \frac{\beta_1\gamma_2 e^{tv}}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2} dv \tag{5.1}$$

In order to simplify Equation(5.1), we use a result from *Wolfram Mathematica* which is stated as follows;

$$\int_0^{\infty} \frac{ae^{tx}}{(ax+b)^2} dx = \left[\frac{te^{(-bt/a)} Ei(xt + \frac{bt}{a})}{a} - \frac{e^{tx}}{(ax+b)} \right]_{x=0}^{\infty},$$

where $Ei(z)$ is the exponential integral defined as

$$Ei(z) = C + \ln(z) + \sum_{k=1}^{\infty} \frac{z^k}{kk!}, z \neq 0,$$

and C is the *Euler-Mascheroni* constant whose value is $C \approx 0.5772156649$.

Applying this result to Equation(5.1), noting that $a = \beta_1\gamma_2$, $b = \beta_2\gamma_1$, and applying the limits from 0 to ∞ we obtain

$$= \beta_2\gamma_1 \left[\frac{t \exp(\frac{-\beta_2\gamma_1 t}{\beta_1\gamma_2}) Ei(tv + \frac{\beta_2\gamma_1 t}{\beta_1\gamma_2})}{(\beta_1\gamma_2)} - \frac{e^{tv}}{(\beta_1\gamma_2 v + \beta_2\gamma_1)} \right]_{v=0}^{\infty} = \infty,$$

which shows that the moment generating function does not exist.
To further prove this, we derive the characteristics function, the raw moments and central moments, and use this information to analyze the shape of the distribution.

5.2 Characteristics function and Moments

By definition, the characteristic function for a random variable V is defined as,

$$\phi_V(t) = E(e^{itv}) = \int_{-\infty}^{\infty} e^{itv} f(v)dv,$$

where $|t| < 1$. For the QBWD, we have,

$$\phi_V(t) = \int_0^{\infty} e^{itv} f(v)dv = \int_0^{\infty} \frac{\beta_1\beta_2\gamma_1\gamma_2 e^{itv}}{(\beta_1\gamma_2 v + \beta_2\gamma_1)^2} dv. \tag{5.2}$$

Consider a special case for which $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$ in Equation(5.2). We then have,

$$\phi_V(t) = \int_0^{\infty} \frac{e^{itv}}{(v+1)^2} dv = \int_0^{\infty} \frac{\cos(tv)}{(v+1)^2} dv + i \int_0^{\infty} \frac{\sin(tv)}{(v+1)^2} dv \tag{5.3}$$

According to Gradshteyn *et.al*([10], pages 187, 406), we have the following standard integrals:

$$\int_0^{\infty} \frac{\sin(kx)}{(x+1)^2} dx = \left[\frac{-\sin(kx)}{x+1}\right]_0^{\infty} + k \int_0^{\infty} \frac{\cos(kx)}{x+1} dx, \tag{5.4}$$

$$\int_0^{\infty} \frac{\cos(kx)}{(x+1)^2} dx = \left[\frac{-\cos(kx)}{x+1}\right]_0^{\infty} - k \int_0^{\infty} \frac{\sin(kx)}{x+1} dx, \tag{5.5}$$

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x+\beta} dx = \pi\cos(a\beta), \qquad [|arg\beta| < \pi, a > 0], \tag{5.6}$$

and

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x+\beta} dx = \pi\sin(a\beta), \qquad [|arg\beta| < \pi, a > 0]. \tag{5.7}$$

From the graphs of the integrands in Equation(5.6) and Equation(5.7), we obtain the following equations

$$\int_0^\infty \frac{\sin(ax)}{x+\beta}dx = \frac{1}{2}\pi\cos(a\beta), \qquad [|arg\beta| < \pi, a > 0] \tag{5.8}$$

$$\int_0^\infty \frac{\cos(ax)}{x+\beta}dx = \frac{1}{2}\pi\sin(a\beta), \qquad [|arg\beta| < \pi, a > 0] \tag{5.9}$$

Substituting Equation(5.8) and Equation(5.9) into Equation(5.4) and Equation(5.5), with $\beta = 1$ and $a = k$, we obtain

$$\int_0^\infty \frac{\sin(kx)}{(x+1)^2}dx = \left[\frac{-\sin(kx)}{x+1}\right]_0^\infty + \frac{k}{2}\pi\sin(k), \qquad k > 0 \tag{5.10}$$

$$\int_0^\infty \frac{\cos(kx)}{(x+1)^2}dx = \left[\frac{-\cos(kx)}{x+1}\right]_0^\infty - \frac{k}{2}\pi\cos(k), \qquad k > 0 \tag{5.11}$$

Now, we apply the identities in Equation(5.10) and Equation(5.11) to Equation(5.3), with $x = v$ and $k = t$ to obtain,

$$\begin{aligned} \phi_V(t) &= \int_0^\infty \frac{\cos(tv)}{(v+1)^2}dv + i\int_0^\infty \frac{\sin(tv)}{(v+1)^2}dv \\ &= \left[\frac{-\cos(tv)}{v+1}\right]_0^\infty - \frac{t}{2}\pi\cos(t) + i\left\{\left[\frac{-\sin(tv)}{v+1}\right]_0^\infty + \frac{t}{2}\pi\sin(t)\right\}, \\ &\quad \text{where} \quad t > 0. \end{aligned} \tag{5.12}$$

We know that $-1 \leq \cos(x) \leq 1$ and $-1 \leq \sin(x) \leq 1$ for any $x \in \Re$. In that case,

$$\begin{aligned} \left[\frac{-\cos(tv)}{v+1}\right]_0^\infty &= 0-(-1)=1, \quad \text{and} \\ \left[\frac{-\sin(tv)}{v+1}\right]_0^\infty &= 0+\frac{0}{1}=0. \end{aligned} \tag{5.13}$$

Equation(5.12) may now be written as

$$\phi_V(t) = 1 - \frac{t}{2}\pi\cos(t) + i\frac{t}{2}\pi\sin(t), \quad t > 0 \tag{5.14}$$

which is the characteristic function of the random variable $V \sim QBWD(1,1,1,1)$.

We now use the characteristic function in Equation(5.14) to compute the raw moments of order n, also known as the n^{th} moment about the origin. The n^{th}raw moment is defined in terms of the characteristic function as follows:

$$\begin{aligned} \phi^{(n)}(0) = \left[\frac{d^n\phi}{dt^n}\right]_{t=0} &= i^n \mu\prime_n = i^n E(x^n) \\ &= i\mu\prime_1 + i^2\mu\prime_2 + i^3\mu\prime_3 + i^4\mu\prime_4 + i^5\mu\prime_5 + i^6\mu\prime_6 + \cdots \\ &= i\mu\prime_1 - \mu\prime_2 - i\mu\prime_3 + \mu\prime_4 + i\mu\prime_5 - \mu\prime_6 - \cdots \end{aligned}$$

Using Equation(5.14), the raw moments are given by

$$\begin{aligned} \phi'(it) &= \frac{-\pi}{2}\cos(t) + \frac{\pi}{2}t\sin(t) + i\frac{\pi}{2}\sin(t) + i\frac{\pi}{2}t\cos(t) \\ \phi'(0) &= \frac{-\pi}{2} \\ \phi''(it) &= \frac{\pi}{2}\sin(t) + \frac{\pi}{2}\sin(t) + \frac{\pi}{2}t\cos(t) + i\frac{\pi}{2}\cos(t) + i\frac{\pi}{2}\cos(t) - i\frac{\pi}{2}t\sin(t) \\ &= \pi\sin(t) - i\frac{\pi}{2}t\sin(t) + \frac{\pi}{2}t\cos(t) + i\pi\cos(t) \\ \phi''(0) &= \pi i \end{aligned}$$

$$\begin{aligned}
\phi'''(it) &= \pi\cos(t) - i\frac{\pi}{2}\sin(t) - i\frac{\pi}{2}t\cos(t) + \frac{\pi}{2}\cos(t) - \frac{\pi}{2}t\sin(t) - i\frac{\pi}{2}\sin(t) - i\frac{\pi}{2}\sin(t) \\
&= \frac{3\pi}{2}\cos(t) - i\frac{3\pi}{2}\sin(t) - i\frac{\pi}{2}t\cos(t) - \frac{\pi}{2}t\sin(t) \\
\phi'''(0) &= \frac{3\pi}{2} \\
\phi^{(4)}(it) &= \frac{-3\pi}{2}\sin(t) - i\frac{3\pi}{2}\cos(t) - i\frac{\pi}{2}\cos(t) + i\frac{\pi}{2}t\sin(t) - \frac{\pi}{2}\sin(t) - \frac{\pi}{2}t\cos(t) \\
&= -2\pi\sin(t) - i2\pi\cos(t) + i\frac{\pi}{2}t\sin(t) - \frac{\pi}{2}t\cos(t) \\
\phi^{(4)}(0) &= -2\pi i \\
\phi^{(5)}(it) &= -2\pi\cos(t) + i2\pi\sin(t) + i\frac{\pi}{2}\sin(t) + i\frac{\pi}{2}t\cos(t) - \frac{\pi}{2}\cos(t) + \frac{\pi}{2}t\sin(t) \\
\phi^{(5)}(it) &= \frac{-5\pi}{2}\cos(t) + i\frac{5\pi}{2}\sin(t) + i\frac{\pi}{2}t\cos(t) + \frac{\pi}{2}t\sin(t) \\
\phi^{(5)}(0) &= \frac{-5\pi}{2} \\
\phi^{(6)}(it) &= \frac{5\pi}{2}\sin(t) + i\frac{5\pi}{2}\cos(t) + i\frac{\pi}{2}\cos(t) - i\frac{\pi}{2}t\sin(t) + \frac{\pi}{2}\sin(t) + \frac{\pi}{2}t\cos(t) \\
&= 3\pi\sin(t) + i3\pi\cos(t) - i\frac{\pi}{2}t\sin(t) + \frac{\pi}{2}t\cos(t) \\
\phi^{(6)}(0) &= 3\pi i,
\end{aligned}$$

and so on.

The sequence of raw moments of the distribution is

$$\phi'(0) = \frac{-\pi}{2}, \quad \phi''(0) = \pi i, \quad \phi'''(0) = \frac{3\pi}{2}, \quad \phi^{(4)}(0) = -2\pi i, \quad \phi^{(5)}(0) = \frac{-5\pi}{2}, \quad \phi^{(6)}(0) = 3\pi i, \cdots$$

In general, we may write the n^{th} moment as,

$$\phi^{(n)}(0) = \begin{cases} \frac{n\pi}{2} i^{n+1}, & \text{if} \quad \text{n is odd;} \\ \\ \frac{n\pi}{2} i^{n-1}, & \text{if} \quad \text{n is even.} \end{cases}$$

We now use the raw moments to analyze the shape of the distribution.

5.3 Shape Characteristics

We discuss the skewness and kurtosis of the QBWD in what follows:

Skewness: The skewness of a distribution is defined as the lack of symmetry. In a symmetrical distribution, the mean, median and, mode are equal to each other, and the ordinate at the mean divides the distribution into two equal parts such that one part is a mirror image of the other. The Karl Pearson's measure of skewness is based upon the divergence of the mean from the mode in a skewed distribution. Pearson's measure of skewness is given by,

$$\gamma_1 = \frac{\beta_3}{\beta_2^{(3/2)}},$$

where β_2 and β_3 are the second and third central moments.

When the distribution is symmetric about the mean; $\beta_3 = 0 \Rightarrow \gamma_1 = 0$. It can also be noted that the measure of skewness γ_1 may take on positive or negetive values depending on whether β_3 is positive or negetive, respectively. Hence, distributions with $\gamma_1 > 0$ are said to be positively skewed distributions, while those with $\gamma_1 < 0$ are said to be negetively skewed.

Kurtosis: Kurtosis is another measure of the shape of a distribution. It is a measure of the relative peakedness of its frequency curve. A measure of skewness is given by

$$\gamma_2 = \frac{\beta_4}{\beta_2^2}.$$

Based on this value, distributions with $\gamma_2 > 3$ are called leptokurtic, distributions with $\gamma_2 < 3$ are called platykurtic, while distributions with $\gamma_2 = 3$ are called mesokurtic. The latter case is similar to the bell shape of the normal distribution.

The central moments or the moments about the mean are expressed in terms of raw moments. By definition,

$$\begin{aligned}\beta_n &= E[X-E(X)]^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}E[E(X)]^K E(X^{n-k})\\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha_1^k\alpha_{n-k}, \quad \alpha_n = E(X^n) = \phi^{(n)}(0).\end{aligned}$$

In particular, we have

$$\begin{aligned}Var(X) &= \beta_2 = \alpha_2 - \alpha_1^2,\\ \beta_3 &= \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3, \quad \text{and}\\ \beta_4 &= \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4.\end{aligned}$$

Note that the first central moment $\beta_1 = 0$.

Now, for the random variable V with the quotient beta-Weibull distribution, the central moments are as follows:

$$\begin{aligned}Var(X) &= \beta_2 = \alpha_2 - \alpha_1^2 = \pi i - \left(\frac{-\pi}{2}\right)^2 \approx -2.467 + 3.412i\\ \beta_3 &= \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3 = \frac{3}{2}\pi - 3\left(\frac{-\pi}{2}\right)(i\pi) + 2\left(\frac{-\pi}{2}\right)^3\\ &\approx -3.039 + 14.804i\\ \beta_4 &= \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4\\ &= -2\pi i - 4\left(\frac{-\pi}{2}\right)\left(\frac{3\pi}{2}\right) + 6\left(\frac{-\pi}{2}\right)^2(i\pi) - 3\left(\frac{-\pi}{2}\right)^4\\ &\approx 11.345 + 40.226i\end{aligned}$$

Thus, the Pearson's measure of skewness is

$$\gamma_1 = \frac{\beta_3}{\beta_2^{(3/2)}} = \frac{-3.039 + 14.804i}{(-2.467 + 3.412i)^{(3/2)}} \approx -0.0204 - 1.893i,$$

and, Pearson's measure of kurtosis is

$$\gamma_2 = \frac{\beta_4}{\beta_2^2} = \frac{11.345 + 40.226i}{(-2.467 + 3.412)^2} \approx -2.617 + 0.093i.$$

The value of the measure of skewness obtained above implies that this distribution is negetively skewed. However, this contradicts the results obtained from the shape of the pdf of the QBWD distribution, as illustrated in the graphs in Figure 3.2. This contradiction further justifies the non-existence of the moments of the QBWD, for the specified parameter values. It would be interesting to investigate the distribution for values of parameters α and c other than $\alpha = 1 = c$. After a number of attempts, this would be either impossible analytically or, in the least, rigorous.

Chapter 6

Entropy and Asymptotic Behaviors

6.1 Rényi Entropy

The Rényi entropy of a random variable is one family of functions for quantifying the uncertainty or randomness in a system (Rényi [1]). The Rényi entropy has been used in various situations in science and engineering (Cordeiro [8]). According to Akinsete [2], the Rényi entropy is defined by,

$$R(s) = \frac{1}{1-s} \log \left[\int_{-\infty}^{\infty} f^{s}(v) dv \right], \tag{6.1}$$

where $s > 0$ and $s \neq 1$, and it is called the Rényi's information measure of order s, or Rényi's S entropy.

Rényi showed that this $R(s)$ also represents the disclosed information (or removed ignorance) after analysing the expression in a close analog with Shannon's theory (Cederlof [11]). The Shannon entropy $H(s)$ is a fundamental measure in information theory, and is a special case of Rényi's entropy. It is defined as $H(s) = \lim_{s\to 1} R(s)$. It is worth noting that, at a deeper level, Rényi's entropy measure is much more flexible due to the parameter s, enabling several measurements of uncertainty (or dissimilarity) within a given distribution.

For the pdf of the QBWD given by Equation(3.6), we have

$$
\begin{aligned}
R(s) &= \frac{1}{1-s}\log\left[\int_0^\infty \left(\frac{\beta_1\beta_2\gamma_1\gamma_2}{(\beta_1\gamma_2 v+\beta_2\gamma_1)^2}\right)^s dv\right] \\
&= \frac{1}{1-s}\log\left[\int_0^\infty \frac{\beta_1^s\beta_2^s\gamma_1^s\gamma_2^s}{(\beta_1\gamma_2 v+\beta_2\gamma_1)^{2s}}dv\right] \\
&= \frac{1}{1-s}\left[\log\left(\frac{(\beta_1\beta_2\gamma_1\gamma_2)^s}{\beta_1\gamma_2}\right)+\log\left(\int_0^\infty \frac{\beta_1\gamma_2}{(\beta_1\gamma_2 v+\beta_2\gamma_1)^{2s}}dv\right)\right] \\
&= \frac{1}{1-s}\left\{s\log(\beta_1\beta_2\gamma_1\gamma_2)-\log(\beta_1\gamma_2)+\log\left[\frac{1}{(-2s+1)(\beta_1\gamma_2 v+\beta_2\gamma_1)^{2s-1}}\right]_0^\infty\right\} \\
&= \frac{1}{1-s}\left\{s\log(\beta_1\beta_2\gamma_1\gamma_2)-\log(\beta_1\gamma_2)+\log\left[\frac{1}{(2s-1)(\beta_2\gamma_1)^{2s-1}}\right]\right\} \\
&= \frac{1}{1-s}\left\{s\log(\beta_1\beta_2\gamma_1\gamma_2)-\log(\beta_1\gamma_2)-\log\left[(2s-1)(\beta_2\gamma_1)^{2s-1}\right]\right\} \\
&= \frac{1}{1-s}\left[s\log(\beta_1\beta_2\gamma_1\gamma_2)-\log(\beta_1\gamma_2)-\log(2s-1)-(2s-1)\log(\beta_2\gamma_1)\right] \\
&= \frac{1}{1-s}\left[\log(\beta_1\beta_2\gamma_1\gamma_2)^s-\log(\beta_1\gamma_2)-\log(2s-1)-\log(\beta_2\gamma_1)^{2s-1}\right] \\
&= \frac{1}{1-s}\log\left[\frac{(\beta_1\beta_2\gamma_1\gamma_2)^s}{(\beta_1\gamma_2)(2s-1)(\beta_2\gamma_1)^{2s-1}}\right] \\
&= \frac{1}{1-s}\log\left[\frac{1}{(2s-1)}\beta_1^{s-1}\beta_2^{-(s-1)}\gamma_1^{-(s-1)}\gamma_2^{s-1}\right] \\
&= \frac{1}{s-1}\log(2s-1)+\frac{1}{1-s}\log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)^{s-1}.
\end{aligned}
$$

The above may now be written as,

$$
R(s) = \frac{1}{s-1}\log(2s-1)-\log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right), \quad s\neq 1.
$$

Therefore, Rényi's entropy for the QBWD random variable is given by

$$
R(s) = \frac{1}{s-1}\log(2s-1)-\log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right), \quad s\neq 1. \tag{6.2}
$$

To analyze the Rényi's entropy for the QBWD random variable, we consider the following three cases:

Case 1: The limit as $s \to 1$:

$$\lim_{s\to 1} R(s) = \lim_{s\to 1}\left(\frac{1}{s-1}\log(2s-1) - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\right).$$

Using the L'Hospital's rule, we have

$$\begin{aligned}\lim_{s\to 1} R(s) &= \lim_{s\to 1}\frac{2/(2s-1)}{1} - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\\ &= \lim_{s\to 1}\frac{2}{2s-1} - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\\ &= 2 - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right). \end{aligned} \tag{6.3}$$

Taking the limit as $s \to 1$ of the Rényi entropy gives the Shannon's entropy, making this a special case of the Rényi's entropy. The results obtained in Equation(6.3) can be interpreted as the needed length, in bits, of a message communicating a measurement that had probability $f(v)$. This makes Shannon's entropy a measure of the expected message length needed to communicate the measured value of a random variable(Cederlof [11]). Therefore, the greater the value of $f(v)$, the greater the value of Shannon's entropy.

Case 2: The limit as $s \to 0$:

$$\begin{aligned}\lim_{s\to 0} R(s) &= \lim_{s\to 0}\left(\frac{1}{s-1}\log(2s-1) - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\right)\\ &= \lim_{s\to 0}\left(\frac{1}{s-1}\log(2s-1)\right) - \lim_{s\to 0}\log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\\ &= -\pi i - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\end{aligned} \tag{6.4}$$

Taking the limit as $s \to 0$ of the Rényi entropy gives the logarithm of the number of nonzero components of the QBWD distribution, generally known in the literature as Hartley's entropy(Cederlof [11]).

Case 3: The limit as $s \to \infty$:

$$\begin{aligned}
\lim_{s\to\infty} R(s) &= \lim_{s\to\infty}\left(\frac{1}{s-1}\log(2s-1) - \log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right)\right) \\
&= \lim_{s\to\infty}\left(\frac{1}{s-1}\log(2s-1)\right) - \lim_{s\to\infty}\log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right) \\
&= -\log\left(\frac{\beta_1\gamma_2}{\beta_2\gamma_1}\right), \quad \beta_1\gamma_2 \le \beta_2\gamma_1
\end{aligned} \tag{6.5}$$

The limit as $s \to \infty$ of $R(s)$ can also be thought of as $\lim_{s\to\infty} R(s) = R_\infty$, with $R_\infty = -\log(\max(f(v)))$, which is called Chebyshev's entropy (Cederlof [11]). We know that $\max(f(v)) = \lim_{v\to 0} f(v) = \frac{\beta_1\gamma_2}{\beta_2\gamma_1}$.

The plots of $R(s)$ are also generated in order to study the behavior of $R(s)$.
These graphs illustrate the behavior of $R(s)$ as $s \to 0$, $s \to 1$ and $s \to \infty$. From the

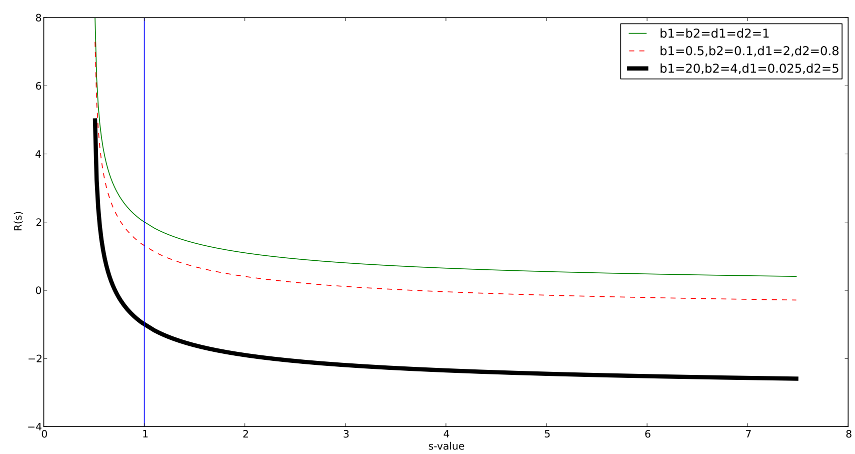

Figure 6.1: The graph of the Renyi entropy of the QBWD

diagram, we see that

$$\lim_{s\to 1^-} R(s) = 2, \quad \text{and} \quad \lim_{s\to 1^+} R(s) = 2,$$

for the graph with parameters $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$, and these numerical results agree with the analytical results computed previously in Equation(6.3).

We also note that the parameters $\beta_1, \beta_2, \gamma_1$and γ_2 are shift parameters. Increasing the value of $\frac{\beta_1\gamma_2}{\beta_2\gamma_1}$ results in a downward vertical shift of the graph of $R(s)$. On the other hand, decreasing the value of $\frac{\beta_1\gamma_2}{\beta_2\gamma_1}$ results in an upward vertical shift of the graph of $R(s)$.

Chapter 7

Parameter Estimation

In this chapter, we consider the process for estimating the parameters of the QBWD by the method of maximum likelihood estimation. Let $V_1, V_2, \ldots, V_n$ be the random sample from n independent and identically distributed random variables, each with density function given by Equation(3.6). Then the likelihood function for the random variables is defined as follows;

$$\begin{aligned} L(V_1, V_2, \ldots, V_n; \beta_1, \beta_2, \gamma_1, \gamma_2) &= \prod_{i=1}^{n} f(v_i) \\ &= \prod_{i=1}^{n} \frac{\beta_1 \beta_2 \gamma_1 \gamma_2}{(\beta_1 \gamma_2 v_i + \beta_2 \gamma_1)^2} \\ &= \frac{(\beta_1 \beta_2 \gamma_1 \gamma_2)^n}{\prod_{i=1}^{n} (\beta_1 \gamma_2 v_i + \beta_2 \gamma_1)^2} \end{aligned} \tag{7.1}$$

The values of the parameters that maximize the likelihood function also maximize the log likelihood, $\log L(V_1, V_2, \ldots, V_n; \beta_1, \beta_2, \gamma_1, \gamma_2)$. Taking the logarithm of Equation(7.1), we have

$$
\begin{aligned}
\ell &= \log L(V_1, V_2, \ldots, V_n; \beta_1, \beta_2, \gamma_1, \gamma_2) \\
&= \log\left(\frac{(\beta_1\beta_2\gamma_1\gamma_2)^n}{\prod_{i=1}^{n}(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}\right) \\
&= \log(\beta_1\beta_2\gamma_1\gamma_2)^n - \log\left(\prod_{i=1}^{n}(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2\right) \\
&= n\log(\beta_1\beta_2\gamma_1\gamma_2) - \left[\log(\beta_1\gamma_2 v_1 + \beta_2\gamma_1)^2 + \ldots + \log(\beta_1\gamma_2 v_n + \beta_2\gamma_1)^2\right] \\
&= n\log(\beta_1\beta_2\gamma_1\gamma_2) - 2\sum_{i=1}^{n}\log(\beta_1\gamma_2 v_i + \beta_2\gamma_1) \qquad (7.2)
\end{aligned}
$$

Now, taking the partial derivatives of Equation(7.2) with respect to $\beta_1, \beta_2, \gamma_1$ and γ_2 respectively, we have,

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n}{\beta_1} - 2\sum_{i=1}^{n}\frac{\gamma_2 v_i}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)} \qquad (7.3)$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{n}{\beta_2} - 2\sum_{i=1}^{n}\frac{\gamma_1}{\beta_1\gamma_2 v_i + \beta_2\gamma_1} \qquad (7.4)$$

$$\frac{\partial \ell}{\partial \gamma_1} = \frac{n}{\gamma_1} - 2\sum_{i=1}^{n}\frac{\beta_2}{\beta_1\gamma_2 v_i + \beta_2\gamma_1} \qquad (7.5)$$

and

$$\frac{\partial \ell}{\partial \gamma_2} = \frac{n}{\gamma_2} - 2\sum_{i=1}^{n}\frac{\beta_1 v_i}{\beta_1\gamma_2 v_i + \beta_2\gamma_1} \qquad (7.6)$$

The maximum likelihood estimates for parameters $\beta_1, \beta_2, \gamma_1$ and γ_2 are obtained by solving Equation(7.3)-(7.6) when equated to zero. Unfortunately, this is analytically impossible.

For interval estimations of the set of $(\beta_1, \beta_2, \gamma_1, \gamma_2)$, and their tests of hypothesis, the Fisher information $I_n(\cdot)$ symmetric matrix is required. The Fisher-information matrix is used to calculate the covariance matrix associated with maximum-likelihood estimates. It is a way of measuring the amount of information that an observable random variable X carries about an unknown parameter θ upon which the probability of X depends. The elements of this matrix consist of the negetives of the expected values of the second partial derivatives of the log likelihood. That is,

$$I_n(\beta_1,\beta_2,\gamma_1,\gamma_2) = \begin{bmatrix} -E\left(\frac{\partial^2\ell}{\partial^2\beta_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_2}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial^2\beta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_2}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial^2\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\gamma_1\partial\gamma_2}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\gamma_1\partial\gamma_2}\right) & -E\left(\frac{\partial^2\ell}{\partial^2\gamma_2}\right) \end{bmatrix}$$

Continuing from the first derivatives in Equation(7.3)-(7.6), the corresponding second partial derivatives are obtained as follows;

$$\frac{\partial^2\ell}{\partial\beta_1^2} = \frac{-n}{\beta_1^2} + 2\sum_{i=1}^{n}\frac{\gamma_2^2 v_i^2}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}; \qquad \frac{\partial^2\ell}{\partial\beta_2^2} = \frac{-n}{\beta_2^2} + 2\sum_{i=1}^{n}\frac{\gamma_1^2}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}$$

$$\frac{\partial^2\ell}{\partial\gamma_1^2} = \frac{-n}{\gamma_1^2} + 2\sum_{i=1}^{n}\frac{\beta_2^2}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}; \qquad \frac{\partial^2\ell}{\partial\gamma_2^2} = \frac{-n}{\gamma_2^2} + 2\sum_{i=1}^{n}\frac{\beta_1^2 v_i^2}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}$$

$$\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2} = 2\sum_{i=1}^{n}\frac{\gamma_1\gamma_2 v_i}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}; \qquad \frac{\partial^2\ell}{\partial\beta_1\partial\gamma_1} = 2\sum_{i=1}^{n}\frac{\beta_2\gamma_2 v_i}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}$$

$$\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_2} = 2\sum_{i=1}^{n}\frac{\beta_1\gamma_2 v_i^2}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}; \qquad \frac{\partial^2\ell}{\partial\beta_2\partial\gamma_1} = 2\sum_{i=1}^{n}\frac{\beta_2\gamma_1}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}$$

$$\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_2} = 2\sum_{i=1}^{n}\frac{\beta_1\gamma_1 v_i}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}; \qquad \frac{\partial^2\ell}{\partial\gamma_1\partial\gamma_2} = 2\sum_{i=1}^{n}\frac{\beta_1\beta_2 v_i}{(\beta_1\gamma_2 v_i + \beta_2\gamma_1)^2}$$

The expressions for the Fisher Information matrix are not simple to handle analytically, and we do not intend to pursue this further. In the next chapter, we discuss the method of simulation by the Markov Chain Monte Carlo and generate random variates of the QBWD from a proposal distribution.

Chapter 8

Simulation

The Markov Chain Monte Carlo (MCMC) methods encompass a general framework for Monte Carlo integration (Roberts [9]) . As outlined in Sun [12], the Monte Carlo estimates the integral

$$\int_A g(t)dt$$

with a sample mean by restating the integration problem as an expectation with respect to some density function $f(\cdot)$. The integration problem is reduced to find a way to generate samples from the target density $f(\cdot)$. According to Maria [13], the MCMC approach to sampling from $f(\cdot)$ is to construct a Markov chain with stationary distribution $f(\cdot)$, and run the chain for a sufficiently long time until the chain converges to its stationary distribution. Simply, the Monte Carlo estimate of

$$E[g(\theta)] = \int g(\theta) f_{\theta|x}(\theta) d\theta$$

is the sample mean

$$\overline{g} = \frac{1}{m}\sum_{i=1}^{m} g(x_i)$$

where $x_1, x_2, \ldots, x_m$ is a sample from the distribution with density $f_{\theta|x}$.

The Metropolis-Hastings (M-H) algorithms are a class of MCMC methods, one of which is the Metropolis sampler. The main idea is to generate a Markov Chain $\{X_t|t = 0, 1, 2, \ldots\}$ such that its stationary distribution is the target distribution. The algorithm must specify, for a given X_t, how to generate X_{t+1}. In all of the Metropolis-Hastings sampling algorithms, there is a candidate point Y generated from a proposal distribution $g(\cdot|X_t)$. If this candidate point is accepted, the chain moves to state Y at time $t + 1$ and $X_{t+1} = Y$; otherwise the chain stays in the state X_t and $X_{t+1} = X_t$. The choice of proposal distribution is very flexible, but the chain generated by this choice must satisfy certain regularity conditions. The proposal distribution must be chosen so that the generated chain will converge to a stationary distribution - the target distribution.

The algorithms or steps required in generating a Markov chain $\{X_0, X_1, X_2, \ldots\}$ by the Matropolis-Hastings sampler are as follows, (See Maria [13] and Sun [12]):

1. Choose a proposal distribution $g(\cdot|X_t)$ (subject to regularity conditions stated above).

2. Generate X_0 from a distribution g.

3. Repeat (until the chain has converged to a stationary distribution according to some criterion):

   (a) Generate Y from $g(\cdot|X_t)$.

   (b) Generate U from Uniform $(0, 1)$.

   (c) If

   $$U \leq \frac{f(Y)g(X_t|Y)}{f(X_t)g(Y|X_t)}$$

   accept Y and set $X_{t+1} = Y$; otherwise set $X_{t+1} = X_t$.

4. Increment t.

Following the procedure described above, we generate a simulation of the quotient beta-Weibull distribution with parameters $\beta_1 = 1, \beta_2 = 4, \gamma_1 = 3\gamma_2 = 2$. The histogram of the simulated data and the curve of the empirical density function of the QBWD with the same parameters are shown in the next diagram. Also, the R script for the MCMC samples of the QBWD is provided in the Appendix.

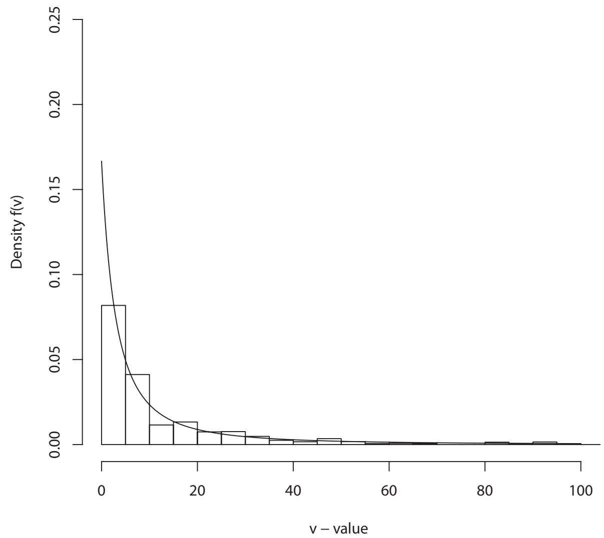

Figure 8.1: The histogram of the simulated QBWD and the pdf of the QBWD

Chapter 9

Conclusion

The quotient beta-Weibull distribution was defined and studied in this work. Various properties of the distribution were discussed, including, for example, the moment generating function, characteristic function, variance, skewness, kurtosis, and the raw moments. Also discussed are the Rényi entropy, asymptotic behaviors, estimation of parameters by the method of maximum likelihood. We pointed out some of the special cases of the distribution and highlighted the similarities of the distribution with few known distributions in the literature. A simulated random variates of the distribution were generated by the method of Markov Chain Monte Carlo (MCMC). The R statistical package and python software were used in implementing our results.

Appendix A

1. The code in R program to generate the graph of the CDF of the Quotient Beta-Weibull distribution with parameters
   $\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2, \gamma_2 = 4.$

```
b1=2
b2=3
d1=2
d2=4
v=seq(0,50,.01)
F=function(x)
(b1*d2*v)/(b1*d2*v + b2*d1)
plot(v,F(v),main="CDF of the QBWD",type="l",xlab="V-value")
```

2. The code in R program to generate the graph of the pdf of the Quotient Beta-Weibull distribution with parameters
   $\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2, \gamma_2 = 4;$
   $\beta_1 = .5, \beta_2 = .4, \gamma_1 = .3, \gamma_2 = .6;$
   $\beta_1 = .5, \beta_2 = .4, \gamma_1 = 5, \gamma_2 = 7.$

```
b1=2
b2=3
```

```
d1=2
d2=4
x=seq(0,15,.01)
bw1.pdf=function(v,b1,b2,d1,d2)
f=(b1*b2*d1*d2)/(b1*d2*v + b2*d1)**2

plot(x,bw1.pdf(x,2,3,2,4),type='l',ylim=c(0,.6),xlab="V-value",ylab="pdf")
lines(x,bw1.pdf(x,.5,.4,.3,.6),lty=2)
lines(x,bw1.pdf(x,.5,.4,5,7),lty=4)
legend("topright",inset=0.02,c("b1=2.0,b2=3.0,d1=2.0,d2=4.0","b1=.5,b2=.4,d1=.3,d2=.6",
"b1=.5,b2=.4,d1=5.0,d2=7.0"),lty=1:2:4)
```

3. The code in R program to generate the graph of the Hazard Rate Function of the Quotient Beta-Weibull distribution with parameters
$\beta_1 = 2, \beta_2 = 3, \gamma_1 = .05, \gamma_2 = .03;$
$\beta_1 = .02, \beta_2 = .15, \gamma_1 = 2, \gamma_2 = 4.$

```
b1=2
b2=3
d1=.05
d2=.03
x=seq(0,20,.01)
bw1.hzd=function(v,b1,b2,d1,d2)
f=(b1*d2)/(b1*d2*v + b2*d1)

plot(x,bw1.hzd(x,2,3,.05,.03),type='l',ylim=c(0,.6),xlab="v-value",ylab="h(v)")
lines(x,bw1.hzd(x,.02,.15,2,4),lty=2)
legend("topright",inset=0.02,c("b1=2,b2=3,d1=0.05,d2=0.03","b1=.02,b2=.15,d1=2,d2=4"),lty=1:
```

4. The code in Python program to generate the graph of the Rényi Entropy of the Quotient Beta-Weibull distribution with parameters
$\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$;
$\beta_1 = .5, \beta_2 = .1, \gamma_1 = 2, \gamma_2 = .8$;
$\beta_1 = 20, \beta_2 = 4, \gamma_1 = .025, \gamma_2 = 5$.

```
from math import ceil, log
from pylab import*

def F1(x): return(log(2*x-1))/(x-1)
def F2(x): return(log(2*x-1))/(x-1) - log(2)
def F3(x): return(log(2*x-1))/(x-1) - 3

S1 = []
S2 = []
S3 = []
R1 = []
R2 = []
R3 = []
N = 30
M = 400
i = 1
x = 0.51
for i in range (N):

S1.append(x)
S2.append(x)
```

```
S3.append(x)
R1.append((log(2*x-1))/(x-1))
R2.append((log(2*x-1))/(x-1) - log(2))
R3.append((log(2*x-1))/(x-1) - 3)
i += 1
x += 0.016
x = 1.1
for i in range (M):

S1.append(x)
S2.append(x)
S3.append(x)
R1.append((log(2*x-1))/(x-1))
R2.append((log(2*x-1))/(x-1) - log(2))
R3.append((log(2*x-1))/(x-1) - 3)
i += 1
x += 0.016

plt.plot(S1, R1,'b-', label="b1=b2=d1=d2=1")
plt.plot(S2, R2,'r–', label="b1=0.5,b2=0.1,d1=2,d2=0.8")
plt.plot(S3, R3,'k-',linewidth =5, label="b1=20,b2=4,d1=0.025,d2=5")
plt.axvline(x=1, ymin=0, ymax=5)
plt.legend(loc="upper right")
xlabel('s-value')
ylabel('R(s)')
title('Renyi Entropy of the QBWD')
show()
```

5. The code in R to generate the histogram of the simulated QBWD and the pdf of the QBWD with parameters

   $\beta_1 = 1, \beta_2 = 4, \gamma_1 = 3, \gamma_2 = 2.$

```
par(mfrow=c(2,2))
K=(b1*b2*r1*r2)
f=function(x,b1,b2,r1,r2)
if(any(x¡0))return(0)
stopifnot(b1¿0,b2¿0,r1¿0,r2¿0)
return(K/(b1*r2*x+b2*r1)**2)

m=10000
b1=1
b2=4
r1=3
r2=2
x=numeric(m)
x[1]=rexp(1,1)
k=0
u=runif(m)
for(i in 2:m)
xt=x[i-1]
z=rexp(1,xt)
num=f(z,b1,b2,r1,r2)*dexp(xt,z)
den=f(xt,b1,b2,r1,r2)*dexp(z,xt)
if(u[i]¡=num/den) x[i]=z else
x[i]=xt
k=k+1
```

```
print(k)
b=1001
z=x[b:m]
hist(z,breaks="scott",freq=F,xlim=c(0,20),ylim=c(0,.25),xlab="V - value")
t=seq(0,18,0.01)
lines(t,K/(b1*r2*t+b2*r1)**2,type="l")
```

6. The code in R program to generate the graph of the pdf of the Beta-Weibull distribution with parameters $\alpha = \beta = c = \gamma = 1$, Versus the pdf of the Quotient Beta-Weibull distribution with parameters $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$;

```
b1=1.0
b2=1.0
d1=1.0
d2=1.0

x=seq(0.1,15,.01)
bw1.pdf=function(v,b1,b2,d1,d2)
f=(b1*b2*d1*d2)/(b1*d2*v + b2*d1)**2

bw2.bwd=function(v)
g=exp(-v)

plot(x,bw1.pdf(x,1.0,1.0,1.0,1.0),type='l',ylim=c(0,.6),xlab="V-value",ylab="pdf")
lines(x,bw2.bwd(x),lty=2)
legend("topright",inset=0.02,c("QBWD-pdf","BWD-pdf"),lty=1:2)
```

7. The code in R program to generate the graph of the pdf of the Pareto distribution with parameters $\alpha = 1, a = -6, h = 6$, Versus the pdf of the Quotient Beta-Weibull

distribution with parameters $\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2, \gamma_2 = .5$;

```
b1=2.0
b2=3.0
d1=2.0
d2=0.5
a=-6.0
b=1.0
h=48
x=seq(1.1,15,.01)
bw1.pdf=function(v,b1,b2,d1,d2)
f=(b1*b2*d1*d2)/(b1*d2*v + b2*d1)**2

bw2.wpd=function(v,a,b,h)
g=(b*h**b)/(v-a)**(b+1)

plot(x,bw1.pdf(x,2.0,3.0,2.0,0.5),type='l',ylim=c(0,.6),xlab="V-value",ylab="pdf")
lines(x,bw2.wpd(x,-6.0,1.0,6.0),lty=2)
legend("topright",inset=0.02,c("QBWD-pdf","Pareto-pdf"),lty=1:2)
```

8. The code in R program to generate the graph of the pdf of the Weibull-Pareto distribution with parameters $c = .5, \beta = 1, \theta = 1$, Versus the pdf of the Quotient Beta-Weibull distribution with parameters $\beta_1 = 2, \beta_2 = 3, \gamma_1 = 2, \gamma_2 = 4$;

```
b1=2
b2=3
d1=2
d2=4
x=seq(1.5,15,.01)
bw1.pdf=function(v,b1,b2,d1,d2)
```

```
f=(b1*b2*d1*d2)/(b1*d2*v + b2*d1)**2

bw2.wpd=function(v)
g=(0.5/v)*((log(v))**(-0.5))*exp(-(log(v))**0.5)

plot(x,bw1.pdf(x,2,3,2,4),type='l',ylim=c(0,.6),xlab="V-value",ylab="pdf")
lines(x,bw2.wpd(x),lty=2)
legend("topright",inset=0.02,c("QBWD-pdf","WPD-pdf"),lty=1:2)
```

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Education

- Master of Arts. Mathematics, Marshall University, May 2012. Thesis Advisor: Dr. Alfred Akinsete.
- Bachelor of Science in Mathematics and Computer Science. University of Swaziland, May 2009. Thesis Advisor: Dr Sandile S. Motsa.

Publications

1. *The quotient of the beta-Weibull distribution.* Master's thesis, Marshall University, May 2012.

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1. *Mathematical Modeling of the SARS Epidemic.* Bachelor of Science Thesis, University of Swaziland, May 2009.