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LIMITING PROBLEMS IN INTEGRATION
AND
AN EXTENSION OF THE REAL NUMBERS SYSTEM

Thesis submitted to
the Graduate College of
Marshall University

In partial fulfillment of
the requirements for the degree of
Master of Arts
in Mathematics

by

Tue Ngoc Ly

Dr Ariyadasa Aluthge, Ph.D., Committee Chairperson
Dr. Ralph Oberste-Vorth, Ph.D., Committee Member
Dr. Bonita Lawrence, Ph.D., Committee Member

Marshall University

May 2008

Abstract

When considering the limit of a sequence of functions, some properties still hold under the limit, but some do not. Unfortunately, the integration is among those not holding. There are only certain classes of functions that still hold the integrability, and the values of the integrals under the limiting process. Starting with Riemann integrals, the limiting integrations are restricted not only on the class of functions, but also on the set on which the integral is taken. By redefining the integration process, Lebesgue integrals successfully and significantly extend both the class of the functions integrated and the sets on which the integral is taken. But the Lebesgue integral still does not hold under the limit even with some simply defined functions. We will attempt to solve this problem by defining a special type of measure to handle the limiting case and introducing an extension the set of real numbers.

Acknowledgment

This project is written under the guidance of Dr. Ariyadasa Aluthge, Dr. Bonita Lawrence, and Dr. Ralph Oberste-Vorth in Mathematics Department at Marshall University, as my advisers.

The project is also the result of my two-year-study in Mathematics Department at Marshall University.

I have to pay my special thanks to my advisers, Dr. Ariyadasa Aluthge, Dr. Bonita Lawrence, and Dr. Ralph Oberste-Vorth, those who advised and commented for my project.

And my thanks to other professors, and colleagues in the Department of Mathematics who helped me during my study at Marshall University.

Finally, I would like to thank my family and my dearest Linlin for their un-measurable support.

Huntington, May 2008

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Introduction

The problem of measuring (or computing) some quantitative properties such as length, area, and volume of geometrical objects has been around for thousands of years. Over two thousand years ago, ancient Greek mathematicians, scientists and philosophers came up with some special ways or formulas to calculate these quantitative areas and volumes of some geometrical objects such as cones, sphere,… But there was still the need of a general concept and method to understand these quantitative properties and to extend the class of computable geometric objects.

About three hundred years ago, Newton and Leibnitz formulated **calculus**, creating the first mechanism on the way to finding a general solution for this problem. Their idea of infinitesimal calculus, dividing an object into infinitely many well-behaved or nearly well-behaved pieces, had a big impact on the other mathematicians at the time. Since then, more detailed and rigorous mathematical concepts and methods have been developed in this field.

In the middle of the 19^{th} century, Bernhard Riemann, a German mathematician, rigorously formulated general solution to this problem; the concept of Riemann integration. It is still so useful and is widely accepted as one of the fundamental concepts of calculus mentioned in most calculus books now.

During one of the most active periods of mathematics around 19^{th} and 20^{th} centuries, many restrains of Riemann's method were recognized, and many attempts were made to extend Riemann integration. One of the most successful extensions of Riemann integration is due to

Henri Lebesgue. Lebesgue's attempt, which includes measure theory, has significantly removed the restrains of Riemann integration.

Lebesgue's measure and integration theory generalized the notion of "length," "area", and "volume" by generalizing the concept of integration over more complex domains rather than a simple interval. This approach successfully generalized the Riemann integration and extended the class of integrable (or *solvable*) functions. But there are still some situations (or functions when dealing with taking then limit) which suggests the need to develop more general ways to integrate a function than Lebesgue integration.

In Chapter 1, we will briefly discuss the Riemann integral, and some of its properties including the process of taking the limit. Concepts of measure and measurability of sets and functions are presented in Chapter 2. The Lebesgue integral is discussed in Chapter 3. Finally in Chapter 4, we will present some extreme cases, and some ideas for extending the system of real numbers.

Chapter 1

The Riemann Integrals

Since any geometrical object can be placed in a coordinate system and then described using functions and/or equations, the problem of finding its area or volume can be reduced to the problem of calculating the area or the volume under the graph of a function. The formulation of the problem finding the area under the graph of a bounded function in the Cartesian coordinate system is presented below using the concept of Riemann integral.

1.1 Riemann Sums and Riemann Integrals

Let f be a bounded real-valued function which is defined on the closed interval $[a, b]$. By a **subdivision** Δ of $[a, b]$, we mean a finite set $\Delta = \{x_i\}_{i=0}^{n}$ for some natural number n such that:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Let y_i be any number between x_{i-1} and x_i, then we define a **Riemann sum** S of f over Δ by:

$$S = \sum_{i=1}^{n} v_i(x_i - x_{i-1}) \text{ with } v_i = f(y_i).$$

Since f is bounded on $[a, b]$, it is bounded above and bounded below on each subinterval of $[a, b]$. Then for each subinterval $[x_i, x_{i+1}]$, if we choose v_i to be the *supremum* (*infimum*) of f over that subinterval, and take the Riemann sum over Δ, we get the **upper Riemann sum** (**lower Riemann sum**) over $[a, b]$. Let $\boldsymbol{U}_{\Delta}(f)$ and $\boldsymbol{L}_{\Delta}(f)$ be the upper and lower Riemann sums of f over $[a, b]$. That is,

$$U_{\Delta}(f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \text{ with } M_i = \sup_{y_i \in [x_{i-1}, x_i]} f(y_i)$$

and

$$L_{\Delta}(f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \text{ with } m_i = \inf_{y_i \in [x_{i-1}, x_i]} f(y_i)$$

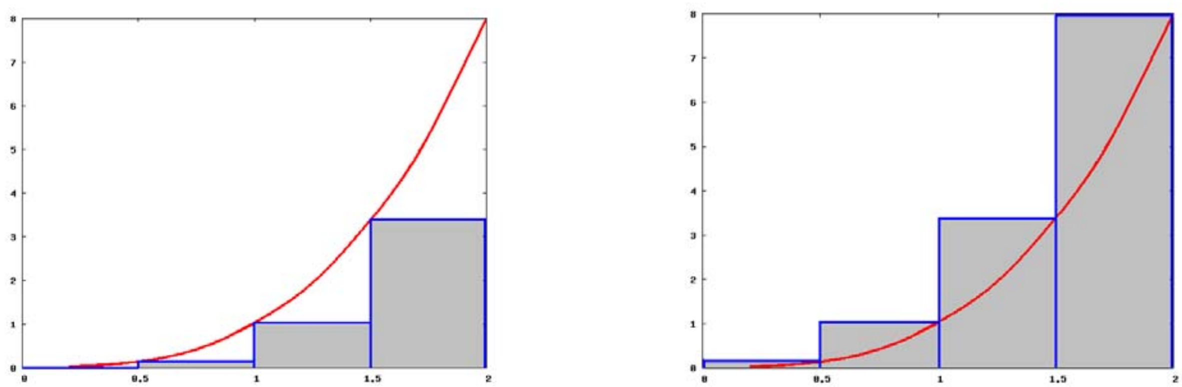

Fig 1.1 - Example of lower Riemann sum and upper Riemann sum

The **upper Riemann integral** of f over $[a, b]$ is defined as the *infimum* of the corresponding upper Riemann sums over all possible subdivisions of Δ of $[a, b]$, that is

$$\bar{R} \int_{a}^{b} f(x)dx = \inf_{\Delta} U_{\Delta}(f)$$

Similarly, we define the **lower Riemann integral** of f over $[a, b]$ as

$$\underline{R} \int_{a}^{b} f(x)dx = \sup_{\Delta} L_{\Delta}(f)$$

If $f(x) \geq 0$ on $[a, b]$, for every subdivision Δ, it is clear that the upper Riemann sum is always greater than or equal to the area under the curve, while the lower Riemann sum is always less than or equal to that area (Apostol [4]). Therefore, we have that:

$$\underline{R}\int_a^b f(x)dx = \sup_{\Delta} \boldsymbol{L}_{\boldsymbol{\Delta}}(f) \leq \text{the area under the graph} \leq \bar{R}\int_a^b f(x)dx = \inf_{\Delta} \boldsymbol{U}_{\boldsymbol{\Delta}}(f)$$

If upper and lower Riemann integrals are equal, we say that the function is **Riemann integrable**, and the common value of those two integrals is called the **Riemann integral** of *f* over [a, b] and is denoted by:

$$R\int_a^b f(x)dx.$$

If $f(x) \geq 0$ on [a, b], the Riemann integral of *f* over [a, b] is equal to the area under the graph of *f* over [a, b]. But in general, the function *f* has both positive and negative values over its domain, and then the Riemann integral is equal to the signed area of the function; that is the area above the horizontal axis minus the area below the horizontal axis.

1.2 Properties of Riemann Integrals

The following properties of the Riemann integral can be easily obtained from the definition:

Proposition 1.2.1 (*Linearity*) *If f and g are two Riemann integrable functions on a closed interval* $[a, b]$*, and c and d are two real numbers, then:*

$$R\int_a^b [cf(x) + dg(x)]dx = c \cdot R\int_a^b f(x)dx + d \cdot R\int_a^b g(x)dx$$

This proposition is useful to calculate the areas or volumes of the objects when we could add or divide the objects into others such that it is easier to solve.

Proposition 1.2.2 *If f is a Riemann integrable function on a closed interval* $[a, c]$, *and b is real number between a and c, then*

$$R\int_a^b f(x)dx + R\int_b^c f(x)dx = R\int_a^c f(x)dx$$

Lemma 1.2.3 *If f is some Riemann integrable function on* $[a, b]$ *then*

$$R\int_a^a f(x)dx = 0$$

It is fact when the length of [a, a] equals 0.

Proposition 1.2.4 *If f and g are two Riemann integrable functions on* $[a, b]$, *and* $f \leq g$, *i.e. for any x in* $[a, b]$, $f(x) \leq g(x)$, *then*

$$R\int_a^b f(x)dx \leq R\int_a^b g(x)dx$$

1.3 Class of Riemann integrable Functions

We have to remind that not all functions are Riemann integrable (Mattuck [5]). We can see it through the following example (example 1.3.1)

Example 1.3.1 *Consider the Dirichlet function defined on* $[a, b]$ *as below:*

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Since there are always at least one rational number and one irrational number between any two real numbers a and b, a < b, it can easily be shown that $\boldsymbol{U}_{\boldsymbol{\Delta}}(f) = 1$, *and* $\boldsymbol{L}_{\boldsymbol{\Delta}}(f) = 0$ *for any subdivision* Δ *of* $[a, b]$. *This implies*

$$\bar{R}\int_a^b f(x)dx = 1 \neq 0 = \underline{R}\int_a^b f(x)dx$$

Thus, the Dirichlet function is not Riemann integrable on any closed interval $[a, b]$■

So, it is important to determine which functions are Riemann integrable and which are not, and most importantly, how "large" the class of Riemann integrable functions is. The following results are well-known:

Theorem 1.3.2 *If a function f is continuous on a closed interval* $[a, b]$, *then f is Riemann integrable on* $[a, b]$.

Theorem 1.3.3 *A function f defined on a closed interval* $[a, b]$ *is Riemann integrable if and only if it is discontinuous at at most finitely many points.*

Corollary 1.3.4 *Let f be a Riemann integrable on* [*a*, *b*], *and let g be a real-valued function defined on the same interval such that the set* $\{x : f(x) \neq g(x)\}$ *is finite, then g is also Riemann integrable, and*

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

1.4 Convergence of Riemann Integrals

The Riemann integral provides a solution to the problem of finding the area under the graph of a function. Intuitively, we can ask the question that if a sequence of functions $\langle f_n \rangle$ converges to a function *f*, then will the areas under the graphs of $\langle f_n \rangle$ also converge to the area under the graph of *f* ? Unfortunately, this convergence property does not hold for the class of Riemann integrable functions in general.

Example 1.4.1 *Since the set of rational numbers is countable, let* $\{r_n\}$ *be an enumeration of the set of rational numbers in* [0, 1]. *Consider the sequence of functions* $\langle f_n \rangle$ *defined on* [0, 1] *as follows:*

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_i\}_{i=1}^{n} \\ 0 & \text{elsewhere} \end{cases}$$

Then we can see that the limit function of this sequence is the Dirichlet function:

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

For each natural number n, the function f_n *has finitely many points of discontinuities, then by Theorem* 1.3.3, *it is Riemann integrable. And moreover, it can easily be shown that* $R\int_0^1 f_n(x)dx = 0$. *But as we saw in Example* 1.3.1 *that the Dirichlet function* $f(x) = \lim_{n\to\infty} f_n(x)$ *is not integrable, therefore the class of Riemann integrable functions is not closed under taking limit.*

Indeed, for the Riemann integrable functions, in order for the convergence property to be held, we need some extra assumptions, which mean that it holds for a smaller class of functions.

Theorem 1.4.2 *Let $\langle f_n \rangle$ be a sequence of Riemann integrable functions on* [*a*, *b*] *that uniformly converges to a function f*, $f(x) = \lim_{n\to\infty} f_n(x)$ $for\ all\ x \in [a, b]$, *then f is Riemann integrable, and*

$$\lim_{n\to\infty} R\int_a^b f_n(x)dx = R\int_a^b f(x)dx$$

<u>*Proof:*</u>

Since $\langle f_n \rangle$ *uniformly converges to f, for any* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$, *such that for every* $n > N$, $|f_n(x) - f(x)| < \epsilon$ *for all* x *in* [*a*, *b*].

Let $\epsilon_n = \sup_{x\in[a,b]}|f_n(x) - f(x)|$, *then* $\langle \epsilon_n \rangle$ *is a decreasing sequence and* $\lim_{n\to\infty} \epsilon_n = 0$, *and for every n*, $f_n(x) - \epsilon_n < f(x) < f_n(x) + \epsilon_n$ *for all x in* [*a*, *b*]. *Taking the upper and lower Riemann integrals, we have that:*

$$\underline{R}\int_a^b [f_n(x) - \epsilon_n]dx \le \underline{R}\int_a^b f(x)dx \le \bar{R}\int_a^b f(x)dx \le \bar{R}\int_a^b [f_n(x) + \epsilon_n]dx$$

Since f_n*'s are Riemann integrable,(* $f_n \pm \epsilon_n$ *)'s are also Riemann integrable, then the previous inequalities become:*

$$R\int_a^b [f_n(x) - \epsilon_n]dx \le \underline{R}\int_a^b f(x)dx \le \bar{R}\int_a^b f(x)dx \le R\int_a^b [f_n(x) + \epsilon_n]dx$$

By Proposition 1.2.1, *it follows that:*

$$R\int_a^b f_n(x)dx - \epsilon_n \cdot R\int_a^b dx \le \underline{R}\int_a^b f(x)dx \le \bar{R}\int_a^b f(x)dx \le R\int_a^b f_n(x)dx + \epsilon_n \cdot R\int_a^b dx$$

$$\Rightarrow R\int_a^b f_n(x)dx - \epsilon_n \cdot (b-a) \leq \underline{R}\int_a^b f(x)dx \leq \bar{R}\int_a^b f(x)dx \leq R\int_a^b f_n(x)dx + \epsilon_n \cdot (b-a)$$

$$\Rightarrow 0 \leq \bar{R}\int_a^b f(x)dx - \underline{R}\int_a^b f(x)dx \leq$$

$$\leq \left[R\int_a^b f_n(x)dx + \epsilon_n \cdot (b-a)\right] - \left[R\int_a^b f_n(x)dx - \epsilon_n \cdot (b-a)\right]$$

That implies

$$0 \leq \bar{R}\int_a^b f(x)dx - \underline{R}\int_a^b f(x)dx \leq 2\epsilon_n(b-a)\ (1)$$

Since (1) *holds for all n, by taking the limit as n approaches to infinity, we have that:*

$$0 \leq \bar{R}\int_a^b f(x)dx - \underline{R}\int_a^b f(x)dx \leq \lim_{n\to\infty} 2\epsilon_n(b-a) = 0$$

So, $\bar{R}\int_a^b f(x)dx = \underline{R}\int_a^b f(x)dx$, *or f is Riemann integrable, and*

$$\lim_{n\to\infty}\left[R\int_a^b f_n(x)dx - \epsilon_n \cdot (b-a)\right] \leq R\int_a^b f(x)dx \leq \lim_{n\to\infty}\left[R\int_a^b f_n(x)dx + \epsilon_n \cdot (b-a)\right],$$

or

$$\lim_{n\to\infty} R\int_a^b f_n(x)dx = R\int_a^b f(x)dx \blacksquare$$

Besides taking the limit of a sequence of functions, it can also be problematic when the domain is infinitely large, i.e. when we take the limit of the domain, the Riemann integral can become unstable most of the time. A further example for this will be shown in Chapter 4.

1.5 Riemann's and Lebesgue's ways of computing areas

When developing the idea of dividing the total area into small pieces, Riemann divided the domain into finitely many intervals (Burrill, Knudsen [3]). Then the area of each piece is bounded above by the upper rectangle in the upper Riemann sum, and is bounded below by the lower rectangle in the lower Riemann sum.

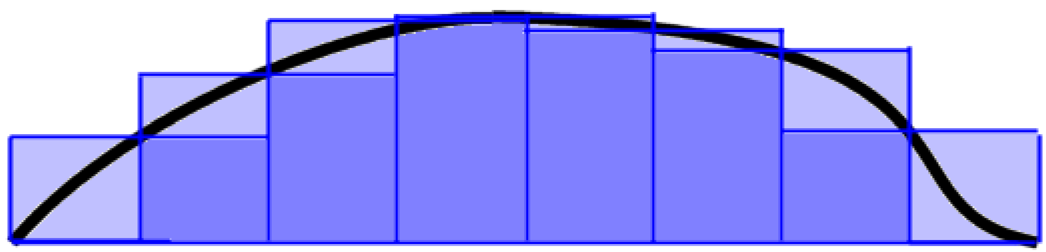

Fig 1.2 - Riemann sums divide the domain into small pieces

Dividing the domain into smaller and smaller subintervals works well if the function is well-behaved in the neighborhoods of each point in the domain. Differentiable functions and continuous functions belong to this category. But many of the functions that we encounter do not always behave nicely. Therefore, the class of Riemann integrable functions is restricted and needs to be extended.

Henri Lebesgue introduced a new attempt to calculate the area under a function in his dissertation in 1902. Instead of subdividing the domain, Lebesgue partitioned the range of the function. And for each interval of the partition, he determined how much of the domain is mapped by the function into this interval. Then using the same idea of Riemann integral, the area under graph of the function is bounded above by the sum of the products of the greater bound of each subinterval and the portion of the domain mapped by the function to that subinterval. Similarly for the lower bound. Finally, by taking the infimum of the sum of the upper bounds, or taking the supremum of the lower bounds over all the partitions of the range, we will have the Lebesgue Integral, if they are equal (Burrill, Knudsen [3]).

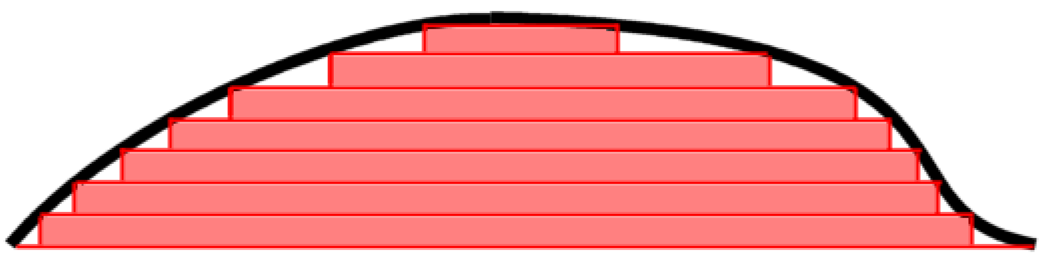

Fig 1.3 – Lebesgue's idea of partitioning the range

The tricky part of this method is to determine how much of the domain contributes to each interval of the subdivision of the range. For this problem, Lebesgue introduced another concept, measure theory. It turns out that with measure theory, the class of integrable functions is larger, and the process of taking the integral is generalized.

Chapter 2
Measure Theory

The core of Lebesgue's idea is measure theory. It gives a general understanding of the "length" of an arbitrary set. Measure theory is a rich subject, related to other areas of mathematics such as set theory, topology, probability theory, and, of course, calculus.

2.1 Algebra of Sets

Given a non-empty set X, let $P(X)$ denote the collection of all subsets of X, which is called the **power set** of X. As mentioned above, one of the difficulties of Lebesgue's approach is how to decide how much of the domain mapped by the function to an interval of the range. That means measuring some subset of the domain. Ideally, we would like to be able to measure any subset of the domain. But, it is impossible to construct a measure that satisfies all of our ideal properties for all the subsets of a non-empty set in general (Royden [1]). Let's identify some characteristics required of a structure on which we can define a measure.

By an **algebra on X**, we mean a non-empty collection of subsets of X, which is *closed under finite union and complement* (Royden [1]). If $\mathcal{A}$ is an algebra, A and B are two subsets of X, then we have that:

i) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

ii) $A \in \mathcal{A} \Rightarrow \tilde{A} \in \mathcal{A}$, with $\tilde{A}$ represents the complement of A

Then it can easily be shown the following properties of an algebra $\mathcal{A}$ on X:

i) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

ii) $\emptyset \in \mathcal{A}, X \in \mathcal{A}$

Example 2.1.1 *i) The collection of open and closed sets of the real numbers is an algebra.*

ii) The power set of X is an algebra on X.

The following useful proposition shows that in an algebra, every countable union can be expressed as a countable union of disjoint sets in the algebra.

Proposition 2.1.2 *Let $\mathcal{A}$ be an algebra on X, and $\langle A_i \rangle$ be a sequence of sets in $\mathcal{A}$, then there is a sequence $\langle B_i \rangle$ of sets in $\mathcal{A}$ such that $B_n \cap B_m = \emptyset$ for $n \neq m$, and*

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

An algebra $\mathcal{A}$ on X is called a ***σ-algebra***, if it is closed under countable union; that is, if $\langle A_i \rangle$ is a sequence of sets in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. De Morgan's laws imply that a σ-algebra is also closed under countable intersection.

Example 2.1.3 *i) The collection of open and closed sets of the real numbers is* **not** *a σ-algebra.*

ii) The power set of X is a σ-algebra on X.

2.2 Measurable Spaces

We are now ready to define the concept of a measure.

A **measurable space** is defined as a pair $(X, \mathcal{B})$, in which X is a non-empty set, and $\mathcal{B}$ is a σ-algebra of subsets of X. Each subset of X in $\mathcal{B}$ is called a **measurable set**. By a **set function**, we mean a function which assigns to some subsets of X an extended real number.

A non-negative set function μ is called a **measure** on a measurable space $(X, \mathcal{B})$, if it is defined for all sets of $\mathcal{B}$ and satisfies the following conditions:

i) $\mu(\emptyset) = 0$

ii) ***Countably additive***: $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ if $A_i \cap A_j = \emptyset$ for $i \neq j$

Three more useful properties of measures are given by the following propositions:

Proposition 2.2.1 *(Monotonicity) If* $A, B \in \mathcal{B}, and\ A \subset B, then\ \mu A \leq \mu B$.

Proposition 2.2.2 *If* $\langle A_i \rangle$ *is a decreasing sequence in* $\mathcal{B}$*, which means* $A_i \supset A_{i+1}$*, and* $\mu(A_1)$ *is finite, then* $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu A_i$.

Proposition 2.2.3 *If* $A_i \in \mathcal{B}$ *then* $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu A_i$.

Below are some special types of measurable sets and measures on X.

A measure μ is **translation invariant** if for each measurable set E and for each $y \in X$, the set $E + y = \{x + y: x \in E\}$ is also measurable and has the same measure as E

$$\mu(E + y) = \mu E, \text{for all } E \in \mathcal{B} \text{ and } y \in X$$

A subset E of X is said to be of **finite measure** if $E \in \mathcal{B}$, and $\mu E < \infty$. Similarly, we define sets of **infinite measure**. A subset E of X is of **σ-finite measure** if E is the union of a countable collection of measurable sets of finite measure; that is: $E = \bigcup_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{B}$ and $\mu E_i < \infty$. It can easily be proved that every measurable subset of a σ-finite measure set is also of σ-finite measure. If every measurable set is of σ-finite measure, then μ is called **σ-finite**.

Another notion that is weaker than σ-finiteness is semi-finiteness. A measure μ is **semi-finite** if every measurable set of infinite measure contains measurable sets of arbitrary large finite measure, which means:

μ is semi-finite $\Leftrightarrow$ If $\mu E = \infty$ then $\forall M > 0, \exists A \subset E, A \in \mathcal{B},$ such that $M < \mu E < \infty$.

So, if a measure is σ-finite, then it is also semi-finite.

An important characteristic that needs to be mentioned is **completeness**. A measure space $(X, \mathcal{B}, \mu)$ is said to be **complete** if $\mathcal{B}$ contains all the subsets of sets of measure zero, that is, if $E \in \mathcal{B}, \mu E = 0, and\ A \subset E$ then $A \in \mathcal{B}$.

Given a measure space $(X, \mathcal{B}, \mu)$, a subset E of X is called **locally measurable** if for every measurable set A of finite measure, $E \cap A$ is measurable. Since $\mathcal{B}$ is a σ-algebra, every measurable set is also locally measurable. Then one can show that the collection $\mathcal{C}$ of locally measurable sets is a σ-algebra and it contains $\mathcal{B}$. A measure μ is called **saturated** if every locally measurable set is measurable, i.e. $\mathcal{C} = \mathcal{B}$. It is not difficult to show that a σ-finite measure is saturated.

We will see two more special types of measures and their properties in Chapter 4. Now we will move on to the concept of outer measure.

2.3 Outer Measure

One of the restrictions of a measure μ on a set X is that not all subsets of X are measurable. In order to measure all subsets of X, we define an outer measure which bypasses this restriction by weakening the countable additive property of a measure.

By an **outer measure μ^***, we mean an extended real-valued set function defined on the power set of X, $P(X)$, and having properties stated below:

i) $\mu^*\emptyset = 0$

ii) $A \subset B \Rightarrow \mu^*A \leq \mu^*B$

iii) $E \subset \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^*E \leq \sum_{i=1}^{\infty} \mu^*E_i$

It follows from (i) and (ii) that μ^* is non-negative. The second property is called **monotonicity**. The third property is a weakened version of countable additivity, called **countable sub-additivity**. By (ii), the third property is equivalent to:

iv) $E = \bigcup_{i=1}^{\infty} E_i, E_i$ disjoint $\Rightarrow \mu^*E \leq \sum_{i=1}^{\infty} E_i$

An outer measure μ^* is **finite** if $\mu^*X < \infty$.

To make an analogy to a measure, a subset E of X is said to be **measurable with respect to an outer measure μ^*** if for every subset A of X, we have that:

$$\mu^*A = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$$

Note *In order to show that a subset E of X is measurable with respect to μ^*, by the countable sub-additivity of μ^* that*

$$\mu^*A \leq \mu^*(A \cap E) + \mu^*\left(A \cap \tilde{E}\right) \text{ since } A = (A \cap E) \cup \left(A \cap \tilde{E}\right)$$

it is only necessary to show the other direction of the inequality:

$$\mu^* A \geq \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$$

Moreover, this inequality is obvious when $\mu^* A = \infty$, so we need only to show it is true for all sets A of finite outer measure $\mu^* A < \infty$.

The following theorem states the relationship between outer measure and measure.

Theorem 2.3.1 *The class $\mathcal{B}$ of μ^*-measurable sets is a σ-algebra. If μ is μ^* restricted to $\mathcal{B}$, then μ is a complete measure on $\mathcal{B}$.*

2.4 Lebesgue Measure

From now on, we will consider the set of real numbers $\mathbb{R}$ as our space. Let's start with our "intuitive measure." The length $l(I)$ of an interval I is defined to be the difference of two endpoints of the interval. Then $l(I)$ is a non-negative set function on $\mathbb{R}$. Now we will extend this set function to construct an outer measure and a measure to measure more subsets of the real numbers.

For each set A of real numbers, consider the *countable* collections $\{I_n\}$ of open intervals that cover A, $A \subset \bigcup_{n=1}^{\infty} I_n$. The **Lebesgue outer measure** $\boldsymbol{m^*}$ of A is defined as the infimum of the sum of the interval lengths of each collection over all the open covers of A:

$$m^*A = \inf_{A \subset \bigcup_{n=1}^{\infty} I_n} \sum_{n=1}^{\infty} l(I_n)$$

It follows immediately from the definition of m^* the first two properties of an outer measure:

i) $m^*\emptyset = 0$

ii) $A \subset B \Rightarrow m^*A \leq m^*B$

Proposition 2.4.1 *The Lebesgue outer measure of an interval is its length.*

There is only the third property of an outer measure left to prove that Lebesgue outer measure m^* is an outer measure. The following proposition shows that the Lebesgue outer measure is actually an outer measure.

Proposition 2.4.2 *Let $\{A_n\}$ be a countable collection of sets of real numbers, then*

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^* A_n.$$

Thus, m is an outer measure.*

Proof:

If one of the sets A_n has infinite outer measure, the right side is infinity, then the inequality is obvious. Let m^*A_n be finite for any n, then given $\epsilon > 0$, there exists a countable disjoint collection of open intervals $\langle I_{n,i}\rangle_{i=1}^{\infty}$ such that $A_n \subset \bigcup_{i=1}^{\infty} I_{n,i} = I_n$ and

$$\sum_{i=1}^{\infty} l(I_{n,i}) < m^* A_n + 2^{-n}\epsilon$$

Since $\{I_n\}$ is a countable collection and its union covers the union of $\{A_n\}$,

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} l(I_{n,i}) \leq \sum_{n=1}^{\infty}(m^* A_n + 2^{-n}\epsilon) = \sum_{n=1}^{\infty} m^* A_n + \epsilon$$

It holds for an arbitrary positive number ϵ, therefore

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^* A_n \blacksquare$$

The following useful results can be derived easily from Proposition 2.4.2:

Corollary 2.4.3 *If A is countable then $m^*A = 0$.*

Corollary 2.4.4 *The set* [0, 1] *is uncountable.*

Proposition 2.4.5 *The Lebesgue outer measure is translation invariant.*

Following the construction we made in the previous section, a set E of real numbers is said to be **Lebesgue measurable** if for each set A, we have that

$$m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Then the theorem below follows directly Theorem 2.3.1 in the last section:

Theorem 2.4.6 *The collection $\mathcal{M}$ of Lebesgue measurable sets is a σ-algebra, and the restriction m of m* to $\mathcal{M}$ is a complete measure on $\mathcal{M}$, called the* ***Lebesgue measure****.*

Thus, by the translation invariance of Lebesgue outer measure, the Lebesgue measure *m* is *a complete translation invariant measure on the real numbers*. Some useful results are stated below

Proposition 2.4.7 *Some Lebesgue measurable sets*:

i) *If* $m^*E = 0$ *then E is Lebesgue measurable.*

ii) *If E is the interval* (a, ∞) *then E is Lebesgue measurable.*

iii) *If E is the union or intersection of a countable collection of open and closed sets, then E is Lebesgue measurable.*

Proposition 2.4.8 *Let E be a given set of real numbers, then the following five statements are equivalent*:

i) *E is Lebesgue measurable.*

ii) *For every* $\epsilon > 0$*, there is an open set O containing E with* $m^*(O \sim E) < \epsilon$.

iii) *For every* $\epsilon > 0$*, there is a closed set F contained in E with* $m^*(E \sim F) < \epsilon$.

iv) *There is a set G which is a countable intersection of open sets such that G contains E and* $m^*(G \sim E) = 0$.

v) *There is a set F which is a countable union of closed sets such that F is contained in E and* $m^*(E \sim F) = 0$.

2.5 Measurable Functions

Now we are in the position to define the concept of measurable functions. Let's start with an abstract definition.

Given two measurable spaces $(X, \mathcal{B})$ and $(Y, \mathcal{C})$, and let f be a function from X to Y. Then f is called a **measurable function from X to Y** if for every measurable subset E of Y, $f^{-1}(E)$ is measurable in X:

$$f^{-1}(E) \in \mathcal{B} \text{ for every } E \in \mathcal{C}.$$

When X and Y are Lebesgue measurable sets, and $\mathcal{B}$ and $\mathcal{C}$ are σ-algebra of measurable sets $\mathcal{M}$ restricted on X and Y respectively, then f is said to be **Lebesgue measurable**. We will use the phrase *measurable function* for *Lebesgue measurable function* when there is no confusion. It is immediately followed from the definition that:

Instead of checking inverse measurability of every measurable set in the range, there is a more efficient way to check measurability of a function with respect to the Lebesgue measure (McShane, Botts [2]).

Proposition 2.5.1 *Let f be an extended real-valued function defined on a* (*Lebesgue*) *measurable domain, then the following statements are equivalent*:

i) *For every real number c, the set $\{x: f(x) > c\}$ is measurable.*

ii) *For every real number c, the set $\{x: f(x) \geq c\}$ is measurable.*

iii) *For every real number c, the set $\{x: f(x) < c\}$ is measurable.*

iv) *For every real number c, the set $\{x: f(x) \leq c\}$ is measurable.*

Corollary 2.5.2 *A function f is Lebesgue measurable if and only if its domain is Lebesgue measurable, and it satisfies one of the statements of Proposition* 2.5.1.

Proposition 2.5.3 *If f is a continuous function, then f is Lebesgue measurable.*

<u>*Proof:*</u>

f is continuous, so for every open sets in the range, the inverse image is also open. Since the set $\{x: f(x) > 0\}$ *is open, its inverse image* $f^{-1}(\{x: f(x) > 0\}$ *is also open. So, by Proposition* 2.4.7 *and Corollary* 2.5.2, *f is measurable* ■

Example 2.5.4 *The Dirichlet function f is measurable. Recall that the Dirichlet function is*

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Then the inverse images:

$$\{x: f(x) > c\} = \begin{cases} \emptyset, & c \geq 1 \\ \mathbb{Q}, & 0 \leq c < 1 \\ \mathbb{R}, & c < 0 \end{cases}$$

which are always measurable. So, by Corollary 2.5.2, *the Dirichlet function is measurable*■

The following proposition can be proved by choosing appropriate sets:

Proposition 2.5.5 *Let c be a constant, and f and g be two* (*Lebesgue*) *measurable real-valued functions defined on the same domain, then the following functions* $f + c$, cf, $f + g$, $g - f$, *and* fg *are also measurable.*

Theorem 2.5.6 *Let $\langle f_n \rangle$ be a sequence of* (*Lebesgue*) *measurable functions defined on the same domain*, *then the following functions* sup $\{f_1,\ldots,f_n\}$, inf $\{f_1,\ldots,f_n\}$, sup f_n, inf f_n, $\overline{\lim}\ f_n$, *and* $\underline{\lim}\ f_n$ *are also measurable*.

Corollary 2.5.7 *Let $\langle f_n \rangle$ be a sequence of* (*Lebesgue*) *measurable functions defined on the same domain and converges (point-wise) to a function f, then f is measurable.*

So, by this result, we have that the class of measurable functions is closed under the process of taking limit (point-wise). However beside the abovementioned functions we could show some stronger results as follows.

A property is said to hold **almost everywhere** (abbreviated **a.e.**) if the set of points where it fails has Lebesgue measure zero.

Proposition 2.5.8 *If f is a* (*Lebesgue*) *measurable function, and f = g a.e., then g is also measurable.*

Corollary 2.5.9 *If $\langle f_n \rangle$ be a sequence of* (*Lebesgue*) *measurable defined on the same domain and converges to a function f a.e., then f is measurable.*

Chapter 3
The Lebesgue Integrals

3.1 Lebesgue Integrals of Simple and Bounded Functions

Now that we have introduced the fundamental concepts of measures, measurable sets and functions, now we are ready to introduce the concept of the Lebesgue integral.

Let A be any set, the **characteristic function χ_A** of the set A is defined as:

$$\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It follows directly from the definition that the function χ_A is measurable if and only if A is measurable (Royden, [1]).

A real-valued function φ is called **simple** if it is measurable and it assumes only finitely many values. If φ is a simple function and has the values $a_1, \ldots, a_n$ then

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}, \text{where } A_i = \{x \colon \varphi(x) = a_i\}$$

Since the simple function φ is measurable, A_i must be also measurable. Another fact is that a_i's are distinct and nonzero values, and A_i's are disjoint sets.

Proposition 3.1.1 *If φ and ψ are two simple functions, then $\varphi + \psi$, $\varphi - \psi$, and $\varphi \cdot \psi$ are also simple functions.*

For the simple function, it is easy to decide the portion of the domain that is mapped by the function into an interval of the range. So intuitively, we define the (**Lebesgue**) **integral** of the simple function φ that **vanishes outside a set of finite measure**, i.e. $m\{x: \varphi(x) \neq 0\} = 0$, by:

$$\int \varphi = \int \varphi(x)dx = \sum_{i=1}^{n} a_i m A_i \text{, when } \varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

If E is a measurable set, then we define:

$$\int_E \varphi = \int \varphi \cdot \chi_E$$

Proposition 3.1.2 (*Linearity*) *Let φ and ψ be simple functions vanishing outside a set of finite measure, then*

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi.$$

Proposition 3.1.3 *If φ and ψ be simple functions vanishing outside a set of finite measure such that $\varphi \geq \psi$ a.e., then*

$$\int \varphi \geq \int \psi$$

Next we move on to more complicated functions. Let f be a bounded function vanishing outside a set of finite measure, then we have the following interesting result which connects measurability, integrals, and simple functions:

Proposition 3.1.4 *Let f be a bounded function vanishing outside a set of finite measure E, then*

$$\inf_{f\le\psi}\int_E \psi(x)dx = \sup_{f\ge\varphi}\int_E \varphi(x)dx$$

if and only if f is measurable.

Thus, we define the (**Lebesgue**) **integral** of a bounded measurable function *f* vanishing outside a set *E* of finite measure by

$$\int_E f(x)dx = \inf\left\{\int_E \psi(x)dx : \psi \text{ is simple and } \psi \ge f\right\}.$$

We will use $\int_E f$ as an abbreviation for $\int_E f(x)dx$.

It is easily to obtain the next result showing that the Lebesgue integral is in fact a generalization of the Riemann integral.

Proposition 3.1.5 *If a bounded function f defined on* [*a*, *b*] *is Riemann integrable, then it is measurable and*

$$R\int_a^b f(x)dx = \int_{[a,b]} f(x)dx.$$

<u>*Proof:*</u>

Since each Riemann sum is the integral of a simple function, we have that:

$$\underline{R}\int_a^b f(x)dx = \sup_{\Delta} \boldsymbol{L}_{\boldsymbol{\Delta}}(f) \le \sup_{f\ge\varphi}\int_E \varphi(x)dx$$

and

$$\bar{R}\int_a^b f(x)dx = \inf_{\Delta} \boldsymbol{U}_{\boldsymbol{\Delta}}(f) \ge \inf_{f\le\psi}\int_E \psi(x)dx$$

So,

$$\underline{R}\int_a^b f(x)dx \le \sup_{f\ge\varphi}\int_E \varphi(x)dx \le \inf_{f\le\psi}\int_E \psi(x)dx \le \bar{R}\int_a^b f(x)dx$$

Since f is Riemann integrable,

$$\underline{R}\int_a^b f(x)dx = \bar{R}\int_a^b f(x)dx = R\int_a^b f(x)dx$$

Therefore

$$\sup_{f\ge\varphi}\int_E \varphi(x)dx = \inf_{f\le\psi}\int_E \psi(x)dx$$

Thus, by Proposition 3.1.4*, f is measurable, and*

$$R\int_a^b f(x)dx = \int_{[a,b]} f(x)dx \;\blacksquare$$

3.2 Lebesgue Integrals of Non-negative and General Functions

In order to integrate unbounded functions, by using the Riemann integrations, we usually integrate the function over a bounded subset of the domain, and then take the limit as the endpoints of the bounded subsets approach the endpoints of the domain. Lebesgue used another way to deal with unbounded problems, working directly with the functions after defining those well-behaved measurable sets and well-integrated functions.

Let's start with non-negative functions. The (**Lebesgue**) **integral** of a non-negative measurable function f on a measurable set E is defined by

$$\int_E f = \sup\left\{\int_E h : h \text{ is bounded measurable on } E, m\{x: h(x) \neq 0\} < \infty, \text{and } h \leq f\right\}$$

A non-negative measurable function f is called (**Lebesgue**) **integrable** over a measurable set E if

$$\int_E f < \infty$$

By the **positive part** f^+ of a function f, we mean function being defined by

$$f^+(x) = \max\{f(x), 0\}.$$

Similarly, we define the **negative part** f^- of a function f by

$$f^-(x) = \max\{-f(x), 0\}.$$

Then it follows from the definitions that:

i) $f = f^+ - f^-$

ii) $|f| = f^+ + f^-$

iii) If f is measurable, then f^+ and f^- are also measurable.

If both f^+ and f^- are (Lebesgue) integrable over a measurable set E, then f is (Lebesgue) integrable over E, and we define

$$\int_E f = \int_E f^+ - \int_E f^-$$

3.3 Properties of Lebesgue Integrals

We will see that, as we expected of a generalization of Riemann integral, Lebesgue integrals still have Riemann integral's properties.

Proposition 3.3.1 *Let f and g be integrable over E, then the followings hold*

i) (*Linearity*) *For any real numbers a and b, the function* ($af + bg$) *is integrable over E, and*

$$\int_E (af + bg) = a\int_E f + b\int_E g$$

ii) *If* $f \leq g$ *a.e., then*

$$\int_E f \leq \int_E g$$

Therefore, if $f = g$ *a.e., then*

$$\int_E f = \int_E g$$

iii) *If A and B are two disjoint measurable subsets of E, then f is integrable over A, B, and* $A \cup B$*, and*

$$\int_{A \cup B} f = \int_A f + \int_B f$$

iv) *If A is a set of measure* 0 *then f is integrable over A, and*

$$\int_A f = 0$$

In the Riemann integration, if we take away finitely many points in the domain, then it is divided into finite number of intervals over which the total of the Riemann integrals of the function on the subintervals is equal to the Riemann integral of the same function over the original domain. So, in the Riemann integration, the sets of points in the domain which does not affect the integration when being removed are finite sets. In the Lebesgue integration, they are generalized to a larger class - the class of sets of measure 0, including countable sets.

Proposition 3.3.2 *If f and g are two functions such that $f = g$ a.e., then f is* (*Lebesgue*) *integrable if and only if g is integrable, and then, their integrals are the same.*

3.4 Convergence of Lebesgue Integrals

As being mentioned before, Lebesgue's idea of integrating a function not only extends the class of integrable functions but also successfully improves the limiting process (Mattuck [5]). The abovementioned will be shown by the hereunder three important theorems and their implications. However, the Lebesgue integral seems not to be closed under limiting process. An example of this will be shown in the next chapter. Let's state convergence properties of Lebesgue integral. The detailed proofs can be found in (Royden, [1]).

Theorem 3.4.1 (*Fatou's Lemma*) *If $\langle f_n \rangle$ is a sequence of non-negative measurable functions then*

$$\int_E \lim_{n\to\infty} \inf_{k\geq n} f_k \leq \lim_{n\to\infty} \inf_{k\geq n} \int_E f_k$$

Theorem 3.4.2 (*Monotone Convergence Theorem*) *If $\langle f_n \rangle$ is an increasing sequence of non-negative measurable functions, and if $f = \lim f_n$ a.e., then f is integrable and*

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$$

Corollary 3.4.3 *Let $\langle f_n \rangle$ be a sequence of non-negative measurable functions. Then*

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

Corollary 3.4.4 *Let f be a non-negative measurable function and $\langle E_n \rangle$ be a disjoint sequence of measurable sets, then*

$$\int_{\cup_{n=1}^{\infty} E_n} f = \sum_{n=1}^{\infty} \int_{E_n} f$$

Theorem 3.4.5 (*Generalized Lebesgue Convergence Theorem*) *Let* $\langle g_n \rangle$ *be a sequence of integrable functions that converges a.e. to an integrable function g. Let* $\langle f_n \rangle$ *be a sequence of measurable functions that converges to f a.e. If* $|f_n| \leq g_n$ *and*

$$\int g = \int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n$$

then

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$$

Chapter 4
Some Limiting Cases and Extending the Real Number System

This chapter presents several examples when the convergence theorems do not hold. We also show how the situations can be treated by introducing some special measures.

4.1 Dirac Delta Function

Example 4.1.1 *Let $\langle f_n \rangle$ be a sequence of functions defined on $\mathbb{R}$ by*

$$f_n(x) = \begin{cases} n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & elsewhere \end{cases}$$

Then, f_n are non-negative simple functions, and are integrable. Taking the integral of f_n over $\mathbb{R}$ gives us

$$\int f_n = n \cdot m\left[-\frac{1}{2n}, \frac{1}{2n}\right] = n\left(\frac{1}{n}\right) = 1, for\ all\ n.$$

By taking the limit as n tends to infinity, we will get

$$\lim_{n\to\infty} \int f_n = 1$$

Now, let f denote the point-wise limit function of $\langle f_n \rangle$, that is

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Since for every number non-zero x, there exists a natural number n_0 such that $\frac{1}{2n_0} < |x|$. *That implies* $\frac{1}{2n} < \frac{1}{2n_0} < |x|$ *for any n > n_0. So,* $f(x) = lim_{n\to\infty} f_n(x) = 0$ *for any non-zero x. Since* $0 < \frac{1}{2n}$ *for any natural number n,* $f_n(0) = n$ *for all n. Therefore* $f(0) = lim_{n\to\infty} f_n(0) = lim_{n\to\infty} n = \infty$.

This shows that f = 0 a.e., so by Proposition 3.3.2, $\int f = \int 0 = 0$. *Thus,*

$$\int \lim_{n\to\infty} f_n = \int f = 0 \neq 1 = \lim_{n\to\infty} \int f_n \ \blacksquare$$

The above example shows that in general the Lebesgue integration does not preserve the integral under limit. In order to preserve the limiting integral for such sequences of functions as above, the British theoretical physicist Paul Dirac introduced a mathematical concept called the ***Dirac delta function*** *δ,* defined informally as

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

with its (Lebesgue) integral: $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

The idea of Dirac delta function is that it is zero everywhere but is so concentrated at one point that the area under the graph of the function (integral) is 1, not zero.

Since, $\delta = 0$ a.e., it is followed from Proposition 3.3.2 that δ is not an extended real-valued function. In the next section, we will see that indeed, δ can be considered as a measure.

Before that, let's consider the Dirac delta function δ as a function on the extended real number and go over some of its properties.

Proposition 4.1.2 (*Weisstein* [6]) *Let f, g be real-valued measurable functions and α, β be non-zero real numbers, then*

i) $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$.

ii) $\int_{-\infty}^{\infty}[\alpha f(x) + \beta g(x)]\delta(x)dx = \int_{-\infty}^{\infty}[\alpha f + \beta g](x)\delta(x)dx = [\alpha f + \beta g](0) = \alpha f(0) + \beta g(0)$.

iii) $\int_{-\infty}^{\infty} f(x)\delta(x-\alpha)dx = f(\alpha)$.

iv) $\int_{-\infty}^{\infty} \delta(\alpha x)dx = \int_{-\infty}^{\infty} \delta(\alpha x)\frac{d(\alpha x)}{\alpha} = \frac{1}{\alpha}\int_{-\infty}^{\infty} \delta(x)dx = \frac{1}{\alpha}$

Example 4.1.3 *Back to the example 4.1.1, if we let the limit*

$$\lim_{n\to\infty} f_n = \delta$$

then, we will have that:

$$\lim_{n\to\infty}\int f_n = 1 = \int \delta = \int \lim_{n\to\infty} f_n$$

Now, let's consider this integration over any set E of real numbers. In order for these integrals to exist, E needs to be measurable. But in fact, if the set E does not contain 0*, there will be a natural number N such that* f_n *vanishes in E for* $n \geq N$*. That means*

$$\lim_{n\to\infty}\int_E f_n = 0 = \int_E \delta = \int_E \lim_{n\to\infty} f_n$$

Similarly, if the set E contains any neighborhood of 0*, then the limit of the integrals of* f_n *over E would be* 1*, which equals to the integral of δ over E:*

$$\lim_{n\to\infty}\int_E f_n = 1 = \int_E \delta = \int_E \lim_{n\to\infty} f_n$$

So, the equality still holds for every subset of the real numbers that we integrate on, not restricted on just measurable sets■

4.2 Dirac measure

Considering the Dirac delta function as a regular function on the real numbers will restrict us to Lebesgue measurable sets and measurable functions. Instead of treating the Dirac delta function as a regular function on the space itself, we define it as a set function or a measure on its power set. With this, we can extend the classes of integrable functions.

Let X be some non-empty set, and a is an element of X. Define a set function μ_a on the power set $P(X)$ by

$$\mu_a(A) = \begin{cases} 1, & a \in A \\ 0, & a \notin A \end{cases} \text{ for given } A \subset X$$

Proposition 4.2.1 *The set function μ_a is a finite measure on the measurable space* (X, $P(X)$).

<u>*Proof:*</u>

It is followed from the definition of μ_a that is it non-negative, and $\mu_a \emptyset = 0$. It remains to prove the countably additive property.

Let $\langle E_n \rangle$ be a disjoint sequence of subsets of X. The proof will be divided into two cases:

Case 1: *If $a \in \bigcup_{n=1}^{\infty} E_n$, then there exists an E_k that contains a. Since E_n's are disjoint, E_i does not contain a for $i \neq k$. So*

$$\mu_a(E_n) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \mu_a(E_n) = \sum_{n \neq k} \mu_a(E_n) + \mu_a(E_k) = 0 + 1 = 1 = \mu_a\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Case 2: *If* $a \notin \bigcup_{n=1}^{\infty} E_n$*, then* $a \notin E_n$ *for all n. That means* $\mu_a(E_n) = 0$ *for any n. So*

$$\sum_{n=1}^{\infty} \mu_a(E_n) = \sum_{n=1}^{\infty} 0 = 0 = \mu_a\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Thus, μ_a *is a finite measure on the measurable space* $(X, P(X))$*, from the fact that* $a \in X$ ■

The measure μ_a is then called a **Dirac measure** on X. Since the σ-algebra of measurable sets is $P(X)$, all subsets of a measurable set of measure 0 is also measurable, μ_a is a *complete and saturated measure*.

A measure μ is called **probability measure** if

$$\int 1 d\mu = \int \chi_X d\mu = 1$$

Hence, μ_a is a probability measure.

Let f be a real-valued function defined on X, then we define the **Dirac delta integral** by

$$\int_E f \, d\mu_a = f(a)\mu_a(E)$$

With this definition, the following properties of this integral will hold:

Proposition 4.2.2

i) *Let* $\langle f_n \rangle$ *be a sequence of real-valued functions defined on X and* $\langle c_n \rangle$ *be a sequence of real numbers, then*

$$\int_E \left(\sum_{n=1}^{\infty} c_n f_n\right) d\mu_a = \left(\sum_{n=1}^{\infty} c_n f_n(a)\right) \mu_a(E)$$

ii) *Let f be a real-valued function defined on X, and* $\langle E_n \rangle$ *be a disjoint sequence of subsets of X, then*

$$\int_{\bigcup_{n=1}^{\infty} E_n} f \, d\mu_a = f(a) \sum_{n=1}^{\infty} \mu_a(E_n)$$

<u>*Proof:*</u>

i) *By the definition of the Dirac delta integral,*

$$\int_E \left(\sum_{n=1}^{\infty} c_n f_n \right) d\mu_a = \left(\sum_{n=1}^{\infty} c_n f_n \right)(a) \mu_a(E) = \left(\sum_{n=1}^{\infty} c_n f_n\,(a) \right) \mu_a(E)$$

ii) *Since* μ_a *is a measure on* $(X, P(X))$*, it follows directly from the definition of the Dirac delta integral:*

$$\int_{\bigcup_{n=1}^{\infty} E_n} f \, d\mu_a = f(a) \mu_a \left(\bigcup_{n=1}^{\infty} E_n \right) = f(a) \sum_{n=1}^{\infty} \mu_a(E_n) \blacksquare$$

4.3 Discrete Measure

Now, let's generalize the Dirac measure to a larger type of measures. Let (X, β, μ) be a measure space. Let **supp** $\boldsymbol{\mu}$ denote the **support set** of measure μ, i.e. the collection of sets of non-zero measure in X. A set function v_S on $P(X)$ is called a **discrete measure** if there exists a countable subset $S = \{s_1, s_2,\ldots\}$ of X such that

i) $\{s_n\} \in \text{supp } v_S$, for all $s_n \in S$

ii) If A is a subset of S, then $v_S(A) = \begin{cases} 0, & A = \emptyset \\ \sum v_S(\{s_i\}), & A \neq \emptyset, s_i \in A \end{cases}$

iii) $v_S(E) = v_S(E \cap S)$ for any subset E of X.

S is called the **core** of v_S (Terekhin [7]).

It is directly followed by the definition that a discrete measure is a measure. And the σ-algebra of measurable sets is the power set $P(X)$ of X. The following results relate the discrete measures and the Dirac measures mentioned before.

Lemma 4.3.1 *The Dirac measure is a discrete measure.*

Proof:

If we let $S = \{a\}$, then the Dirac delta measure μ_a clearly satisfies three conditions of a discrete measure with the core S■

Theorem 4.3.2 The measure *v_S is a discrete measure on X with the core $S = \{s_1, s_2,\ldots\}$ if and only if there is a sequence of real numbers $\langle\alpha_n\rangle$ such that*

$$v_S = \sum_{n=1}^{\infty} \alpha_n \mu_{s_n} \text{ with } \mu_{s_n} \text{are Dirac delta measures.}$$

Proposition 4.3.3 *Given two countable subsets S_1 and S_2 of X, and two positive real numbers α_1 and α_2. If ν_{S_1} and ν_{S_2} are two discrete measures on X with the cores S_1 and S_2 respectively, then $\alpha_1\nu_{S_1} + \alpha_2\nu_{S_2}$ is a discrete measure on X with the core $S_1 \cup S_2$.*

<u>*Proof:*</u>

Assume $S_1 = \{s_{1,n}\}_{n=1}^{\infty}$ and $S_2 = \{s_{2,n}\}_{n=1}^{\infty}$ are two countable subsets of X, and ν_{S_1} and ν_{S_2} are two discrete measures on X with the cores S_1 and S_2 respectively. Let α_1 and α_2 are two positive real numbers. Define the set function ν as

$$\nu(E) = \alpha_1\nu_{S_1}(E) + \alpha_2\nu_{S_2}(E)$$

And let $S = S_1 \cup S_2$, then for any x in S, x is either in S_1 or S_2, that is,

$$\nu_{S_1}(\{x\}) > 0 \text{ or } \nu_{S_2}(\{x\}) > 0$$

So,

$$\nu(\{x\}) = \alpha_1\nu_{S_1}(\{x\}) + \alpha_2\nu_{S_2}(\{x\}) > 0$$

or $\{x\} \in \operatorname{supp} \nu$.

$$\nu(\emptyset) = \alpha_1\nu_{S_1}(\emptyset) + \alpha_2\nu_{S_2}(\emptyset) = 0$$

If $A \subset S = S_1 \cup S_2$, then

$$\nu(A) = \alpha_1\nu_{S_1}(A) + \alpha_2\nu_{S_2}(A) = \alpha_1\nu_{S_1}(A \cap S_1) + \alpha_2\nu_{S_2}(A \cap S_2)$$

$$= \alpha_1 \sum_{x\in A\cap S_1} \nu_{S_1}(\{x\}) + \alpha_2 \sum_{x\in A\cap S_2} \nu_{S_2}(\{x\}) = \sum_{x\in A} \alpha_1\nu_{S_1}(\{x\}) + \sum_{x\in A} \alpha_2\nu_{S_2}(\{x\})$$

$$= \sum_{x \in A} [\alpha_1 \nu_{S_1}(\{x\}) + \alpha_2 \nu_{S_2}(\{x\})] = \sum_{x \in A} \nu(\{x\})$$

This proves the second property of a discrete measure. The third property is proved by the same argument as above. Thus, $\nu = \alpha_1 \nu_{S_1} + \alpha_2 \nu_{S_2}$ *is a discrete measure with the core* $S = S_1 \cup S_2$ ■

4.4 Another Example

In Example 4.1.1, the Lebesgue integral does not hold under the limit when the upper bounds of functions approach infinity, while the measure of the support sets approach zero. In this section we will give another example that the sequence of functions uniformly converges to 0 and their support sets expand to a set of infinity measure.

Let $\lfloor x \rfloor$ denote the largest integer that smaller than or equal to *x*, called the floor function. Consider the sequence of function $\langle f_n \rangle$ defined as below

$$f_n(x) = \begin{cases} \dfrac{\binom{2n}{n+\lfloor x \rfloor}}{2^{2n}}, & -n \le x < n+1 \\ 0, & elsewhere \end{cases} = \begin{cases} 2^{-2n}\dbinom{2n}{n+\lfloor x \rfloor}, & -n \le x < n+1 \\ 0, & elsewhere \end{cases}$$

Lemma 4.4.1 *For any real number x, the sequence $\langle f_n(x) \rangle$ converges.*

<u>*Proof:*</u>

For any real number x, there exists a positive integer n such that $|x| \le n$, or $-n \le x < n+1$. So, for n large enough, we have that:

$$f_n(x) = 2^{-2n}\binom{2n}{n+\lfloor x \rfloor} = \frac{1}{2^{2n}} \cdot \frac{(2n)!}{(n+\lfloor x \rfloor)!\,[2n-(n+\lfloor x \rfloor)]!} = \frac{1}{2^{2n}} \cdot \frac{(2n)!}{(n+\lfloor x \rfloor)!\,(n-\lfloor x \rfloor)!}$$

Since $-(n+1) < -n \le x < n+1 < (n+1)+1$,

$$f_{n+1}(x) = 2^{-2(n+1)}\binom{2(n+1)}{(n+1)+\lfloor x \rfloor} = \frac{1}{2^{2n+2}} \cdot \frac{(2n+2)!}{(n+1+\lfloor x \rfloor)!\,[2n+2-(n+1+\lfloor x \rfloor)]!}$$

$$= \frac{1}{2^{2n+2}} \cdot \frac{(2n+2)!}{(n+1+\lfloor x \rfloor)!\,(n+1-\lfloor x \rfloor)!}$$

Then the difference between $f_{n+1}(x)$ and $f_n(x)$ is:

$$f_{n+1}(x) - f_n(x) = \frac{1}{2^{2n+2}} \cdot \frac{(2n+2)!}{(n+1+\lfloor x \rfloor)!\,(n+1-\lfloor x \rfloor)!} - \frac{1}{2^{2n}} \cdot \frac{(2n)!}{(n+\lfloor x \rfloor)!\,(n-\lfloor x \rfloor)!}$$

$$= \frac{1}{2^{2n+2}} \cdot \frac{(2n)!}{(n+1+\lfloor x \rfloor)!\,(n+1-\lfloor x \rfloor)!} [(2n+1)(2n+2) - 4(n+1+\lfloor x \rfloor)(n+1-\lfloor x \rfloor)]$$

$$= \frac{1}{2^{2n+2}} \cdot \frac{(2n)!}{(n+1+\lfloor x \rfloor)!\,(n+1-\lfloor x \rfloor)!} [4n^2 + 6n + 2 - 4(n+1)^2 + 4\lfloor x \rfloor^2$$

$$= \frac{1}{2^{2n+2}} \cdot \frac{(2n)!}{(n+1+\lfloor x \rfloor)!\,(n+1-\lfloor x \rfloor)!} (4n^2 + 6n + 2 - 4n^2 - 8n - 4 + 4\lfloor x \rfloor^2$$

$$= \frac{1}{2^{2n+2}} \cdot \frac{(2n)!}{(n+1+\lfloor x \rfloor)!\,(n+1-\lfloor x \rfloor)!} (4\lfloor x \rfloor^2 - 2n - 2)$$

For any real number x, there exist an integer n_0 such that $2n_0 > 4\lfloor x \rfloor^2 - 2$, *then for* $n \geq n_0$, $2n > 4\lfloor x \rfloor^2 - 2$, *or* $f_{n+1}(x) - f_n(x) < 0$

So, the sequence $\{f_n(x) | n \geq n_0\}$ *is monotonic decreasing, and it is bounded below by 0.*

Therefore, the sequence $\{f_n(x) | n \geq n_0\}$ *converges, that implies the sequence* $\{f_n(x)\}$ *converges as n tends to infinity for any real number x.*

Thus, $f_\infty = \lim_{n\to\infty} f_n$ *is a well-defined function*■

Lemma 4.4.2 *For any positive integer n,*

$$\int_{-\infty}^{\infty} f_n(x)dx = 1$$

<u>*Proof:*</u>

$$\int_{-n}^{n+1} f_n(x)dx = \sum_{i=-n}^{n} \int_{i}^{i+1} f_n(x)dx = \sum_{i=-n}^{n} \int_{i}^{i+1} 2^{-2n}\binom{2n}{n+\lfloor x \rfloor} dx = \sum_{i=-n}^{n} \int_{i}^{i+1} 2^{-2n}\binom{2n}{n+i} dx$$

$$= 2^{-2n} \sum_{i=-n}^{n} \binom{2n}{n+i} \int_{i}^{i+1} dx = 2^{-2n} \sum_{i=-n}^{n} \binom{2n}{n+i} = 2^{-2n} \sum_{i=0}^{2n} \binom{2n}{i} = 2^{-2n} 2^{2n}$$

$$= 1$$

So, by taking the integral over the whole set of real numbers, we have that:

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{-n} f_n(x)dx + \int_{-n}^{n+1} f_n(x)dx + \int_{n+1}^{\infty} f_n(x)dx$$

$$= \int_{-\infty}^{-n} 0dx + \int_{-n}^{n+1} f_n(x)dx + \int_{n+1}^{\infty} 0dx = 1 \blacksquare$$

Lemma 4.4.3 *For every positive integer n:*

$$\frac{f_{n+1}(0)}{f_n(0)} = \frac{2n+1}{2n+2}$$

<u>*Proof:*</u>

$$f_n(0) = 2^{-2n}\binom{2n}{n+\lfloor 0 \rfloor} = 2^{-2n}\binom{2n}{n} = 2^{-2n} \cdot \frac{(2n)!}{n! \cdot n!}$$

$$f_{n+1}(0) = 2^{-2n-2} \cdot \frac{(2n+2)!}{(n+1)! \cdot (n+1)!}$$

So,

$$\frac{f_{n+1}(0)}{f_n(0)} = \frac{2^{-2n-2}}{2^{-2n}} \cdot \frac{(2n+2)!}{(2n)!} \cdot \frac{n! \cdot n!}{(n+1)! \cdot (n+1)!} = \frac{1}{4} \cdot \frac{(2n+1)(2n+2)}{(n+1)^2} = \frac{2n+1}{2n+2} \blacksquare$$

Proposition 4.4.4

$$f_{\infty}(0) = \lim_{n\to\infty} f_n(0) = 0$$

<u>*Proof:*</u>

$$f_1(0) = \frac{\binom{1}{2}}{2^2} = \frac{2}{4} = \frac{1}{2}$$

$$f_2(0) = f_1(0) \cdot \frac{2 \cdot 1 + 1}{2 \cdot 1 + 2} = \frac{1}{2} \cdot \frac{3}{4}$$

Assume that for some positive integer k:

$$f_k(0) = \prod_{i=1}^{k} \frac{2i-1}{2i} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k}$$

Then,

$$f_{k+1}(0) = f_k(0) \cdot \frac{2k+1}{2k+2} = \left(\prod_{i=1}^{k} \frac{2i-1}{2i}\right) \cdot \frac{2(k+1)-1}{2(k+1)} = \prod_{i=1}^{k+1} \frac{2i-1}{2i}$$

So, for any positive integer n, we have that:

$$f_n(0) = \prod_{i=1}^{n} \frac{2i-1}{2i}$$

Now, consider the subsequence $\langle f_{kn}(0) \rangle$, *for some k,*

$$f_{kn}(0) = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2kn-1}{2kn}$$

$$= \left(\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\right)\cdot\left(\frac{2n+1}{2n+2}\cdots\frac{4n-1}{4n}\right)\cdots\left(\frac{2(kn-1)-1}{2(kn-1)}\cdots\frac{2kn-1}{2kn}\right)$$

$$< \left(\frac{2n-1}{2n}\right)^n\cdot\left(\frac{4n-1}{4n}\right)^n\cdots\left(\frac{2kn-1}{2kn}\right)^n = \left(1-\frac{1}{2n}\right)^n\cdot\left(1-\frac{1}{4n}\right)^n\cdots\left(1-\frac{1}{2kn}\right)^n$$

$$= \prod_{i=1}^{k}\left(1-\frac{1}{2in}\right)^n$$

As we proved above, the sequence $\langle f_n(x)\rangle$ converges for any real number x, so the sequence $\langle f_n(0)\rangle$ and all of its subsequences $\langle f_{kn}(0)\rangle$ also converge, and

$$f_\infty(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} f_{kn}(0) \le \lim_{n\to\infty}\left(\prod_{i=1}^{k}\left(1-\frac{1}{2in}\right)^n\right) = \prod_{i=1}^{k}\lim_{n\to\infty}\left(1-\frac{1}{2in}\right)^n = \prod_{i=1}^{k} e^{-\frac{1}{2i}}$$

$$= e^{-\frac{1}{2}\sum_{i=1}^{k} 1/i}$$

This holds for any positive integer k, therefore,

$$0 \le f_\infty(0) \le inf\, e^{-\frac{1}{2}\sum_{i=1}^{k} 1/i} = e^{-\frac{1}{2}\sum_{i=1}^{\infty} 1/i} = e^{-\infty} = 0$$

Thus, $f_\infty(0) = 0$■

Since max $\left\{\binom{2n}{m}, 0 \le m \le 2n\right\} = \binom{2n}{n}$, $f_n(x) \le f_n(0)$ for any real number *x* and positive integer *n*. So, the sequence $\langle f_n\rangle$ uniformly converges to $f_\infty = 0$. It proves that

$$\int \lim_{n\to\infty} f_n = \int f_\infty = \int 0 = 0 \ne 1 = \lim_{n\to\infty}\int f_n \;■$$

4.5 Extending System of Real Numbers

In 4.1 and 4.2, we discussed several examples where the point-wise limit does not hold the integrals, even though we already extended the real numbers to infinity. It is a consequence of property (convention) that $0 \cdot \infty = 0$. In order to solve this problem, making the point-wise limit still holds the integrals, it is suggested to extend the real numbers not only to infinity-direction, but also to 0-direction, and reconsider the limiting process in the new extended real numbers system. Let's denote the extension of real numbers system by $\mathfrak{R}$ with addition, subtraction and comparison in the sense as real numbers, and satisfying these assumptions:

1) $\mathbb{R} \subset \mathfrak{R}$

2) The set $\mathcal{E}^* = \{z \in \mathfrak{R}: -x \leq z \leq x, z \neq 0, \forall x \in \mathbb{R}^+\}$ is not empty, and the positive, negative and the union are $\mathcal{E}^+ = \{z \in \mathfrak{R}: 0 < z < x, \forall x \in \mathbb{R}^+\}$, $\mathcal{E}^- = \{z \in \mathfrak{R}: -x < z < 0, \forall x \in \mathbb{R}^+\}$, and $\mathcal{E} = \{z \in \mathfrak{R}: -x < z < x, \forall x \in \mathbb{R}^+\} = \mathcal{E}^* \cup \{0\} = \mathcal{E}^+ \cup \mathcal{E}^- \cup \{0\}$. They will be called the **epsilums**.

3) The set $+\infty = \{z \in \mathfrak{R}: x < z, \forall x \in \mathbb{R}\}$ is not empty, and similarly,

$-\infty = \{z \in \mathfrak{R}: z < x, \forall x \in \mathbb{R}\}$ and the union $\infty = (+\infty) \cup (-\infty)$. They will be called the **infinitums**.

Those elements that are not in the infinitums are finite.

4) $0 \cdot z = z \cdot 0 = 0$ *for all* $z \in \mathfrak{R}$.

Given two subsets A and B of $\mathfrak{R}$, and an extended real number a, let denote:

i) $A + B = \{x + y: x \in A, y \in B\}$ and $A \cdot B = \{xy: x \in A, y \in B\}$

ii) $A + a = a + A = \{x + a : x \in A\}$ and $A \cdot a = a \cdot A = aA = \{ax : x \in A\}$

Based on these notations, the followings are properties of the epsilums and infinitums

Theorem 4.5.1 *Properties of epsilum:*

a) $\mathcal{E}^{\pm} + \mathcal{E}^{\pm} = \mathcal{E}^{\pm}$

b) $a\mathcal{E}^{\pm} = \mathcal{E}^{\pm}$ *for any* $a \in \mathbb{R}^{+}$, *and* $a\mathcal{E}^{\pm} = \mathcal{E}^{\mp}$ *for any* $a \in \mathbb{R}^{-}$

c) $\mathcal{E}^{\pm} \cdot \mathcal{E}^{\pm} = \mathcal{E}^{+}$ *and* $\mathcal{E}^{\pm} \cdot \mathcal{E}^{\mp} = \mathcal{E}^{-}$

Theorem 4.5.2 *Properties of infinitums:*

a) $(\pm\infty) + z = z + (\pm\infty) = \pm\infty$ *for all z finite*

b) $(\pm\infty) + (\pm\infty) = \pm\infty$

c) $a(\pm\infty) = \pm\infty$ *for any* $a \in \mathbb{R}^{+}$, *and* $a(\pm\infty) = \mp\infty$ *for any* $a \in \mathbb{R}^{-}$

d) $(\pm\infty) \cdot (\pm\infty) = +\infty$ *and* $(\pm\infty) \cdot (\mp\infty) = -\infty$

Next, let *a* be any real number, and *z* be any finite extended real number, then the **casting operators** are defined as:

$(\boldsymbol{\mathfrak{R}})(a) = a + \mathcal{E}$ and

$(\mathbb{R})(z) = x$ such that $x \in \mathbb{R}$ and $z \in (\boldsymbol{\mathfrak{R}})(x)$

Then we state the properties of casting operators as following

Lemma 4.5.3 *If z is any finite extended real number, then* $(\mathbb{R})(z)$ *is unique.*

Proof:

Assume that there exist two real number a and b such that $(\mathbb{R})(z) = a$ *and* $(\mathbb{R})(z) = b$. *Then by the definition of casting operator, we have that* $z \in (\mathfrak{R})(a)$ *and* $z \in (\mathfrak{R})(b)$. *So there are* $\epsilon_1, \epsilon_2 \in \mathcal{E}$ *such that* $a + \epsilon_1 = z = b + \epsilon_2$. *Or,* $a - b = \epsilon_2 - \epsilon_1 \in \mathcal{E}$. *By the definition of the infinitesimal, that means* $-x < a - b < x$ *for any* $x \in \mathbb{R}^+$. *Therefore,* $a - b = 0$, *or* $a = b$ ■

The following results follow directly from the definitions of the casting operations and the extension of the real number system.

Proposition 4.5.4 *Let x, y be any two finite extended real numbers, and a, b be any two real numbers, then*

i) $(\mathfrak{R})(0) = \mathcal{E}$ *and* $(\mathbb{R})(\epsilon) = 0$ *for any* $\epsilon \in \mathcal{E}$.

ii) *x is real if and only if* $(\mathbb{R})(x) = x$.

iii) *If* $a \neq b$ *then* $(\mathfrak{R})(a) \cap (\mathfrak{R})(b) = \emptyset$.

iv) *If* $x \leq y$ *then* $(\mathbb{R})(x) \leq (\mathbb{R})(y)$.

v) $(\mathfrak{R})(a + b) = (\mathfrak{R})(a) + (\mathfrak{R})(b)$

$(\mathfrak{R})(ab) = (\mathfrak{R})(a) \cdot (\mathfrak{R})(b)$

vi) $(\mathbb{R})(x + y) = (\mathbb{R})(x) + (\mathbb{R})(y)$

$(\mathbb{R})(xy) = (\mathbb{R})(x) \cdot (\mathbb{R})(y)$

Lemma 4.5.5 $\mathcal{E}$ *is a field.*

Proposition 4.5.6 $\mathfrak{R} = (\mathbb{R} + \mathcal{E}) \cup \infty$ *is a field.*

Conclusion

Arising from the problem of finding the area or volume as an efficient mechanism, the integration has been refined for over a hundred year. Among those earliest and most acceptable integration techniques is Riemann's integration. Riemann integration works well with most functions that have such well-behaved properties as continuity, but it fails when dealing with functions not in that well-behaved class of functions, especially when taking the limit (Burrill, Knudsen [3]). Attempting to solve this problem in a different way, Lebesgue integration extended not only the class of integrable functions, but also the sets on which the function is integrated. Moreover, Lebesgue integration works better with the limiting process, resulting in Fatou's lemma, Monotone Convergence Theorem, and Generalized Lebesgue Convergence Theorem.

But as we see in Chapter 4, the Lebesgue integral still fails with the limit of some easily defined functions. One attempt for this problem is defining a special type of function to handle the limit. The Dirac delta function and its extension, Dirac measure and discrete measure, are well-known examples. Another attempt is to extend the real number system on which the functions and the limit are defined on. Some ideal properties of this system are presented in Chapter 4. For the future work, there will be a more detailed and careful construction for this idea of extending the real number system and how the limiting process will be adopted to the new number system.

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