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Mechanical Visualization of a Second Order Dynamic Equation on Varying Time Scales

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MECHANICAL VISUALIZATION OF A SECOND ORDER DYNAMIC EQUATION ON VARYING TIME SCALES

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ABSTRACT

In this work, we give an introduction to Time Scales Calculus, the properties of the exponential function on an arbitrary time scale, and use it to solve linear dynamic equation of second order. Time Scales Calculus was introduced by Stefan Hilger in 1988. It brings together the theories of difference and differential equations into one unified theory. By using the properties of the delta derivative and the delta anti-derivative, we analyze the behavior of a second order linear homogeneous dynamic equation on various time scales. After the analytical discussion, we will graphically evaluate the second order dynamic equation in Marshall's Differential Analyzer Lab. Differential analyzers (DA) are mechanical machines designed to solve differential equations through a process called mechanical integration. The DA can be used to demonstrate to students and science educators a mechanical visualization of integration, specifically, Riemann sums. The DA can be used to solve nonlinear differential equations of interest to mathematics researchers in the broad field of dynamic equations on time scales.

CHAPTER 1

INTRODUCTION

In 1988, Stefan Hilger's PhD dissertation laid out the framework for the theory of Time Scales Calculus which unified continuous and discrete analysis. His dissertation offered the unification of the theory of differential equations with that of difference equations. For example, if we differentiate a function defined with domain $\mathbb{R}$, then the definition of derivative and integration is equal to that of standard calculus textbooks. However, if the function is defined on the integers then it is equal to the forward difference operator from the study of difference equations. Dr. Martin Bohner and Dr. Allan Peterson extended and combined the work of Stefan Hilger with that of many others in the field in a book, *Dynamic Equations on Time Scales – An Introduction with Applications*, which has allowed the field of time scales calculus to grow and expand. Instead of proving results for differential equations and then for difference equations, the study of dynamic equations on time scales allows us to prove the result for dynamic equations in which the domain of the unknown function is a time scale, an arbitrary nonempty closed subset of the real numbers.

In this work, we will discuss the basic terms related to the theorems dealing with differentiation and integration on time scales. It is also necessary to define the generalized exponential function and properties related to the first and second order linear dynamic equations. The main goal of this project is to analyze how the second order linear homogeneous equation performs on various time scales. In order to do this we will provide analytical discussion of a particular equation and we will provide graphical solutions on the given time scales obtained by Art, Marshall University's four integrator differential analyzer.

CHAPTER 2
DIFFERENTIAL ANALYZERS

2.1 History of Differential Analyzers

The DA was designed and first built in the late 1920s to solve nonlinear differential equations that could not be solved by other methods [4]. This machine is a precursor to the modern computer and consider by many to be the first analogue computer.

2.1.1 DAs in the 1920s-30s

Dr. Vannevar Bush, Professor of Electrical Engineering at Massachusetts Institute of Technology (M.I.T.), designed and built the first machine. The first machine had six integrators and the second one called the Rockefeller Differential Analyzer had 16 integrators and was built in 1941. The idea traveled to England in the early 1930s. Dr. Douglas Hartree of the University of Manchester visited Dr. Bush to learn more about his machine in the hopes to take the idea back to Manchester with him. During this visit, Hartree developed the idea to build a working model of Bush's machine from Meccano parts (the British version of Erector Set) [4]. Hartree's idea became a reality with the assistance of Arthur Porter, Hartree's research student majoring in physics. Porter took Hartree's idea and began to construct the first DA at the University of Manchester, primarily built from Meccano parts [7]. Porter's machine had four integrators and was used to solve many different problems and in particular Porter analyzed the atomic structure for the chromium atom and other problems involving control theory. Porter finished his doctoral thesis and then was selected for the Commonwealth Fund Fellowship allowing him to study at M.I.T. with Bush using the Rockefeller Differential Analyzer [7]. With the advancement in electronics most DAs were destroyed or became static displays, residing in museums across the world.

2.1.2 Present Day DAs

In the last ten years, there has been increased interest in DAs as a tool to develop our understand of the first computers. This project was initiated by Dr. Bonita Lawrence, mathematics professor at Marshall University, in 2004. Dr. Lawrence was inspired by a static display of a portion of the University of Manchester's machine built by the Metropolitan-Vickers Company at the London Science Museum while visiting that summer [1]. Dr. Lawrence returned to Marshall with the plan to get students involved in researching this machine because she was interested in how the machine offers a physical model of a mathematics equation and the educational benefits that can result from this aspect of the machine [1]. The Marshall DA Team started to make contacts with several people in the U.S. that had experience with the machine and learned that Dr. Porter was residing in North Carolina. Dr. Lawrence and a few of her students visited him to gain additional insight about the machine and learn about the construction of the first machine built in England. The team first built a two-integrator machine called Lizzie in 2006. Then, in May of 2009, the team finished the construction of Art, a four integrator machine named after Dr. Arthur Porter. Art has the capacity to run nonlinear differential equations.

My connection with this historical work started when I saw a presentation by Dr. Lawrence at the Rocky Mountain Mathematics Consortium Summer School in 2009 at the University of Wyoming. Dr. Lawrence and her husband, Dr. Clayton Brooks, gave a presentation on Lizzie. After the presentation, Dr. Lawrence, turned the DA on to show us how it worked. I was so intrigued by the machine because it was something that I have never seen before. I was attending the conference with Dr. Heidi Berger, Simpson College mathematics professor. Dr. Berger suggested that I talk to Dr. Lawrence about the machine and that conversation led to me actually building a DA for Simpson College the following summer. Dr. Berger and Dr. Lawrence planned a trip for me to Marshall University to learn about the mechanics of a DA and to build a machine in Marshall's DA Lab and then bring it back to Simpson for my senior research.

During the fall semester of 2009, I studied abroad in London and was able to go to the London Science Museum where Dr. Lawrence saw the piece of the Metropolitan-Vickers Company DA. I spent time in the mathematics section where they have many different pieces from differential analyzers built in the early 1930s. With internal grants from Simpson and funding from Marshall it was possible for me to visit Marshall for two weeks to complete the construction of Simpson's DA. Building the machine in two weeks by myself would have been impossible, even with the help from the graduate students at Marshall. I asked another Simpson math student, Dani Peterson, to come with me and help with the construction. As a team we were able to complete the building of the machine in less than two weeks.

Dani and I built a two-integrator DA for Simpson College in 11 days and named it Miles-Diffy (MD), since we had to drive miles to West Virginia and it is a differential analyzer. MD is a novel design for a DA; Dr. Lawrence designed this DA to have a large interconnect, which is the section of the machine that contains all the gears that are needed to program the equation the machine is solving. This also allows for the possibility of doing more complex equations by building additional integrators to model higher order differential equations. Simpson College is the second location in the U.S. to have a publicly accessible DA. The machines are currently not the most advanced technology, but they provide a great resource as a teaching tool for many math courses and research in differential and dynamic equations.

During my senior year at Simpson I wrote my honors capstone paper on Miles-Diffy and gave presentations to local high schools, at the Science Center of Iowa, the Midwest Undergraduate Mathematics Symposium, Simpson College's Math Day, and to Simpson math and computer science classes. I founded the Simpson College DA Club which won Outstanding Poster Presentation at Simpson's Honor Research Symposium. Having a strong connection with Marshall and Dr. Lawrence, I knew it was the best fit for me when attending graduate school. Marshall's DA Team has added a third machine similar to Miles-Diffy, called DA Vinci. During my first year at Marshall I studied the mechanics specific to Art and worked to streamline DA Vinci's interconnect system. Dr. Lawrence, Kayode Olumoyin, my DA lab partner, and I have conducted labs for a variety of classes at Marshall and traveled to Simpson College to assist in troubleshooting Miles-Diffy to insure smooth operation.

CHAPTER 3

MECHANICS AND MATHEMATICS OF DA, ART

In this chapter we offer insight concerning the mechanics and mathematics of differential analyzers. Since the machines are mechanical we must consider the turns of gears and rods to calculate the values of the associated mathematical expressions throughout the machines.

3.1 Mechanics of DA

The main mathematical component of a DA is the integrator, consisting of a wheel and disk combination, *Figure 3.1*.

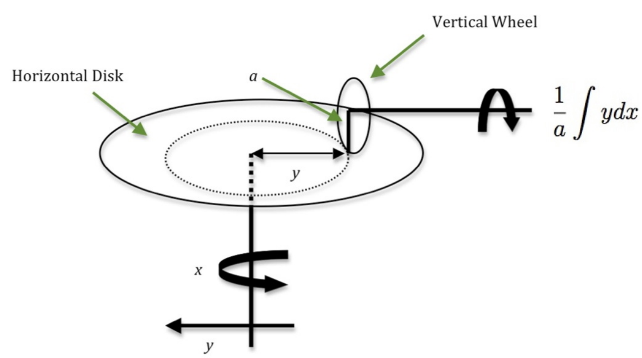

Figure 3.1: Setup of the wheel and disk for each integrator.

The point of contact of the wheel and the disk creates the frictional force that causes torque on the rod connected to the wheel on the integrator [5]. This torque is needed to move the rest of the machine, since the motion of the wheel is calculating the result from each integrator and these rotations must continue through the various components of the machine [4]. For instance, if the equation involves several integrators, this movement would have to go through all of them. Also, the torque created on a particular integrator might be required to move the output table, if that part of the equation is being plotted.

The torque amplifiers do not affect the mathematics of the machine because they do not have gears that affect the motion. Throughout history, different types of torque amplifiers have been used, some more efficient than others. The DA used in this thesis, Art, is a four-integrator differential analyzer, which uses a computer-programmed servo-type motor to amplify torque.

An adder is necessary to model more complex equations that require adding or subtracting of the output from two or more integrators. The DA uses a mechanical adder that consists of a bevel gear connected in such a way as to allow rotations from two independent rods to be connected in a manner that creates the sum of the two motions on a single rod [4].

An output table is required to graphically visualize solutions created by the particular differential equation programed on the DA. Note that an output table connected to the DA is an accessory and is not necessary for the operation of the DA. The integrators and interconnect of the DA are setup to run the equation without an output table. If the operator desires a specific plot of an equation he or she must connect the appropriate rods to the output table.

The output table plots the desired solution when fed two sources of motion that moves a pen in both vertical and horizontal directions. Depending on the equation being modeled by the DA, the operator has the option to choose the source of motion going into each direction of movement [4]. For example, if the operator wants to plot a derivative represented by a turning rod, this rod can be connected to the vertical axis. If the independent variable motion is connected to the horizontal axis, this connection to the output table would result in a plot of that particular derivative.

3.2 Mathematics of the Differential Analyzer

The DA performs mechanical integration within each of its integrators. Because of this there are three types of motion involved: the input, output, and independent variable. The DA's mechanical integrator consists of a horizontal disk located on a carriage that turns a vertical wheel [5]. The horizontal disk (referred to as the independent variable) rotates under the wheel on the carriage and the rotations of the disk turn the wheel [4]. A rod is connected to the wheel and the turns of the rod (referred to as the integral) are added together to create motion that moves to other components, in some cases the solution curve for a particular differential equation. The carriage of each integrator moves along a track fed by a lead screw, which changes the location of the wheel from the center of the horizontal disk. The position of the wheel on the disk measured in inches (referred to as the integrand) determines the number of turns of the wheel. For example, if the wheel is situated on the center of the disk this means it is at zero and the wheel will not rotate.

For another example, if the radius of the wheel is denoted by a and the wheel is located a units from the center of the disk then the wheel will turn one time for each turn of the disk. Ultimately, all variables in the machine will be measured in shaft rotations. Therefore, we have to transform the units from inches to rotations.

We must consider the embedded gears that are within Art. There is an integrator constant that reflects a geardown of the output on Art. Its presence is due to the radius of the wheel, the thread of the lead screw, and imbedded gear ratios. The integrator constant is important because each time motion passes through the integrator the output is $\frac{p \cdot k}{a}$ of the integral, where p represents pitch of the lead screw, a is the radius of the wheel in inches, and k is the ratio of reduction gears.

To calculate the movement of output relative to input to the integrator we evaluate:

$$\frac{p \cdot k}{a} = \frac{\frac{1}{32} \text{ inch per rotation} \cdot \left(\frac{50}{57} \cdot \frac{2}{5} \text{ rotations}^2\right)}{\frac{15}{16} \text{ inches}} = .0116959 \text{ rotations}$$

Therefore, one turn of each of the input rods will result in a factor of 0.0116959 rotations of the output rod. This result is from the following particular data: the wheel has a radius of $\frac{15}{16}$ inches, the lead screw has thread of 32 per inch, the helical gears in the carriage have a gear ratio of $\frac{2}{5}$ and the gearing near the clutch that moves the lead screw has a gear ratio of $\frac{50}{57}$. From this we calculate our unit of measure on Art, excluding the counters, as

$$\frac{1}{\frac{p \cdot k}{a}} = \frac{1}{0.0116959} = 85.5.$$

Thus, one turn of input we get 85.5 turns of output.

Now, we need to consider the rotation on the integrator counter and the gearing within the counter. To determine the unit of one on the machine we need to take into account the gear ratio of $\frac{50}{57}$ that moves the lead screw at the counter and the fact that one rotation of the gear on the counter makes the counter spin $\dfrac{10}{3}$ turns. Thus, on our counter our unit measure is equivalent to

$$85.5 \cdot \frac{10}{1} \cdot \frac{1}{3} \cdot \frac{50}{57} = 250.$$

That is, 250 turns on the counters equals one unit on Art. This allow us to use the counters to set initial conditions on each integrator. For example, if we want initials conditions of 1/2 and 1/4, we would set the counter on one integrator at 125 and the other at 62.5.

3.2.1 Mechanical Integration – Riemann Sums

To explain how integration is occurring within the machine, consider the case when the carriage is not moving along the track. Rotations of the disk will cause the wheel to create a circle on the disk. We know that the circumference of the circle created by the wheel on the disk is $C_d = 2\pi y$ where y is the distance the wheel is from the center of the disk. The circumference of the wheel is $C_w = 2\pi a$. From this we know then that one turn of the disk equals $\frac{y}{a}$ turns of the wheel.

Once the carriage starts to move along the track from the input of motion from the lead screw, the arc lengths being created by the rotations of the wheel on the disk will change according to the change in radius (y) that defines the circle. The equation for arc length is $s = r\theta$, where $r = y$ and $\theta = 2\pi\Delta x$, with Δx being the portion of a circle traced by the wheel at radius y. Also, let n represent the number of portions of a turn.

From this we can describe the total distance the wheel rotates on the disk for n portions of a turn of the disk as the sum of the arc lengths at the different positions of the wheel on the disk [4]. The total distance can be written mathematically as

$$\sum_{i=1}^{n} y(x_i) 2\pi \Delta x_i \tag{3.1}$$

where $y(x_i) = r_i$ and $2\pi\Delta x_i = \theta_i$.

To obtain the number of rotations of the wheel, we must divide the sum (3.1) by $2\pi a$, the distance of one rotation of the wheel. Units of length for the radius a and the radii r_i are the same and the result is a sum of rotations since (3.1) is measured in units of length and $2\pi a$ is measured in length/rotation of the wheel, the product is measured in rotations. So, the turns of the wheel can be written as

$$\begin{aligned}
\text{Turns of the wheel} &= \frac{\text{Distance traveled by wheel}}{\text{Distance of one rotation of the wheel}} \\
&= \sum_{i=1}^{n} \frac{y(x_i)2\pi\Delta x_i}{2\pi a} \\
&= \frac{1}{a}\sum_{i=1}^{n} y(x_i)\Delta x_i
\end{aligned} \tag{3.2}$$

When we allow the portions of a rotation of the disk to shrink, the sum becomes the integral,

$$\frac{1}{a}\int y(x)dx,$$

with integrand $y(x)$ multiplied by the constant $\frac{1}{a}$.

As one can see this sum is a Riemann sum [4]. The following is a formal definition of Riemann sum.

Definition 1 ([2]). *If P is the tagged partition, which is a partition of a given interval together with a finite sequence of numbers $i = 1, 2, \ . \ . \ . \ , n$, we define the Riemann sum of a function $f : [a, b] \to \mathbb{R}$ corresponding to P to be the number*

$$S(f; P) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

If the function f is positive on $[a, b]$, then the Riemann sum is the sum of the areas of n rectangles whose bases are subintervals $I_i = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$.

To be able to visualize the continuous movement of the integrators we can think about the shape being inscribed by the wheel on the disk. When the DA is running one period of a differentiable periodic function the wheel and disk movement creates four spirals because the wheel crosses each side of the disk twice. The shape of these spirals is determined by the particular equation programmed on Art. The image of these spirals shows the path of the wheel on the disk as the

DA runs a positive increasing function. These spirals display the definite integral represented in the mechanical integrator on Art, but we can connect this to the traditional description of definite integrals representing areas under the curve. The four spirals can be graphed on a xy-plane where the x-axis is the number of rotations of the wheel and the y-axis is the distance of the wheel from the center of the disk.

The Riemann sum being represented in a mechanical form provides a physical representation of integration, where the rotations of the rod that carries the wheel on top of the disk is the value of the integral. Just as a graph has positive and negative values this is represented on the DA when the wheel passes through the center of the disk (the zero location) [4]. This is because the motion of the wheel reverses directions, offering one direction to represent positive values and the other negative values.

In general, if the location of the wheel on the disk of a given integrator defines the nth derivative of a desired function, the turns of the wheel moving on the disk represents the $(n-1)$st derivative of a selected function [4]. Once the movement passes through n integrators the result is the desired function.

CHAPTER 4

TIME SCALES CALCULUS

In this chapter, we will provide an introduction to time scales calculus. This will include the definitions of basic terms and specific properties needed to work with second order linear dynamic equations. Time scales calculus was initiated in 1988 by Stefan Hilger. It bridges the gap between continuous and discrete analysis and expands on both theories [3]. The following introductory material can be found in [3] where complete proofs are provided.

4.1 Basic Terms

A *time scale*, denoted by $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers, $\mathbb{R}$. For example, the real numbers, $\mathbb{R}$, the integers, $\mathbb{Z}$, the natural numbers, $\mathbb{N}$ and the nonnegative integers $\mathbb{N}_0$ are all time scales. However, the rational numbers, $\mathbb{Q}$, complex numbers, $\mathbb{C}$, and the open interval $(7, 11)$ are *not* time scales.

We are concerned with the classification of points in a time scale. One main operation is moving forward or backwards on a time scale. Specifically, we are often concerned with moving to the next point or previous point in the set when possible. Let us start by defining the forward and backward jump operators.

Definition 2 ([3]). *1. For $t \in \mathbb{T}$ we define the* **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \ inf\{s \in \mathbb{T} : s > t\}.$$

2. For $t \in \mathbb{T}$ we define the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ *by*

$$\rho(t) := \ sup\{s \in \mathbb{T} : s < t\}.$$

The forward jump operator gives you the next point in the time scale when there is a gap. Similarly, the backward jump operator provides you the previous point in the time scale at a point after a gap.

We use jump operators to classify points in the time scale. We define t as *right-scattered*, if $\sigma(t) > t$. This means that such points have a measurable gap between them and the next point in the time scale. Similarly, we define t as *left-scattered*, if $\rho(t) < t$. These particular points have a measurable gap between them and the previous point in the time scale. Points that are both left and right scattered are called *isolated*. In contrast, if a point t is arbitrarily close to the next point in $\mathbb{T}$ this implies that $\sigma(t) = t$. In this case, t is called *right-dense*. Likewise, if a point t is arbitrarily close to the previous point in $\mathbb{T}$ this implies that $\rho(t) = t$. Here t is called *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*. We also will need to know the distance between points, calculated using the *graininess function*.

Definition 3 ([3])**.** *We define the change in position between consecutive points as $\mu(t) := \sigma(t) - t$ and we call $\mu(t)$ the* **graininess function***.*

Note that $\mu(t)$ will always be an element of the interval $[0, \infty)$ and this function provides the distance from a point t to $\sigma(t)$.

We now define the set $\mathbb{T}^k$, which will be important in later sections dealing with differentiation and integration.

Definition 4 ([3])**.** *If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. So,*

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - (\rho(sup\mathbb{T}), sup\mathbb{T}] & if\ sup\mathbb{T} < \infty \\ \mathbb{T} & if\ sup\mathbb{T} = \infty. \end{cases}$$

Now, let's us examine a couple of examples to better understand the operators, $\sigma(t)$ and $\rho(t)$. Also, we will determine the graininess, $\mu(t)$ for the given time scales.

Example 1. Let us consider the three examples.

1. If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ is

$$\mu(t) = \sigma(t) - t = 0 \text{ for all } t \in \mathbb{T}.$$

2. If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf(t+1, t+2, t+3, \dots) = t+1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) = (t-1) - t = 1 \text{ for all } t \in \mathbb{T}.$$

3. Refer to Figure 4.1. If $\mathbb{T} = [0,1] \cup [2,3]$, then for $t \in (0,1) \cup (2,3)$

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = t$$

and similarly $\rho(t) = t$. The graininess function μ is

$$\mu(t) = 0 \text{ for all } t \in (0,1) \cup (2,3).$$

Hence every point $t \in (0,1) \cup (2,3)$ is dense. If $t = 0$, then

$$\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = 0$$

and

$$\rho(0) = \sup\{s \in \mathbb{T} : s < 0\} = 0.$$

The graininess function μ in this case yields

$$\mu(0) = \sigma(0) - 0 = 0 - 0 = 0.$$

Hence $t = 0$ is dense. If $t = 1$, then

$$\sigma(1) = \inf\{s \in \mathbb{T} : s > 1\} = 2$$

and

$$\rho(1) = \sup\{s \in \mathbb{T} : s < 1\} = 1.$$

The graininess function μ at $t = 1$ offers

$$\mu(1) = \sigma(1) - 1 = 2 - 1 = 1.$$

Hence $t = 1$ is left-dense and right-scattered. If $t = 2$, then

$$\sigma(2) = \inf\{s \in \mathbb{T} : s > 2\} = 2$$

and

$$\rho(2) = \sup\{s \in \mathbb{T} : s < 2\} = 1.$$

The graininess function μ at $t = 2$ offers

$$\mu(2) = \sigma(2) - 2 = 2 - 2 = 0.$$

Hence $t = 2$ is left-scattered and right-dense. If $t = 3$, then

$$\sigma(3) = \inf\{s \in \mathbb{T} : s > 3\} = 3$$

and

$$\rho(3) = \sup\{s \in \mathbb{T} : s < 3\} = 3.$$

The graininess function μ at $t = 3$ offers

$$\mu(3) = \sigma(3) - 3 = 3 - 3 = 0.$$

Hence $t = 3$ is dense.

Figure 4.1: Visual representation of Example 1 Part 3, $\mathbb{T} = [0, 1] \cup [2, 3]$

The above example shows how the operators and graininess function perform on various time scales. Using our knowledge of the above terms, in the next section we will examine how differentiation works in Time Scale Calculus.

4.2 Differentiation

In this section, we will provide definitions, theorems, and examples to assist us in gaining an understanding of differentiation in Time Scale Calculus. This will allow us to see the similarities and differences between this and traditional calculus definitions and theorems on differentiation.

Let us start with the definition of the derivative of a function from Time Scale Calculus. We call this derivative the *delta derivative*.

Definition 5 ([3]). *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$, the* **delta derivative** *of f at t, to be the number (provided it exists) with the property that given any $\epsilon > 0$, there exist a neighborhood U of t (that is, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$), such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \ \textit{for all} \ s \in U.$$

Note that throughout this work when we refer to *derivative* we mean the *delta derivative* and will use the symbol $f^{\Delta}(t)$ to denote the derivative of f at t.

The following theorem provides a useful characterizations of delta differentiable functions.

Theorem 2 ([3]). *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^k$. Then we have the following:*

1. *If f is differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

3. *If t is right-dense, then f is differentiable at t iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

We refer to this formula as the "simple useful formula."

Now let us consider some examples of how this differentiation works on a time scale.

Example 3. Let $f : \mathbb{T} \to \mathbb{R}$.

1. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$ and if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta} = \Delta f(t) \equiv f(t+1) - f(t)$.

2. Refer to our previous Example (1) part 3, $\mathbb{T} = [0, 1] \cup [2, 3]$. We desire to compute the derivative at points 0.5, 1, 2, and 2.5. By using the definition of derivative we have:

   (a) At dense point $t = 0.5$
   $$f^{\Delta}(0.5) = \lim_{s \to 0.5} \frac{f(0.5) - f(s)}{0.5 - s}.$$

   (b) The point $t = 1$ is right-scattered, so
   $$f^{\Delta}(1) = \frac{f(\sigma(1)) - f(1)}{\mu(1)} = \frac{f(2) - f(1)}{1} = f(2) - f(1).$$

(c) The point $t = 2$ is right-dense, so

$$f^{\Delta}(2) = \lim_{s \to 2} \frac{f(2) - f(s)}{2 - s}.$$

(d) The point $t = 2.5$ is dense, so

$$f^{\Delta}(2.5) = \lim_{s \to 2.5} \frac{f(2.5) - f(s)}{2.5 - s}.$$

These examples show us how in $\mathbb{R}$ we can use our traditional understanding of the derivative from calculus, and with other time scales, we must use the theorems involving the delta derivative. The following theorem shows the linearity of the derivative, as well as the product and quotient rules for delta differentiation.

Theorem 4 ([3])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in T^k$. Then:*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

4. *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

5. *If* $g(t)g(\sigma(t)) \neq 0$*, then* $\frac{f}{g}$ *is differentiable at* t *and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Note that we use the notation f^{σ} and $f \circ \sigma$ interchangeably in the next sections. Thus, if $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is a function, it is defined by

$$f^{\sigma}(t) = f(\sigma(t)), \text{for all } t \in \mathbb{T}.$$

4.3 Integration

To be able to describe classes of functions that are "integrable", we introduce the following two concepts. We must start with the definitions of regulated and right-dense continuous functions before we define the delta antiderivative.

Definition 6 ([3])**.** *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called* **regulated** *provided its right-sided limits exist (finite) at all right-dense points in* $\mathbb{T}$ *and its left-sided limits exist (finite) at all left-dense points in* $\mathbb{T}$*.*

Definition 7 ([3])**.** *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called* **rd-continuous** *provided it is continuous at each right-dense point in* $\mathbb{T}$ *and its left-side limits exist (finite) at all left-dense points in* $\mathbb{T}$*. The set of rd-continuous functions* $f : \mathbb{T} \to \mathbb{R}$ *is denoted by*

$$C_{rd} \equiv C_{rd}(\mathbb{T}) \equiv C_{rd}(\mathbb{T}, \mathbb{R}).$$

This next theorem provides us with results that focus on regulated, right-dense continuous, and continuous functions.

Theorem 5 ([3])**.** *Assume* $f : \mathbb{T} \to \mathbb{R}$*.*

1. *If* f *is continuous, then* f *is rd-continuous.*

2. *If* f *is rd-continuous, then* f *is regulated.*

3. *The jump operator* σ *is rd-continuous.*

4. *If f is regulated or rd-continuous, then so is f^{σ}.*

5. *Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.*

Now with the knowledge of properties of regulated and right-dense continuous functions we need the concept of a pre-differentiable function and the existence of pre-antiderivatives before defining our delta antiderivative.

Theorem 6 ([3]). *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D [[3]] such that*

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in D.$$

Definition 8 ([3]). *Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 6 is called a* **pre-antiderivative** *of f. We define the* **indefinite integral** *of a regulated function f by*

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for all } r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^k.$$

4.4 Hilger's Complex Plane

Hilger's complex plane is necessary for calculating the generalization of the exponential function on a time scale. In Hilger's complex plane, the Hilger imaginary circle is tangent to the imaginary axis and the diameter of the circle is the reciprocal of the graininess, h.

Definition 9 ([3]). *For $h > 0$ we define the* **Hilger complex numbers***, the* **Hilger real axis***,* *the* **Hilger alternating axis**, *and the* **Hilger imaginary circle** *as*

$$\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -\frac{1}{h}\}$$
$$\mathbb{R}_h := \{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h}\}$$
$$\mathbb{A}_h := \{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h}\}$$
$$\mathbb{I}_h := \{z \in \mathbb{C}_h : |z + \frac{1}{h}| = \frac{1}{h}\}$$

respectively. Note that for $h = 0$, *let* $\mathbb{C}_0 := \mathbb{C}, \mathbb{R}_0 := \mathbb{R}, \mathbb{I}_0 := i\mathbb{R}$, *and* $\mathbb{A}_0 := \emptyset$. *Refer to Figure 4.2 to see a visualization of Hilger's complex plane.*

Now, we need the definition of the Hilger real part and imaginary part of a complex number z.

Definition 10 ([3]). *Let* $h > 0$ *and* $z \in \mathbb{C}_h$. *We define the* **Hilger real part** *of* z *by*

$$\mathbb{R}e_h(z) := \frac{|zh + 1| - 1}{h}$$

and the **Hilger imaginary part** *of* z *by*

$$\mathbb{I}m_h(z) := \frac{\arg(zh + 1)}{h}$$

where $-\pi < \arg(z) \leq \pi$.

In Figure 4.2, we can see how Hilger's complex plane is visualized.

Note that as the graininess, h, decreases then $-\frac{1}{h}$ becomes large causing the Hilger circle to become large as well.

Definition 11 ([3]). *The* **Hilger purely imaginary number** $\mathring{\imath}\omega$ *is stated by*

$$\mathring{\imath}\omega = \frac{e^{i\omega h} - 1}{h}$$

where $-\frac{\pi}{h} < \omega \leq \frac{\pi}{h}$.

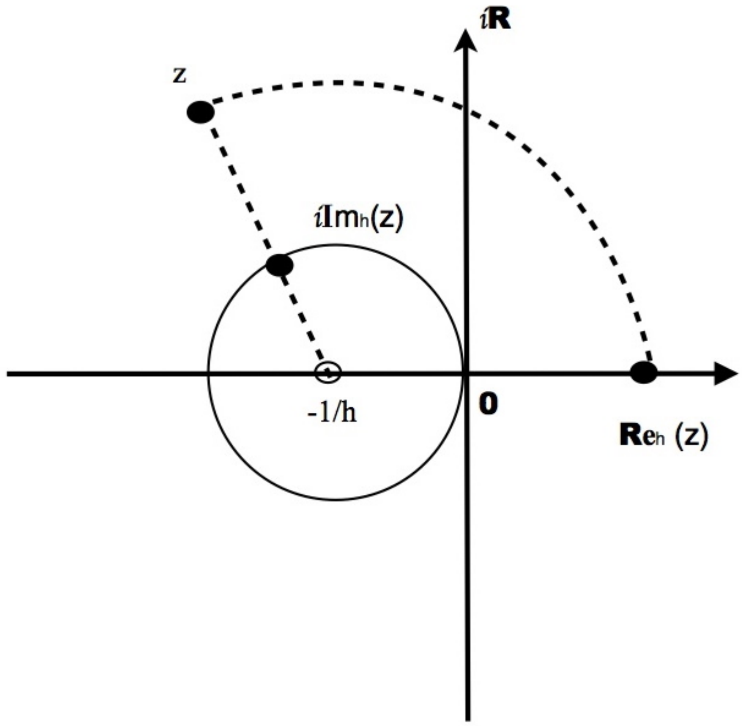

Figure 4.2: Hilger's Complex Plane

The Hilger imaginary circle has a radius of $\frac{1}{h}$ and is positioned $\frac{1}{h}$ units to the left of the origin.

Now we need to define circle plus and circle minus, which are both defined on $\mathbb{C}_h$. Also, we must define the cylinder transformation and the inverse cylinder transformation in order to define and understand our exponential function on a time scale.

Definition 12 ([3]). *The formula for* **circle plus addition** $\oplus$ *on* $\mathbb{C}_h$ *is*

$$z \oplus w := z + w + zwh.$$

Note that $(\mathbb{C}_h, \oplus)$ *forms an Abelian group.*

Additionally, we use the definition of circle plus extensively when dealing with the exponential function on time scales.

Theorem 7 ([3])**.** *For $z \in \mathbb{C}_h$ we have*

$$z = \mathbb{R}e_h z \oplus i\mathbb{I}m_h z.$$

As one would guess, we also have a definition for circle minus, which uses the concept of circle plus.

Definition 13 ([3])**.** *The* **circle minus subtraction** $\ominus$ *on* $\mathbb{C}_h$ *has the form*

$$z \ominus w := z \oplus (\ominus w)$$

where

$$\ominus w := \frac{-w}{1+wh}$$

Note that if $z, w \in \mathbb{C}_h$ with $h \geq 0$ then the following properties hold

1. $z \ominus z = 0$
2. $z \ominus w = \frac{z-w}{1+wh}$
3. $z \ominus w = z - w$ if $h = 0$.

Next we provide the important definition of the cylinder transformation that is used in our construction of the exponential function.

Definition 14 ([3])**.** *For $h > 0$, we define the* **cylinder transformation** $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h} Log(1 + zh)$$

where Log is the principle logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$. Note that $\mathbb{Z}_h$ is the strip defined by $\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \mathbb{I}m(z) \leq \frac{\pi}{h}\}$ for all $h > 0$.

Definition 15 ([3])**.** *The* **inverse transformation** *of the cylinder transformation ξ_h when $h > 0$ is given by*

$$\xi_h^{-1}(z) = \frac{1}{h}(e^{zh} - 1)$$

for $z \in \mathbb{Z}_h$.

Note that we call ξ_h, the cylinder transformation, since when $h > 0$ we can view $\mathbb{Z}_h$ as a cylinder if we join the boundary lines $Im(z) = -\frac{\pi}{h}$ and $Im(z) = \frac{\pi}{h}$ of $\mathbb{Z}_h$ together to form a cylinder.

4.5 Exponential Function

Now using our knowledge of the cylinder transformation we can generalize the exponential function for a time scale. To arrive at this result we need a few additional definitions.

Definition 16 ([3])**.** *We say that a function* $p : \mathbb{T} \to \mathbb{R}$ *is* **regressive** *provided*

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^K$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ *will be denoted by* $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

We need the conditions of regressive and rd-continuous in order to evaluate the second order linear dynamic equation. Now, we will define the general exponential function on a time scale.

Definition 17 ([3])**.** *If* $p \in \mathcal{R}$, *then we define the* **exponential function** *by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\right) \Delta\tau \tag{4.1}$$

Note that

$$\xi_{\mu(\tau)} = \frac{1}{\mu(\tau)} Log(1 + z\mu(\tau)) \tag{4.2}$$

is the cylinder transformation with respect to $\mu(\tau)$ *and Log is the principle logarithm.*

One property of the exponential function is that if $p \in \mathbb{R}$, then the semigroup property

$$e_p(t, r)e_p(r, s) = e_p(t, s)$$

for all $r, s, t \in \mathbb{T}$ holds.

The following theorem provides us with some additional properties of the exponential function.

Theorem 8 ([3]). *If $p, q \in \mathcal{R}$, then*

1. $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

2. $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$*;*

3. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

4. $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

5. $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

6. $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

7. $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$*;*

8. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$

9. $e_p^{\Delta}(t,t_0) = p(t) \cdot e_p(t,t_0)$*.*

Now, we have arrived at the understanding of the basic terms of time scales calculus in order to be able to analytically and graphically evaluate the second order linear dynamic equation!

4.6 Second Order Linear Dynamic Equations

In this section, we will define the second order linear dynamic equation. We will offer both the general and particular solutions to this dynamic equation along with the necessary definitions and theorems leading up to these results. Note that the second order linear dynamic equation has the form

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t) \tag{4.3}$$

where $p, q, f \in C_{rd}$. Let the linear operator $L_2 : C_{rd}^2 \to C_{rd}$ be defined

$$L_2 y(t) = y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y$$

for $t \in (T^K)^K$.

Within the proofs of existence and uniqueness of solutions we need the regressive and rd-continuous conditions.

Definition 18 ([3]). *Equation 4.3 is* **regressive** *given $p, q, f \in C_{rd}$ such that the regressive condition is*

$$1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0 \quad \textit{for all}\, t \in \mathbb{T}^{K^2}. \tag{4.4}$$

Before we can discuss conditions that insure uniqueness and existence of solutions of the second order linear dynamic equation we need the concept of the Wronskian. The Wronskian is used in the proof of the existence of solutions.

Definition 19 ([3]). *Let y_1 and y_2 be two differentiable functions. We define the* **Wronskian** *$W = W(y_1, y_2)$ by*

$$W(t) = det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1^{\Delta}(t) & y_2^{\Delta}(t) \end{pmatrix}.$$

Note that when $W(y_1, y_2) \neq 0$ for all $t \in \mathbb{T}^K$, then the solutions y_1 and y_2 form a fundamental system for the homogeneous equation $L_2 y = 0$.

The next theorem provides us with conditions that insure the existence and uniqueness to solutions.

Theorem 9 ([3]). *Suppose the dynamic equation 4.3 is regressive. If $t_0 \in \mathbb{T}^K$, then the initial value problem*

$$L_2 y = f(t), \quad y(t_0) = y_0, \quad y^{\Delta}(t_0) = y_0^{\Delta},$$

where y_0 and y_0^{Δ} are given constants, has an unique solution, and this solution is defined on the whole time scale $\mathbb{T}$.

One of our main goals is to use the following two theorems to be able to evaluate the analytical behavior of the second order linear homogeneous dynamic equation. Also, note that when we state general solutions we mean every function of this form is a solution and every solution is in this form.

Theorem 10 ([3]). *If the pair of functions y_1, y_2 forms a fundamental system of solutions for $L_2 y = 0$, then $y(t) = \alpha y_1(t) + \beta y_2(t)$, where α and β are constants, is a general solution of $L_2 y = 0$. In particular, the solution of the initial value problem ($L_2 y = 0$, $y(t_0) = y_0$, $y^{\Delta}(t_0) = y_0^{\Delta}$)*

is given by

$$y(t) = \frac{y_2^\Delta(t_0)y_0 - y_2(t_0)y_0^\Delta}{W(y_1,y_2)(t_0)} y_1(t) + \frac{y_1(t_0)y_0^\Delta - y_1^\Delta(t_0)y_0}{W(y_1,y_2)(t_0)} y_2(t)$$

More specifically, our focus is the second order linear dynamic homogeneous equation with constant coefficients

$$y^{\Delta\Delta} + \alpha y^\Delta + \beta y = 0 \tag{4.5}$$

where $\alpha, \beta \in \mathbb{R}$ on a time scale $\mathbb{T}$.

Similar to our traditional understanding of the second order differential equation we need the characteristic equation in order to obtain a solution. Note that if $y(t) = e_\lambda(t,t_0)$, then

$$\begin{aligned} y^{\Delta\Delta}(t) + \alpha y^\Delta(t) + \beta y(t) &= \lambda^2 e_\lambda(t,t_0) + \alpha\lambda e_\lambda(t,t_0) + \beta e_\lambda(t,t_0) \\ &= (\lambda^2 + \alpha\lambda + \beta)e_\lambda(t,t_0). \end{aligned}$$

The above uses the fact that $e_\lambda^\Delta(t,t_0) = \lambda(t) \cdot e_\lambda(t,t_0)$, which is shown in the proof of Theorem 2.33 in [3]. Thus, the characteristic equation is

$$\lambda^2 + \alpha\lambda + \beta = 0 \tag{4.6}$$

and the solutions λ_1 and λ_2 of equation (4.6) are

$$\lambda_1 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \text{ and } \lambda_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \tag{4.7}$$

by using the quadratic formula.

The following theorem provides us with the general solution of the homogeneous linear dynamic equation and the solution of the initial value problem.

Theorem 11 ([3]). *Suppose $\alpha^2 - 4\beta \neq 0$. If $\mu\beta - \alpha \in \mathcal{R}$, then a fundamental system of equations (4.5) is given by*

$$e_{\lambda_1}(t,t_0) \textit{ and } e_{\lambda_2}(t,t_0)$$

where $t_0 \in \mathbb{T}^K$ and λ_1 and λ_2 are given in (4.7). The solution of the initial value problem

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0, \;\; y(t_0) = y_0, \; y^{\Delta}(t_0) = y_0^{\Delta} \tag{4.8}$$

is given by

$$y(t) = y_0 \frac{e_{\lambda_1}(t,t_0) + e_{\lambda_2}(t,t_0)}{2} + \frac{\alpha y_0 + 2y_0^{\Delta}}{\sqrt{\alpha^2 - 4\beta}} \, \frac{e_{\lambda_2}(t,t_0) - e_{\lambda_1}(t,t_0)}{2}.$$

Now we have three theorems dealing with the solutions of the second order linear dynamic equation. Just as in traditional differential equations we will either have distinct, repeated or complex roots. Let us first consider the case when $\alpha^2 - 4\beta < 0$. However, before we do this we need definitions of $\sin_p$ and $\cos_p$ in Time Scale Calculus .

Definition 20 ([3])**.** *If $p \in C_{rd}$ and $\mu p^2 \in \mathcal{R}$, then we define the trigonometric functions $\cos_p$ and $\sin_p$ by*

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} \;\; \textit{and} \;\; \sin_p = \frac{e_{ip} - e_{-ip}}{2i}.$$

We begin with the case when $\alpha^2 - 4\beta < 0$.

Theorem 12 ([3])**.** *Suppose $\alpha^2 - 4\beta < 0$. Define*

$$p = \frac{-\alpha}{2} \;\; \textit{and} \;\; q = \frac{\sqrt{4\beta - \alpha^2}}{2}$$

If p and $\mu\beta - \alpha$ are regressive, the a fundamental system of (4.5) is given by

$$\cos_{\frac{q}{1+\mu p}}(t,t_0)e_p(t,t_0) \;\textit{and}\; \sin_{\frac{q}{1+\mu p}}(t,t_0)e_p(t,t_0)$$

where $t_0 \in \mathbb{T}$, and the Wronskian of these two solutions is $q\, e_{\mu\beta-\alpha}(t,t_0)$. The solution of the initial value problem (4.8) is given by

$$y(t) = \left[y_0 \cos_{\frac{q}{1+\mu p}}(t,t_0) + \frac{y_0^{\Delta} - p y_0}{q} \, \sin_{\frac{q}{1+\mu p}}(t,t_0) \right] e_p(t,t_0).$$

Next, let us consider the case where $\alpha^2 - 4\beta = 0$.

Theorem 13 ([3])**.** *Suppose $\alpha^2 - 4\beta = 0$. Define $p = \frac{-\alpha}{2}$. If $p \in \mathcal{R}$, then a fundamental system of*

(4.5) is given by

$$e_p(t,t_0) \text{ and } e_p(t,t_0)\int_{t_0}^{t}\frac{1}{1+p\mu(\tau)}\Delta\tau$$

where $t_0 \in \mathbb{T}$*, and the Wronskian of these two solutions is* $e_{\frac{\mu\alpha^2}{4}}(t,t_0)$*. The solution of the initial value problem (4.8) is given by*

$$y(t) = e_p(t,t_0)\left[y_0 + (y_0^{\Delta} - py_0)\int_{t_0}^{t}\frac{\Delta\tau}{1+p\mu(\tau)}\right].$$

Finally, let us consider the case when $\alpha^2 - 4\beta > 0$.

Theorem 14 ([3])**.** *Suppose* $\alpha^2 - 4\beta > 0$*. Define*

$$p = \frac{-\alpha}{2} \text{ and } q = \frac{\sqrt{\alpha^2 - 4\beta}}{2}$$

If p *and* $\mu\beta - \alpha$ *are regressive, the a fundamental system of (4.5) is given by*

$$\cosh_{\frac{q}{1+\mu p}}(t,t_0)e_p(t,t_0) \text{ and } \sinh_{\frac{q}{1+\mu p}}(t,t_0)e_p(t,t_0)$$

where $t_0 \in \mathbb{T}$*, and the Wronskian of these two solutions is* $qe_{\mu\beta-\alpha}(t,t_0)$*. The solution of the initial value problem (4.8) is given by*

$$y(t) = \left[y_0 \cosh_{\frac{q}{1+\mu p}}(t,t_0) + \frac{y_0^{\Delta} - py_0}{q}\ \sinh_{\frac{q}{1+\mu p}}(t,t_0)\right]e_p(t,t_0).$$

The hyperbolic functions used in the above theorem are defined as follows and are very similar to $\cos_p$ and $\sin_p$.

Definition 21 ([3])**.** *If* $p \in C_{rd}$ *and* $-\mu p^2 \in \mathcal{R}$*, the we define the* **hyperbolic functions** $\cosh_p$ *and* $\sinh_p$ *by*

$$\cosh_p = \frac{e_p + e_{-p}}{2} \text{ and } \sinh_p = \frac{e_p - e_{-p}}{2}.$$

CHAPTER 5

A SECOND ORDER LINEAR HOMOGENEOUS DYNAMIC EQUATION: ANALYTICAL SOLUTIONS

In this chapter, we will discuss the solutions of a particular second order linear dynamic equation. We will provide analytical solutions on varying time scales. This section mainly uses theorems and definitions from Chapter 4, such as Theorem 11 and Theorem 12, along with Definition 17 and Definition 20.

5.1 Our Initial Value Problem

This work's focus is a study of simple harmonic motion using dynamic equations on varying time scales. Recall, that in differential equations (when $\mathbb{T} = \mathbb{R}$) this problems looks like $y'' + y = 0$ with initial conditions $y''(0) = 0$ and $y'(0) = 1$. We know that this initial value problem has the general solution $y(t) = c_1 \cos(t) + c_2 \sin(t)$ where c_1 and c_2 are constants. Thus, the particular solution to our example is $y(t) = sin(t)$. Also, note that the roots of the associated characteristic equation are complex.

Our initial value problem, IVP,

$$y^{\Delta\Delta} = -y \qquad y^{\Delta\Delta}(0) = 0,\ y^{\Delta}(0) = 1$$

has a unique solution Theorem 9. The time scale that we will use starts with one large gap and we will slowly close the gap, thus, showing how the solution converges to the solution of the original time scale on the limiting time scale. Our time scale is a union of two closed intervals, $[0, \pi] \cup [2\pi, 3\pi]$. We will slowly decrease the gap between the intervals and see how the solution tends toward the solution of our second order dynamic equation on $\mathbb{T} = [0, 3\pi]$. Be sure to note that for this particular IVP has only one set of initial conditions are necessary. Time Scales Calculus allows us to have disjoint sets in our domain, but requires only the one set of initial conditions. This is because when we have a gap the value of the solution after the gap comes from information that precedes the gap.

Now, lets look at $[0, \pi] \cup [2\pi, 3\pi]$ analytically and note that it has an unique solution. Using theorems and definitions from Chapter 4 we know the dynamic equation

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0 \text{ for } y(t_0) = y_0,\ y^{\Delta}(t_0) = y_0^{\Delta}$$

has the general solution

$$y(t) = \left[y_0 \cos_{\frac{q}{1+\mu p}}(t, t_0) + \frac{y_0^{\Delta} - p y_0}{q} \sin_{\frac{q}{1+\mu p}}(t, t_0) \right] e_p(t, t_0).$$

For this particular case we have $\alpha^2 - 4\beta = 0^2 - 4(1) = -4 < 0$. Since $\alpha = 0$, $\beta = 1$ then $p = -\frac{\alpha}{2} = 0$ and $q = \frac{\sqrt{4\beta - \alpha^2}}{2} = 1$. Furthermore the characteristic equation of the dynamic equation $y^{\Delta\Delta} = -y$ is $\lambda^2 + 1 = 0$. So, its roots are, $\lambda_1 = i$ and $\lambda_2 = -i$.

Now, we will look at $t \in [0, \pi]$, $t = 2\pi$, and $t \in [2\pi, 3\pi]$. We will be using both Hilger's sine, cosine, and exponential functions as well as sine, cosine, and exponential functions from our traditional calculus (when $\mathbb{T} = \mathbb{R}$). Also, we will need to use Euler's Formula to evaluate $e^{i\pi}$ and e^{it}. Note that $e_p(t, t_0)$ is the Hilger exponential function and has the property $e_0(t, t_0) = 1$. Also, the Hilger cosine and sine functions are defined as follows: $\cos_p = \frac{e_{ip}(t,t_0) + e_{-ip}(t,t_0)}{2}$ and $\sin_p = \frac{e_{ip}(t,t_0) - e_{-ip}(t,t_0)}{2i}$. The analytical solution follows.

- For $t \in [0,\pi]$, since $y_0 = 0$, $y_0^\Delta = 1$ and $\mu(t) = 0$ we have

$$
\begin{aligned}
y(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t,t_0) + \frac{y_0^\Delta - p y_0}{q} \sin_{\frac{q}{1+\mu p}}(t,t_0) \right] e_p(t,t_0) \\
&= \left[(0) \cos_{\frac{1}{1+0}}(t,0) + \frac{1+0}{1} \sin_{\frac{1}{1+0}}(t,0) \right] e_0(t,0) \\
&= \sin_1(t,0) \\
&= \frac{e_i(t,0) - e_{-i}(t,0)}{2i} \\
&= \frac{\exp(\int_0^t i\Delta\tau) - \exp(\int_0^t (-i)\Delta\tau)}{2i} \\
&= \frac{\exp(it) - \exp(-it)}{2i} \\
&= \frac{\cos(t) + i\sin(t) - (\cos(-t) + i\sin(-t))}{2i} \\
&= \frac{2i\sin(t)}{2i} \\
&= \sin(t)
\end{aligned}
$$

- For $t = 2\pi$, since $\mu(\pi) = \pi$ we have two methods to determine the solution. We can use the simple useful formula such that

$$
\begin{aligned}
y(\sigma(t)) &= y(t) + \mu(t) y^\Delta(t) \\
y(\sigma(\pi)) &= y(\pi) + \mu(\pi) y^\Delta(\pi) \\
y(2\pi) &= 0 + \pi(-1) \\
&= -\pi.
\end{aligned}
$$

Or, we can use the general solution,

$$
\begin{aligned}
y(2\pi) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t,t_0) + \frac{y_0^{\Delta} - p y_0}{q} \sin_{\frac{q}{1+\mu p}}(t,t_0)\right] e_p(t,t_0) \\
&= \left[(0) \cos_{\frac{1}{1+(\pi)0}}(2\pi,0) + \frac{1+0}{1} \sin_{\frac{1}{1+(\pi)0}}(2\pi,0)\right] e_0(2\pi,0) \\
&= \sin_1(2\pi,0) \\
&= \frac{e_i(2\pi,0) - e_{-i}(2\pi,0)}{2i} \\
&= \frac{\exp\left(\int_0^{\pi} \xi_0(i)\Delta\tau + \int_{\pi}^{2\pi} \xi_\pi(i)\Delta\tau\right)}{2i} \\
&\quad - \frac{\exp\left(\int_0^{\pi} \xi_0(-i)\Delta\tau + \int_{\pi}^{2\pi} \xi_\pi(-i)\Delta\tau\right)}{2i} \\
&= \frac{\exp\left(\int_0^{\pi} (i)\Delta\tau + \int_{\pi}^{2\pi} \frac{1}{\pi}\mathrm{Log}(1+i\pi)\Delta\tau\right)}{2i} \\
&\quad - \frac{\exp\left(\int_0^{\pi} (-i)\Delta\tau + \int_{\pi}^{2\pi} \frac{1}{\pi}\mathrm{Log}(1-i\pi)\Delta\tau\right)}{2i} \\
&= \frac{e^{i\pi} e^{\mathrm{Log}(1+i\pi)} - e^{-i\pi} e^{\mathrm{Log}(1-i\pi)}}{2i} \\
&= \frac{(1+i\pi)(-1) - (-1)(1-i\pi)}{2i} \\
&= -\pi
\end{aligned}
$$

- For $t \in (2\pi, 3\pi]$, since $\mu(t) = 0$ we have

$$\begin{aligned}
y(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t, t_0) + \frac{y_0^\Delta - p y_0}{q} \sin_{\frac{q}{1+\mu p}}(t, t_0)\right] e_p(t, t_0) \\
&= \left[(0) \cos_{\frac{1}{1+0}}(t, 0) + \frac{1+0}{1} \sin_{\frac{1}{1+0}}(t, 0)\right] e_0(t, 0) \\
&= [\sin_1(t, 0)]\,(1) \\
&= \frac{e_i(t, 0) - e_{-i}(t, 0)}{2i} \\
&= \frac{\exp\left(\int_0^\pi \xi_0(i)\Delta\tau + \int_\pi^{2\pi} \xi_\pi(i)\Delta\tau + \int_{2\pi}^t \xi_0(i)\Delta\tau\right)}{2i} \\
&\quad - \frac{\exp\left(\int_0^\pi \xi_0(-i)\Delta\tau + \int_\pi^{2\pi} \xi_\pi(-i)\Delta\tau \int_{2\pi}^t \xi_0(-i)\Delta\tau\right)}{2i} \\
&= \frac{\exp((i)[\pi - 0] + \frac{1}{\pi}\mathrm{Log}(1 + i\pi)[2\pi - \pi] + i[t - 2\pi])}{2i} \\
&\quad - \frac{\exp((-i)[\pi - 0] + \frac{1}{\pi}\mathrm{Log}(1 - i\pi)[2\pi - \pi] + (-i)[t - 2\pi])}{2i} \\
&= \frac{e^{i\pi}(1 + i\pi)e^{it}e^{-2i\pi} - e^{-i\pi}(1 - i\pi)e^{-it}e^{2i\pi}}{2i} \\
&= \frac{(-1 - i\pi)e^{it} + (1 - i\pi)e^{-it}}{2i} \\
&= \frac{-2i(\sin(t) + \pi\cos(t))}{2i} \\
&= -(\sin(t) + \pi\cos(t)).
\end{aligned}$$

Thus, the particular solution for $\mathbb{T} = [0, \pi] \cup [2\pi, 3\pi]$ is

$$y(t) = \begin{cases} \sin(t), & \text{for } t \in [0, \pi] \\ -\pi, & \text{for } t = 2\pi \\ -(\sin(t) + \pi\cos(t)), & \text{for } t \in (2\pi, 3\pi] \end{cases}.$$

5.2 More Insight Into Different Time Scales

The above example uses the time scale $\mathbb{T} = [0, \pi] \cup [2\pi, 3\pi]$, so $y^{\Delta\Delta}(t)$, at $t = \pi$, has value 0. This reduces easily and hides some of the analytical structure of the solution, so we decided to try different time scales to seek more insight. Note that to get $y^\Delta(\sigma(t))$ we use the simple useful formula, $y(\sigma(t)) = y(t) + \mu(t)y^{\Delta}(t)$. Now for $y^{\Delta}(\sigma(t))$ the simple useful formula is $y^{\Delta}(\sigma(t)) = y^{\Delta}(t) + \mu(t)y^{\Delta\Delta}(t)$. Here we have $y^{\Delta\Delta}(\pi) = 0$, so the simple useful formula becomes $y^{\Delta}(\sigma(t)) = y^{\Delta}(t) = 0$. The following example does not have this simplification.

We decided to look at $\mathbb{T} = [0, 3\pi/4] \cup [2\pi, 3\pi]$ with the same particular dynamic equation, $y^{\Delta\Delta} = -y,\ y^{\Delta\Delta}(0) = 0,\ y^{\Delta}(0) = 1$. This has the piecewise solution

$$y(t) = \begin{cases} \sin(t), & \text{for } t \in [0, 3\pi/4] \\ \frac{(-5\pi+4)\sqrt{2}}{8}, & \text{for } t = 2\pi \\ -(\frac{\sqrt{2}}{2})(\frac{5\pi}{4})\cos(t-2\pi) + (\frac{\sqrt{2}}{2})\cos(t-2\pi) & \\ -(\frac{\sqrt{2}}{2})\sin(t-2\pi) - (\frac{\sqrt{2}}{2})(\frac{5\pi}{4})\sin(t-2\pi), & \text{for } t \in (2\pi, 3\pi] \end{cases}.$$

Then we looked at the piecewise solution for this same particular IVP if we make the gap smaller or make $\mu(t)$ smaller.

We decided to look at $\mathbb{T} = [0, 3\pi/4] \cup [3\pi/2, 3\pi]$. This has the piecewise solution

$$y(t) = \begin{cases} \sin(t), & \text{for } t \in [0, 3\pi/4] \\ \frac{(-3\pi+4)\sqrt{2}}{8}, & \text{for } t = 3\pi/2 \\ -(\frac{\sqrt{2}}{2})(\frac{3\pi}{4})\cos(t-\frac{3\pi}{2}) + (\frac{\sqrt{2}}{2})\cos(t-\frac{3\pi}{2}) & \\ -(\frac{\sqrt{2}}{2})\sin(t-\frac{3\pi}{2}) - (\frac{\sqrt{2}}{2})(\frac{3\pi}{4})\sin(t-\frac{3\pi}{2}), & \text{for } t \in (3\pi/2, 3\pi] \end{cases}.$$

By comparing these two time scales with the same IVP we can see that we have the same form of the piecewise solution. In the first interval the solution is $\sin(t)$ and when $t = \frac{3\pi}{2}$ or $t = 2\pi$ the solution value is a negative number. The value at $t = \frac{3\pi}{2}$ is $\frac{(-3\pi+4)\sqrt{2}}{8}$ and $t = 2\pi$ is $\frac{(-5\pi+4)\sqrt{2}}{8}$. For the second interval we have the form

$$(\text{constant})\mu(t)[-\cos(t - \text{shift}) - \sin(t - \text{shift})] + (\text{constant})[\cos(t - \text{shift}) - \sin(t - \text{shift})].$$

Thus, as we close the gap the only values that are changing are the constant and the shift, where the shift takes the value of the start of the next interval.

5.3 The General Case

Now, let's consider the general case of $y^{\Delta\Delta} = -y$ with initial conditions $y(t_0) = y_0$, $y^{\Delta}(t_0) = y_0^{\Delta}$ with $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$. From Theorem 12 with $\alpha = 0$, $\beta = 1$, $p = 0$, and $q = 1$ our solution takes the form:

- For $t \in [t_0, t_1]$ with $\mu(t) = 0$,

$$\begin{aligned}
y(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t, t_0) + \frac{y_0^{\Delta} - p \cdot y_0}{q} \sin_{\frac{q}{1+\mu p}}(t, t_0) \right] e_p(t, t_0) \\
&= \left[y_0 \cos_{\frac{1}{1+0}}(t, t_0) + \frac{y_0^{\Delta} + 0 \cdot y_0}{1} \sin_{\frac{1}{1+0}}(t, t_0) \right] e_0(t, t_0) \\
&= \left[y_0 \cos_1(t, t_0) + y_0^{\Delta} \sin_1(t, t_0) \right] (1) \\
&= y_0 \frac{e_i + e_{-i}}{2} + y_0^{\Delta} \frac{e_i - e_{-i}}{2i} \\
&= y_0 \frac{\exp(\int_{t_0}^{t} i \Delta\tau) + \exp(\int_{t_0}^{t} -i \Delta\tau)}{2} + y_0^{\Delta} \frac{\exp(\int_{t_0}^{t} i \Delta\tau) - \exp(\int_{t_0}^{t} -i \Delta\tau)}{2i} \\
&= y_0 \frac{(e^{it} e^{-it_0} + e^{it_0} e^{-it})}{2} + y_0^{\Delta} \frac{(e^{it} e^{-it_0} - e^{it_0} e^{-it})}{2i} \\
&= y_0 \left(\frac{(\cos(t - t_0) + i \sin(t - t_0)) + (\cos(-(t - t_0)) + i \sin(-(t - t_0))}{2} \right) \\
&\quad + y_0^{\Delta} \left(\frac{(\cos(t - t_0) + i \sin(t - t_0)) - (\cos(-(t - t_0)) + i \sin(-(t - t_0))}{2i} \right) \\
&= y_0 \cos(t - t_0) + y_0^{\Delta} \sin(t - t_0)
\end{aligned}$$

- For $t = t_2$ with $\mu(t_1) = t_2 - t_1$,

$$\begin{aligned}
y(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t_2, t_0) + \frac{y_0^{\Delta} - p \cdot y_0}{q} \sin_{\frac{q}{1+\mu p}}(t_2, t_0) \right] e_p(t_2, t_0) \\
&= \left[y_0 \cos_{\frac{1}{1+0}}(t_2, t_0) + \frac{y_0^{\Delta} + 0 \cdot y_0}{1} \sin_{\frac{1}{1+0}}(t_2, t_0) \right] e_0(t_2, t_0) \\
&= \left[y_0 \cos_1(t_2, t_0) + y_0^{\Delta} \sin_1(t_2, t_0) \right] (1) \\
&= y_0 \frac{\exp(\int_{t_0}^{t_2} \xi_{\mu(\tau)} p(\tau) \Delta\tau) + \exp(\int_{t_0}^{t_2} \xi_{\mu(\tau)} p(\tau) \Delta\tau)}{2} \\
&\quad + y_0^{\Delta} \frac{\exp(\int_{t_0}^{t_2} \xi_{\mu(\tau)} p(\tau) \Delta\tau) - \exp(\int_{t_0}^{t_2} \xi_{\mu(\tau)} p(\tau) \Delta\tau)}{2i} \\
&= y_0[\cos(t_1 - t_0) - \sin(t_1 - t_0)\mu(t_1)] + y_0^{\Delta}[\sin(t_1 - t_0) + \cos(t_1 - t_0)\mu(t_1)]
\end{aligned}$$

- For $t \in (t_2, t_3]$ with $\mu(t) = 0$,

$$\begin{aligned}
y(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t,t_0) + \frac{y_0^\Delta - p \cdot y_0}{q} \sin_{\frac{q}{1+\mu p}}(t,t_0)\right] e_p(t,t_0) \\
&= \left[y_0 \cos_{\frac{1}{1+0}}(t,t_0) + \frac{y_0^\Delta + 0 \cdot y_0}{1} \sin_{\frac{1}{1+0}}(t,t_0)\right] e_0(t,t_0) \\
&= \left[y_0 \cos_1(t,t_0) + y_0^\Delta \sin_1(t,t_0)\right] (1) \\
&= y_0 \frac{\exp(\int_{t_0}^t \xi_{\mu(\tau)} p(\tau)\Delta\tau) + \exp(\int_{t_0}^t \xi_{\mu(\tau)} p(\tau)\Delta\tau)}{2} \\
&\quad + y_0^\Delta \frac{\exp(\int_{t_0}^t \xi_{\mu(\tau)} p(\tau)\Delta\tau) - \exp(\int_{t_0}^t \xi_{\mu(\tau)} p(\tau)\Delta\tau)}{2i} \\
&= y_0 \left[\frac{e^{i(t_1-t_0)}(1+i\mu(t_1))e^{i(t-t_2)} + e^{-i(t_1-t_0)}(1-i\mu(t_1))e^{-i(t-t_2)}}{2}\right] \\
&\quad + y_0^\Delta \left[\frac{e^{i(t_1-t_0)}(1+i\mu(t_1))e^{i(t-t_2)} - e^{-i(t_1-t_0)}(1-i\mu(t_1))e^{-i(t-t_2)}}{2i}\right] \\
&= y_0[\cos(t_1-t_0)[\cos(t-t_2) - \sin(t-t_2)\mu(t_1)] \\
&\quad + \sin(t_1-t_0)[-\sin(t-t_2) - \cos(t-t_2)\mu(t_1)]] \\
&\quad + y_0^\Delta[\cos(t_1-t_0)[\sin(t-t_2) + \cos(t-t_2)\mu(t_1)] \\
&\quad + \sin(t_1-t_0)[-\sin(t-t_2)\mu(t_1) + \cos(t-t_2)]].
\end{aligned}$$

Thus, the general solution for $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$ is

$$y(t) = \begin{cases}
y_0 \cos(t-t_0) + y_0^\Delta \sin(t-t_0), & \text{for } t \in [t_0, t_1] \\
y_0[\cos(t_1-t_0) - \sin(t_1-t_0)\mu(t_1)] & \\
+y_0^\Delta[\sin(t_1-t_0) + \cos(t_1-t_0)\mu(t_1)], & \text{for } t = t_2 \\
y_0[\cos(t_1-t_0)[\cos(t-t_2) - \sin(t-t_2)\mu(t_1)] & \\
+\sin(t_1-t_0)[-\sin(t-t_2) - \cos(t-t_2)\mu(t_1)]] & \\
+y_0^\Delta[\cos(t_1-t_0)[\sin(t-t_2) + \cos(t-t_2)\mu(t_1)] & \\
+\sin(t_1-t_0)[-\sin(t-t_2)\mu(t_1) + \cos(t-t_2)]], & \\
& \text{for } t \in (t_2, t_3]
\end{cases}.$$

Note that $\cos(t_1 - t_0)$ and $\sin(t_1 - t_0)$ are constant terms since they do not depend on t.

5.4 Sequence of Time Scales

Now, we create a sequence of time scales, $\mathbb{T}_n$. This allows us to analyze what is happening as we close the gap. Let $y^{\Delta\Delta} = -y$ with initial conditions $y^{\Delta\Delta}(t_0) = y_0$ and $y^{\Delta}(t_0) = y_0^{\Delta}$ where y_0, y_0^{Δ} are constants. The time scale we are considering is $\mathbb{T}_n = [m,a] \cup [a+\delta_n, b]$ where m, a, b are constants and $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let $0 \le \delta_n \le b - (a + \frac{1}{n})$. In particular, we will use $\delta_n = \frac{1}{n}$ and define $t_n = a + \frac{1}{n}$. Recall that the general solution to this dynamic equations has the form

$$\begin{aligned} y(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t, t_0) + \frac{y_0^{\Delta} - p \cdot y_0}{q} \sin_{\frac{q}{1+\mu p}}(t, t_0) \right] e_p(t, t_0) \\ &= \left[y_0 \cos_{\frac{1}{1+0}}(t, m) + \frac{y_0^{\Delta} + 0 \cdot y_0}{1} \sin_{\frac{1}{1+0}}(t, m) \right] e_0(t, m) \end{aligned}$$

Now, we want to analyze the solution on the intervals and at the jump to see the solution's behavior. It is important to note that we use the same initial conditions and general solution for each calculation. Time Scale Calculus provides us with a general solution that requires only one set of initial conditions.

For $n \in \mathbb{N}$ consider time scale, $\mathbb{T}_n = [m,a] \cup [a + \frac{1}{n}, b]$.

- When $t \in [m, a]$.

$$\begin{aligned} y_n(t) &= \left[y_0 \cos_{\frac{1}{1+0}}(t, m) + \frac{y_0^{\Delta} + 0 \cdot y_0}{1} \sin_{\frac{1}{1+0}}(t, m) \right] e_0(t, m) \\ &= \left[y_0 \cos_1(t, m) + y_0^{\Delta} \sin_1(t, m) \right] (1) \\ &= y_0 \frac{e_i(t, m) + e_{-i}(t, m)}{2} + y_0^{\Delta} \frac{e_i(t, m) - e_{-i}(t, m)}{2i} \\ &= y_0 \frac{\exp(\int_m^t i \Delta\tau) + \exp(\int_m^t -i\Delta\tau)}{2} + y_0^{\Delta} \frac{\exp(\int_m^t i\Delta\tau) - \exp(\int_m^t -i\Delta\tau)}{2i} \\ &= y_0 \frac{(e^{i(t-m)} + e^{-i(t-m)})}{2} + y_0^{\Delta} \frac{(e^{i(t-m)} - e^{-i(t-m)})}{2i} \\ &= y_0 \left(\frac{(\cos(t-m) + i\sin(t-m)) + (\cos(t-m) - i\sin(t-m)}{2} \right) \\ &\quad + y_0^{\Delta} \left(\frac{(\cos(t-m) + i\sin(t-m)) - (\cos(t-m) - i\sin(t-m)}{2i} \right) \\ &= y_0 \cos(t-m) + y_0^{\Delta} \sin(t-m) \end{aligned}$$

- For $t_n = a + \frac{1}{n}$ with $\mu(a) = \frac{1}{n}$.

$$
\begin{aligned}
y_n(t) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}\left(a+\frac{1}{n},m\right) + \frac{y_0^\Delta - p\cdot y_0}{q}\sin_{\frac{q}{1+\mu p}}\left(a+\frac{1}{n},m\right)\right] e_p\left(a+\frac{1}{n},m\right)\\
&= \left[y_0 \cos_1\left(a+\frac{1}{n},m\right) + y_0^\Delta \sin_1\left(a+\frac{1}{n},m\right)\right](1)\\
&= y_0 \frac{e_i(a+\frac{1}{n},m)+e_{-i}(a+\frac{1}{n},m)}{2} + y_0^\Delta \frac{e_i(a+\frac{1}{n},m)-e_{-i}(a+\frac{1}{n},m)}{2i}\\
&= y_0 \frac{\exp(\int_m^{a+\frac{1}{n}} \xi_{\mu(\tau)} i\Delta\tau) + \exp(\int_m^{a+\frac{1}{n}} \xi_{\mu(\tau)} - i\Delta\tau)}{2}\\
&\quad + y_0^\Delta \frac{\exp(\int_m^{a+\frac{1}{n}} \xi_{\mu(\tau)} i\Delta\tau) - \exp(\int_m^{a+\frac{1}{n}} \xi_{\mu(\tau)} - i\Delta\tau)}{2i}\\
&= y_0 \left[\frac{\exp(\int_m^a i\Delta\tau + \int_a^{a+\frac{1}{n}} \frac{1}{\frac{1}{n}} Log(1+i(\frac{1}{n}))\Delta\tau)}{2}\right]\\
&\quad + y_0 \left[\frac{\exp(\int_m^a -i\Delta\tau + \int_a^{a+\frac{1}{n}} \frac{1}{\frac{1}{n}} Log(1-i(\frac{1}{n}))\Delta\tau)}{2}\right]\\
&= y_0^\Delta \left[\frac{\exp(\int_m^a i\Delta\tau + \int_a^{a+\frac{1}{n}} \frac{1}{\frac{1}{n}} Log(1+i(\frac{1}{n}))\Delta\tau)}{2i}\right]\\
&\quad - y_0^\Delta \left[\frac{\exp(\int_m^a -i\Delta\tau + \int_a^{a+\frac{1}{n}} \frac{1}{\frac{1}{n}} Log(1-i(\frac{1}{n}))\Delta\tau)}{2i}\right]\\
&= y_0 \left[\frac{e^{i(a-m)}(1+i(\frac{1}{n})) + e^{-i(a-m)}(1-i(\frac{1}{n}))}{2}\right]\\
&\quad + y_0^\Delta \left[\frac{e^{i(a-m)}(1+i(\frac{1}{n})) - e^{-i(a-m)}(1-i(\frac{1}{n}))}{2i}\right]\\
&= y_0 \left[\cos(a-m) - \left(\frac{1}{n}\right)\sin(a-m)\right]\\
&\quad + y_0^\Delta \left[\sin(a-m) + \left(\frac{1}{n}\right)\cos(a-m)\right]
\end{aligned}
$$

Note that this can also we found by using the simple useful formula, $y(\sigma(t)) = y(t) + \mu(t)y^\Delta(t)$.

- For $t_n \in (a+\frac{1}{n}, b]$ with $\mu(t_n)=0$.

$$\begin{aligned}
y_n(t_n) &= \left[y_0 \cos_{\frac{q}{1+\mu p}}(t,m) + \frac{y_0^\Delta - p\cdot y_0}{q}\sin_{\frac{q}{1+\mu p}}(t,m)\right] e_p(t,m) \\
&= \left[y_0 \cos_{\frac{1}{1+0}}(t,m) + \frac{y_0^\Delta + 0\cdot y_0}{1}\sin_{\frac{1}{1+0}}(t,m)\right] e_0(t,m) \\
&= \left[y_0 \cos_1(t,m) + y_0^\Delta \sin_1(t,m)\right](1) \\
&= y_0 \frac{\exp(\int_m^t \xi_{\mu(\tau)} p(\tau)\Delta\tau) + \exp(\int_m^t \xi_{\mu(\tau)} p(\tau)\Delta\tau)}{2} \\
&\quad + y_0^\Delta \frac{\exp(\int_m^t \xi_{\mu(\tau)} p(\tau)\Delta\tau) - \exp(\int_m^t \xi_{\mu(\tau)} p(\tau)\Delta\tau)}{2i} \\
&= y_0\left[\frac{e^{i(a-m)}(1+i(\frac{1}{n}))e^{i(t-(a+\frac{1}{n}))} + e^{-i(a-m)}(1-i(\frac{1}{n}))e^{-i(t-(a+\frac{1}{n}))}}{2}\right] \\
&\quad + y_0^\Delta\left[\frac{e^{i(a-m)}(1+i(\frac{1}{n}))e^{i(t-(a+\frac{1}{n}))} - e^{-i(a-m)}(1-i(\frac{1}{n}))e^{-i(t-(a+\frac{1}{n}))}}{2i}\right] \\
&= y_0\left[\cos(a-m)\left[\cos\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\sin\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] \\
&\quad + y_0\left[\sin(a-m)\left[-\sin\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\cos\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] \\
&\quad + y_0^\Delta\left[\cos(a-m)\left[\sin\left(t-\left(a+\frac{1}{n}\right)\right) + \left(\frac{1}{n}\right)\cos\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] \\
&\quad + y_0^\Delta\left[\sin(a-m)\left[\cos\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\sin\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right].
\end{aligned}$$

Thus, the solution can be written as a piecewise solution.

$$y_n(t) = \begin{cases}
y_0\cos(t-m) + y_0^\Delta \sin(t-m), & \text{for } t\in[m,a] \\
y_0\left[\cos(a-m) - \left(\frac{1}{n}\right)\sin(a-m)\right] & \\
+y_0^\Delta\left[\sin(a-m) + \left(\frac{1}{n}\right)\cos(a-m)\right], & \text{for } t_n = a+\frac{1}{n} \\
y_0\left[\cos(a-m)\left[\cos\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\sin\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] & \\
+y_0\left[\sin(a-m)\left[-\sin\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\cos\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] & \\
+y_0^\Delta\left[\cos(a-m)\left[\sin\left(t-\left(a+\frac{1}{n}\right)\right) + \left(\frac{1}{n}\right)\cos\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] & \\
+y_0^\Delta\left[\sin(a-m)\left[\cos\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\sin\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right], & \text{for } t_n\in(a+\frac{1}{n}, b]
\end{cases}.$$

Now, let's look at particular initial conditions and assign m a value for our sequence case. Let $y^{\Delta\Delta} = -y$ with initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta}(0) = 1$. The time scale we are considering is $\mathbb{T}_n = [0, a] \cup [a + \frac{1}{n}, b]$ where a, b are constants and $n \in \mathbb{N}$. Let $t_n \equiv a + \frac{1}{n}$. To calculate the solution for this case is very similar to general case shown above, but with our particular initial conditions. Thus, the particular solution is

$$y(t) = \begin{cases} \sin(t), & \text{for } t \in [0, a] \\ \sin(a) + (\frac{1}{n}) \cos(a), & \text{for } t_n = a + \frac{1}{n} \\ \cos(a) \sin(t - (a + \frac{1}{n})) + \sin(a) \cos(t - (a + \frac{1}{n})) & \\ +(\frac{1}{n}) \cos(a) \cos(t - (a + \frac{1}{n})) - (\frac{1}{n}) \sin(a) \sin(t - (a + \frac{1}{n})), & \text{for } t_n \in (a + \frac{1}{n}, b] \end{cases}.$$

5.5 Convergence of Solutions

We want to talk about the sequence of time scales, $\mathbb{T}_n$, that converges to a time scale $\mathbb{T}$ in the Hausdorff metric. Our goal is to show that

$$\lim_{n \to \infty} \mathbb{T}_n = \mathbb{T},$$

that is, the distance between $\mathbb{T}_n$ and $\mathbb{T}$ goes to 0. Let $CL(\mathbb{R})$ be the space of closed nonempty subsets of $\mathbb{R}$. Define a metric

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \;\; d(x, y) = \min\{|x - y|, 1\}.$$

Using the metric we define the Hausdorff metric, which measures distance between two sets.

Definition 22 ([6])**.** *Let $H(\mathbb{T}, \mathbb{T}_n)$ represent the Hausdorff metric, then*

$$H(\mathbb{T}, \mathbb{T}_n) = \max\{\sup_{s \in \mathbb{T}} d'(s, \mathbb{T}_n), \sup_{t \in \mathbb{T}_n} d'(\mathbb{T}, t)\}$$

where

$$d'(s, \mathbb{T}_n) = \inf_{t \in \mathbb{T}_n} d(s, t)$$

and

$$d'(\mathbb{T}, t) = \inf_{s \in \mathbb{T}} d(s, t).$$

Since $|s - t| \geq 0$,

$$\inf_{t \in \mathbb{T}_n} d(s, t) = \min_{t \in \mathbb{T}_n} d(s, t)$$

and

$$\inf_{s \in \mathbb{T}} d(s, t) = \min_{s \in \mathbb{T}_n} d(s, t).$$

Also, $\sup_{s \in \mathbb{T}} d'(s, \mathbb{T}_n)$ and $\sup_{t \in \mathbb{T}_n} d'(\mathbb{T}, t)$ are bounded above. Therefore,

$$\sup_{s \in \mathbb{T}} d'(s, \mathbb{T}_n) = \max_{s \in \mathbb{T}} d'(s, \mathbb{T}_n)$$

and

$$\sup_{t \in \mathbb{T}_n} d'(\mathbb{T}, t) = \max_{t \in \mathbb{T}_n} d'(\mathbb{T}, t)$$

and the Hausdorff metric reduces to

$$H(\mathbb{T}, \mathbb{T}_n) = \max\{\max_{s \in \mathbb{T}} d'(s, \mathbb{T}_n), \max_{t \in \mathbb{T}_n} d'(\mathbb{T}, t)\}.$$

Utilizing the Hausdorff metric and the analytical work with $y^{\Delta\Delta} = -y$ we state the following proposition.

Proposition 1. *Let $\mathbb{T}_n$ denote the time scale $\mathbb{T}_n = [m, a] \cup [a + \frac{1}{n}, b]$ with $n \in \mathbb{N}$ and $\mathbb{T} = [m, b]$. Consider the second order dynamic equation*

$$y^{\Delta\Delta} = -y \ \ \textit{with} \ \ y^{\Delta\Delta}(0) = 0, y^{\Delta}(0) = 1. \tag{5.1}$$

Let $y_n = y_n(t)$ be a solution of (5.1) on $\mathbb{T}_n$. Then, for all $t \in [m, b]$,

$$\lim_{n \to \infty} y_n(t) = \sin(t - m).$$

That is, as $\mathbb{T}_n \to \mathbb{T}$, $y_n(t) \to y(t) = \sin(t - m)$ for all $t \in [m, b]$.

Proof. Let $\mathbb{T}_n$ denote the time scale $\mathbb{T}_n = [m,a] \cup [a+\frac{1}{n}, b]$ with $n \in \mathbb{N}$ and $\mathbb{T} = [m,b]$. For all $n \in \mathbb{N}$ the solution of (5.1) on $\mathbb{T}_n$ is

$$y_n(t) = \begin{cases} \sin(t-m), & \text{for } t \in [m,a] \\ \left[\sin(a-m) + \left(\frac{1}{n}\right)\cos(a-m)\right], & \text{for } t_n = a+\frac{1}{n} \\ \left[\cos(a-m)\left[\sin\left(t-\left(a+\frac{1}{n}\right)\right) + \left(\frac{1}{n}\right)\cos\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right] & \\ +\left[\sin(a-m)\left[\cos\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\sin\left(t-\left(a+\frac{1}{n}\right)\right)\right]\right], & \text{for } t_n \in (a+\frac{1}{n}, b] \end{cases}.$$

Using the Hausdorff metric, we have $H(\mathbb{T}, \mathbb{T}_n) = \max\{0, \frac{1}{n}\} = \frac{1}{n}$.

If $t \in [m,a]$, then

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \sin(t-m).$$

Now, let's consider when $t \in \left(a + \dfrac{1}{n}, b\right]$. Since $\mathbb{T}_n \to \mathbb{T}$, there exists $n_1 \in \mathbb{N}$ such that $t \in \mathbb{T}_{n_1}$. Since $\mathbb{T}_n \subset \mathbb{T}_{n+1}$, $t \in \mathbb{T}_n$ for all $n \geq n_1$. Then

$$\begin{aligned} \lim_{n\to\infty} y_n(t) &= \lim_{n\to\infty} \cos(a-m)\left[\sin\left(t-\left(a+\frac{1}{n}\right)\right) + \left(\frac{1}{n}\right)\cos\left(t-\left(a+\frac{1}{n}\right)\right)\right] \\ &\quad + \lim_{n\to\infty} \sin(a-m)\left[\cos\left(t-\left(a+\frac{1}{n}\right)\right) - \left(\frac{1}{n}\right)\sin\left(t-\left(a+\frac{1}{n}\right)\right)\right] \\ &= \sin(t-m). \end{aligned}$$

Thus, $y_n(t) \to y(t) = \sin(t-m)$ for all $t \in [m,b]$. $\square$

Note that $y(t) = \sin(t-m)$ is the solution of the DE on $[m,b]$. Therefore, using Hausdorff metric and the dynamic equation (5.1) we were able to prove that, as $\mathbb{T}_n$ converges to $\mathbb{T}$, the solution to (5.1) converges as well, that is $y_n(t) \to y(t) = \sin(t-m)$.

5.6 More Than One Gap

The preceding discussion focuses different cases involving time scales with one gap. Now, let's consider more than one gap. We analyzed a time scale with two gaps for particular initial conditions and a particular time scale. Let $y^{\Delta\Delta} = -y$ with initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta}(0) = 1$. The time scale we considered is $\mathbb{T} = [0, \frac{\pi}{2}] \cup [\pi. \frac{3\pi}{2}] \cup [2\pi, \frac{5\pi}{2}]$. To calculate the solution for this case is very similar to our first particular case, but with an additional gap. Thus, the particular solution is

$$y(t) = \begin{cases} \sin(t), & \text{for } t \in [0, \frac{\pi}{2}] \\ 1, & \text{for } t = \pi \\ \cos(t-\pi) - (\frac{\pi}{2})\sin(t-\pi), & \text{for } t \in (\pi, \frac{3\pi}{2}] \\ -\pi, & \text{for } t = 2\pi \\ -\sin(t-2\pi) - (\pi/2)\cos(t-2\pi) + (\frac{\pi}{2})^2 \sin(t-2\pi), & \text{for } t \in (2\pi, \frac{5\pi}{2}] \end{cases} .$$

Throughout our journey of analyzing different time scales, initial conditions, and methods of solving the dynamic equation $y^{\Delta\Delta} = -y$, we concluded that the solution is a linear combination of sine and cosine with a shift depending on the gaps. We also concluded, when analyzing one gap, that when we close the gap the solution tends towards the solution on the interval without the gap. We will present some visual representations of this in Chapter 6.

CHAPTER 6

A SECOND ORDER LINEAR HOMOGENEOUS DYNAMIC EQUATION: GRAPHICAL SOLUTIONS ON THE DIFFERENTIAL ANALYZER

With our high-tech society there are many ways to solve differential equations graphically, such as the use of MATLAB. However, in this study we will focus on evaluating the particular IVP $y^{\Delta\Delta} = -y$ in Marshall's Differential Analyzer Lab. The graphical representation of solutions created with the DA inspired our analytical study.

6.1 Graphing Solution of Time Scale with One Gaps

We will start graphing the solution on $[0, \pi] \cup [3\pi/2, 3\pi]$ (note that we are using this specific time scale because it works well when graphing on Art's output table) and slowly close the gap between the two closed intervals until we are near a single closed interval. Before we can program Art to run our IVP, we must determine the set-up for the interconnect. As stated in Chapter 2, we need to draw a Bush Schematic Diagram (Refer to Figure 6.1).

Now, we can begin evaluating the graphical solution of the IVP,

$$y^{\Delta\Delta} = -y,\ y^{\Delta\Delta}(0) \ = \ 0, y^{\Delta}(0) \ = \ 1$$

on varying time scales using the Differential Analyzer, Art. Note, that as we closed the gap between π and 2π, $t \in [2\pi, 3\pi]$ our solution converged to the solution $y(t) = \sin(t)$ on $[0, 3\pi]$.

There is a specific method for running Art when dealing with a time scale domain of this type. In particular, we will explain how to run Art for a second order dynamic equation,

$$y^{\Delta\Delta} = -y,\ y^{\Delta\Delta}(0) = 0, y^{\Delta}(0) = 1$$

and $\mathbb{T} = [0, \pi] \cup [3\pi/2, 3\pi]$. We start by setting our initial conditions for the problem ($y^{\Delta\Delta}(0) = 0,\ y^{\Delta}(0) = 1$). These are set using counters on Art; a counter set at 250 is equivalent to 1 unit in the domain of the DE. Note that we only need one set of initial conditions to start the machine. Then, with the mechanics of Art and time scale calculus, the following steps take us to the next

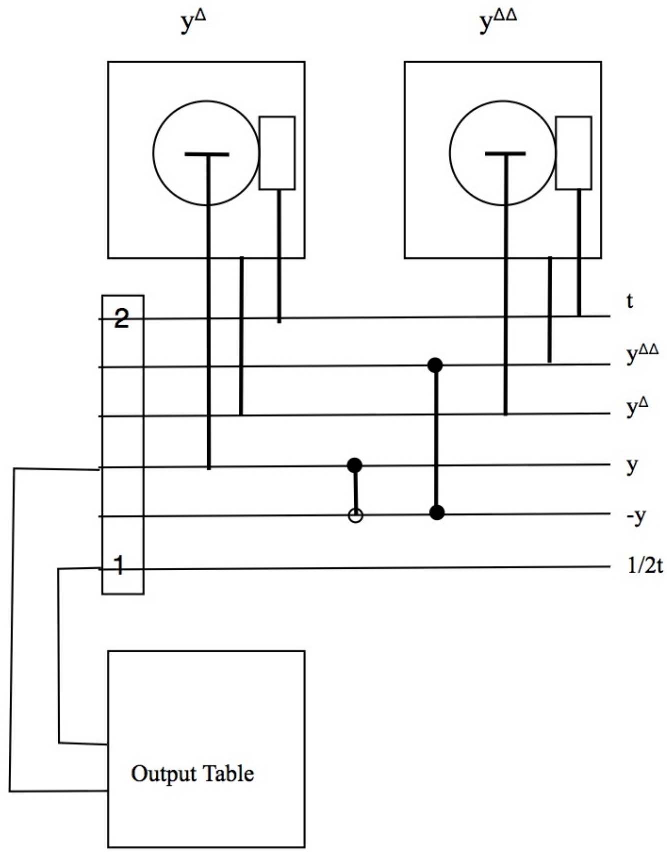

Figure 6.1: Bush Schematic Diagram for $y^{\Delta\Delta} = -y$.

interval. Our interconnect is programed as our Bush schematic diagram directs us (refer to Figure 6.1). Once Art is programmed, the following steps are used to run the machine for a second order dynamic equation with one gap:

1. Run to the first gap and stop the machine.

2. Disconnect the lead screw to y^{Δ}, lift pen and run across the time gap ($\mu(\pi)$). Here we know y and $y^{\Delta\Delta}$ at $\sigma(t)$. We record $y^{\Delta\Delta}$ placement on the disk using the counter and we mark the spot of y on the output table.

3. Run $y^{\Delta\Delta}$ back to the position before the gap. Note that y^{Δ} value has not changed.

4. We need $y^{\Delta}(\sigma(t))$ We can use our knowledge of the simple useful formula for y^{Δ} which is $y^{\Delta}(\sigma(t)) = y^{\Delta}(t) + \mu(t)y^{\Delta\Delta}(t)$. We want to fix $y^{\Delta\Delta}$, so we disconnect the lead screw to $y^{\Delta\Delta}$.

Also we need to move y^{Δ}, so we reconnect the lead screw to y^{Δ}.

5. We run back across the time gap. Now, we have the $y^{\Delta}(\sigma(t))$ term and we reset $y^{\Delta\Delta}$ term found in Step 2.

6. Finally, we put the pen down and turn the machine back on!

We need all of the above steps to insure that y, $y^{\Delta\Delta}$ and y^{Δ} are in the correct position at $\sigma(t)$ when we turn the machine back on to plot the solution on $[3\pi/2, 3\pi]$. Note that this process can be used when there is more than one gap. One would need to repeat the process for each gap. Table 6.1 shows the recording of the counter for the first three run of this time scale. We record the counter at Steps 2 and 5.

$\mu(t)$	$y^{\Delta}(\sigma(t))$	$y^{\Delta\Delta}(\sigma(t))$
$3\pi/2$	376	762
$4\pi/3$	205	756
$5\pi/4$	251	758

Table 6.1: Readings from Art from $\mathbb{T} = [0, \pi] \cup [2\pi, 3\pi]$.

Figure 6.2 shows $y^{\Delta\Delta} = -y$ with one gap and reducing the gap gradually five times. This plot uses the initial time scale of $\mathbb{T} = [0, \pi] \cup [2\pi, 3\pi]$. Note that after the gap we let the machine run until the output table runs out of paper.

As stated in the analytical section, when we were calculating the solution on a given time scale, we had $y(\sigma(t)) = y(t)$ since at $t = \pi$, $y(t) = 0$. We recognized this graphically as well since, when running the problem we do not have to change the values of $y(t)$ at $y(\sigma(t))$ for $t = \pi$. So, we can skip Steps 3 to 5. Thus, we changed our time scale, to get more insight, to $\mathbb{T} = [0, 3\pi/4] \cup [3\pi/2, 2\pi]$ and ran the problem on Art to get a graphically view.

Table 6.2 was used to determine $y^{\Delta}(\sigma(t))$ and $y^{\Delta\Delta}(\sigma(t))$ in Steps 4 and 6 and the plot (Figure 6.3) shows solutions of $y^{\Delta\Delta} = -y$ with one gap and gradually closing the gap two times. We started with the time scale $\mathbb{T} = [0, 3\pi/4] \cup [13\pi/4, 2\pi]$ and reduced the gap first to $\pi/6$ and then to $\pi/8$. We can see from the plots on Art that as we shrink $\mu(t)$ the function is converging to the continuous curve described by $y = \sin(t)$, which is what we analytically proved in Chapter 5.

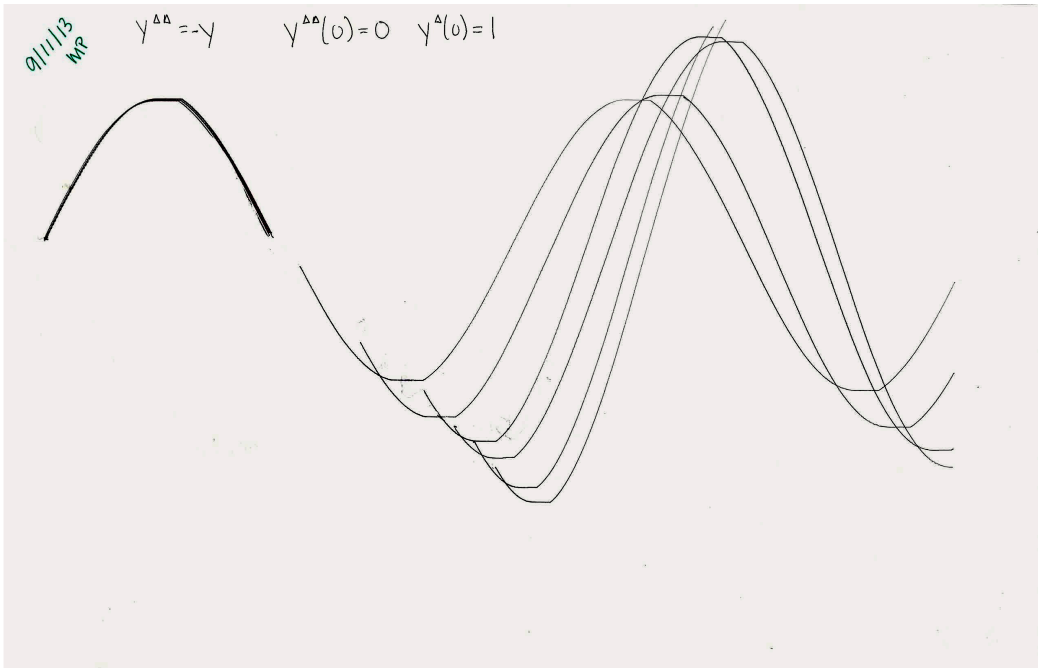

Figure 6.2: Plot of $y^{\Delta\Delta} = -y$ with $\mathbb{T} = [0, \pi] \cup [2\pi, 3\pi]$. Shows the convergence of the graph as the gap closes.

Table 6.2: Readings from Art from $\mathbb{T} = [0, 3\pi/4] \cup [13\pi/4, 2\pi]$.

$\mu(t)$	$y^{\Delta}(\sigma(t))$	$y^{\Delta\Delta}(\sigma(t))$
$\pi/3$	494	990
$\pi/6$	597	901
$\pi/8$	640	887

6.2 Graphing Solution of Time Scale with Two Gaps

Finally, we wanted to graphically look at a time scale with two gaps. The particular time scale we used to graph with Art was $\mathbb{T} = [0, \frac{3\pi}{4}] \cup [\pi, \frac{3\pi}{2}] \cup [2\pi, \frac{5\pi}{2}]$. We followed the process of running Art on a time scale to the first gap and, when we stopped the machine for the second gap we repeated the steps. Table 6.3 shows the readings from the counters on Art at $y(\sigma(t)) = y(\pi)$ and $y(\sigma(t)) = y(2\pi)$. Figure 6.4 shows the output of Art with two gaps.

Thus, looking at solutions of $y^{\Delta\Delta} = -y$ graphically we can see that, as you close the gap, the graph of the solution converges to the solution in the limit interval case, which is that same result we discovered analytically. Being able the graph the solutions of $y^{\Delta\Delta} = -y$ on various time scales

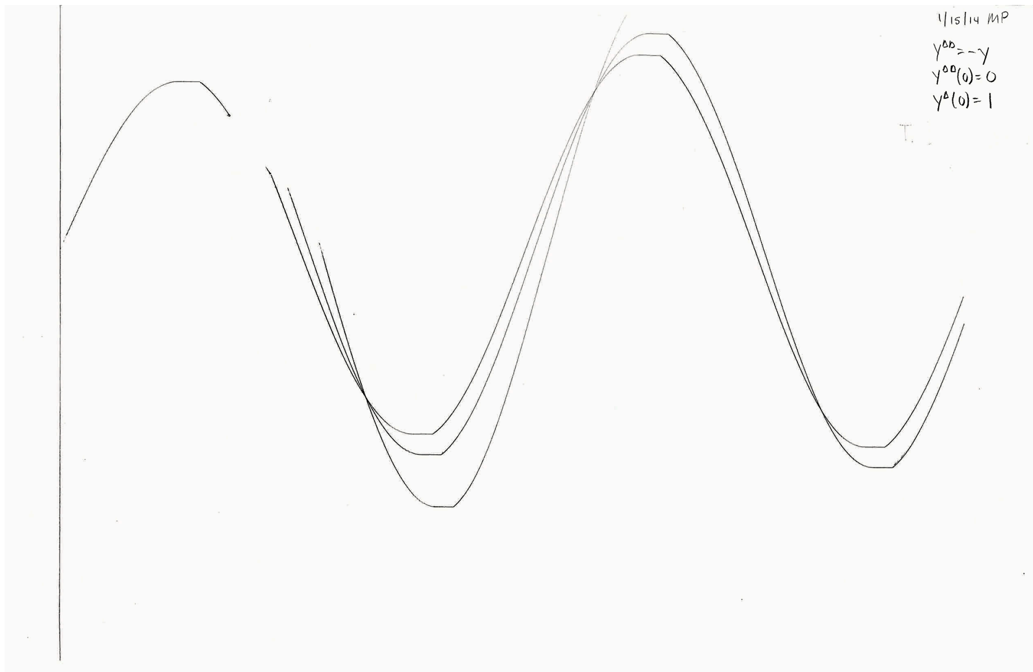

Figure 6.3: Plot of $y^{\Delta\Delta} = -y$ with $\mathbb{T} = [0, 3\pi/4] \cup [13\pi/4, 2\pi]$. Shows the convergence of the graph as the gap closes.

Table 6.3: Readings from Art from $\mathbb{T} = [0, \frac{3\pi}{4}] \cup [\pi, \frac{3\pi}{2}] \cup [2\pi, \frac{5\pi}{2}]$.

$\mu(t)$	$y^{\Delta}(\sigma(t))$	$y^{\Delta\Delta}(\sigma(t))$
$\pi/4$	656	942
$\pi/4$	758	251

using Art provides us with great insight about how the dynamic equation's solution is changed from its value on a single closed interval to its value after the jump. Performing the steps presented earlier in the chapter allows the machine to take us to the next point when we jump over a gap. Therefore, as you close the gap of dynamic equation $y^{\Delta\Delta} = -y$ with initial conditions $y^{\Delta\Delta}(0) = 0$ and $y^{\Delta}(0) = 1$, the solution converges to $y = \sin(t)$, which is the solution to this dynamic equation with $\mathbb{T} = \mathbb{R}$ on the limit interval.

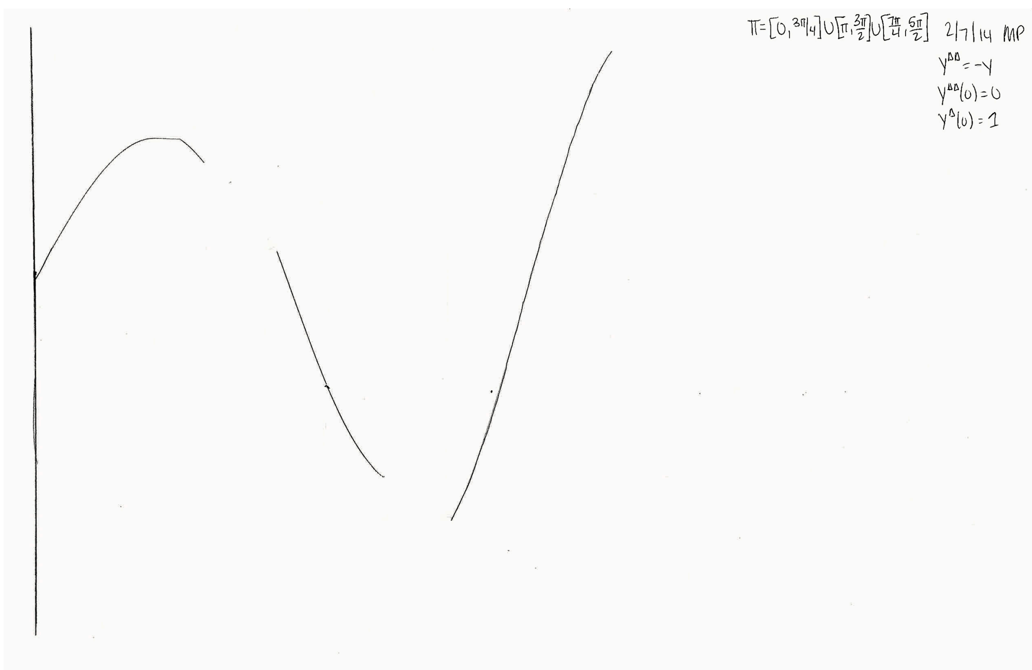

Figure 6.4: Plot of $y^{\Delta\Delta} = -y$ with $\mathbb{T} = [0, \frac{3\pi}{4}] \cup [\pi, \frac{3\pi}{2}] \cup [2\pi, \frac{5\pi}{2}]$.

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APPENDIX A
LETTER FROM INSTITUTIONAL RESEARCH BOARD

MARSHALL UNIVERSITY®
www.marshall.edu

Office of Research Integrity

January 24, 2014

Molly Peterson
1028 8th Street #6
Huntington WV, 25701

Dear Ms. Peterson:

This letter is in response to the submitted thesis abstract on the Marshall Differential Analyzer Lab. After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day

Bruce F. Day, ThD, CIP

Molly Kathryn Peterson
Curriculum Vitæ
April 14, 2014

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Mechanical Visualization of Second Order Dynamic Equation on Varying Time Scales.

Undergraduate Research
Simpson College, 2010–2011. *Advisor Dr. Heidi Berger.*
Senior Capstone on Simpson College's Differential Analyzer.

Undergraduate Research
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Research on Marshall's Differential Analyzer Lab.

Undergraduate Research
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Undergraduate Research
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Research on applying Time Scale Calculus to model the Emerald Ash Borer.

Presentations

Co-presenter with Dr. Bonita Lawrence and Alex Amorim.
Emmanuel College, Boston, MA. March 2014.
Demonstrated the use of the Differential Analyzer programmed to solve simple harmonic motion.

Co-presenter with Dr. Bonita Lawrence.
Missouri University of S & T, Rolla, MO. October 2013.
Demonstrated the use of the Differential Analyzer programmed to solve simple harmonic motion.

Co-presenter with Dr. Bonita Lawrence.
Lindenwood University, St. Charles, MO. October 2013.
Demonstrated the use of the Differential Analyzer programmed to solve simple harmonic motion.

AMS Sectional Meeting Invited Co-Presenter with Dr. Bonita Lawrence.
Washington University, St. Louis, MO. October 2013.
Solutions of dynamic equations on a sequence of converging time scales. Demonstrated the use of the Differential Analyzer on a time scale.

Co-presenter with Dr. Lawrence and Kayode Olumoyin.
Simpson College, Indianola, IA. March 2013.
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Publications

1. *Mechanical Visualization of Second Order Dynamic Equation.* Master's thesis, Marshall University, May 2014.

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