2015

Solutions of a Logistic Equation on Varying Time Scales: A Quantitative and Qualitative Analysis

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SOLUTIONS OF A LOGISTIC EQUATION ON VARYING TIME SCALES: A QUANTITATIVE AND QUALITATIVE ANALYSIS

A thesis submitted to
the Graduate College of
Marshall University
In partial fulfillment of
the requirements for the degree of
Master of Arts
in
Mathematics
by
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Marshall University
May 2015

ACKNOWLEDGEMENTS

I will never be able to repay Dr. Bonita Lawrence for what she has given me - knowledge, confidence, and respect, just to name a few. I have matured as a student and as an adult as a result of her guidance and leadership. It has been a privilege working in her in the Marshall University Differential Analyzer Lab.

A special thank you is in order for my committee members, Dr. Clayton Brooks and Dr. Ralph Oberste-Vorth. I greatly appreciate their patience and wisdom.

I would also like to thank my fellow DA Team members, especially Molly Peterson, Chad Lott, and Richie Merritt. Molly, thank you for introducing me to the world of differential analyzers. Chad, thank you for your endless assistance and creativity. Alas, Richie, thank you for your questions and advice.

Finally, I would like to express my gratitude towards the faculty, staff, and other graduate students in the Mathematics department at Marshall University for all of their support, encouragement, and motivation.

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ABSTRACT

Time Scale Calculus, introduced by Dr. Stefan Hilger in 1988, combines the study of differential and difference equations into a single topic. We begin with an introduction of sets used in this field, time scales, and build up to the definition of the exponential function on a time scale. The main focus of this work is a study of the solutions of a particular logistic dynamic equation on varying time scales. We study both the analytical and graphical solutions of this equation. Analytical solutions are worked out using theorems from Time Scale Calculus, including properties of the exponential function. Graphical solutions are obtained using the Marshall University Differential Analyzer (DA), fondly known as Art. Differential analyzers, such as Art, are machines that perform mechanical integration to solve differential equations. Both the analytical and graphical solutions offer the same conclusions about the convergence of solutions as the time scales converge.

CHAPTER 1

INTRODUCTION

Over the past few decades, Time Scale Calculus has become a growing field of study in mathematics. As opposed to studying differential and difference equations separately, Time Scale Calculus, introduced by Dr. Stefan Hilger, unites these two studies. With the use of Time Scale Calculus, we are not only able to solve dynamic equations defined on domains such as $\mathbb{R}$ and $\mathbb{Z}$ but also on domains that are the unions of disjoint sets. Since its introduction in 1988, others such as Dr. Allan Peterson and Dr. Martin Bohner have gathered and continued the work of Dr. Stefan Hilger. Their work *Dynamic Equations on Time Scales – An Introduction with Applications*, referenced throughout this work, has broadened the field of Time Scale Calculus.

The goal of this work is to analyze the solutions of a particular logistic dynamic equation on a variety of time scales. Our focus is to show that as the gap between intervals of a time scale closes, the solution of the dynamic equation converges to its solution when the time scale is equivalent to the real numbers. We begin by introducing definitions and theorems which involve differentiation, integration, the exponetial function, and the logistic dynamic equation with respect to time scales. These are used to study the solutions of our logistic dynamic equation analytically while Art, Marshall University's four-integrator differential analyzer, is used to study the solutions graphically.

CHAPTER 2

DIFFERENTIAL ANALYZERS

2.1 History

First built to solve nonlinear differential equations which could not be solved by any other method at the time, the differential analyzer has quite a history [4]. From rumors of helping Allied forces in World War II to becoming obsolete after the invention of electronic computers, DAs undeniably remain a learning tool and a visualization of differential equations.

2.1.1 Early Days

The late 1920s brought with it the development of the first differential analyzer. Dr. Vannevar Bush, a professor of Electrical Engineering at Massachusetts Institute of Technology (MIT), designed and constructed the first differential analyzer which consisted of six integrators. By the early 1930s, word of the machine had spread to England and captured the interest of a University of Manchester professor, Dr. Douglas Hartree. Dr. Hartree, during a visit to MIT, formulated a plan to create a working model of Bush's machine out of Meccano parts, the British version of Erector Set [4]. By the mid-1930s, Dr. Hartree's undergraduate research assistant majoring in physics, Arthur Porter, had successfully built a four-integrator working model of Bush's machine predominately out of Meccano parts.

Porter used his machine to solve problems relating to control theory and, particularly, to study the atomic structure of the chromium atom. Porter's University of Manchester machine was the first differential analyzer outside of the United States, but little did he know, he would soon end up continuing his work in America under the leadership of Dr. Bush. After completing his doctoral thesis, Dr. Porter was chosen for the Commonwealth Fund Fellowship. This led him to MIT to work with Dr. Bush on the Rockefeller DA which was built by Dr. Bush in 1941 and had 16 integrators. While the differential analyzer may have been the dawning of the computer era, most have been destroyed. However, a lucky few exist as static displays in museums serving as a reminder of advancements in electronics and even inspiring an occasional visitor.

2.1.2 Recent Past

While differential analyzers lost their vitality decades ago, in the fall of 2004 a professor of mathematics in Huntington, West Virginia brought life to the Marshall Univesity Differential Analyzer Project. Dr. Bonita Lawrence paid a visit to the London Science Museum that summer where she discovered the static display of a portion of the Manchester DA, built by the Metropolitan-Vickers Company [8]. She immediately saw the educational benefits that an active differential analyzer could offer current students of mathematics. Upon returning to Marshall for the fall semester, Dr. Lawrence created a team of undergraduate and graduate student researchers and the project began.

Students studied everything they could on the first and most well-known DAs. They read papers and contacted people with first-hand experience. Finally, they discovered that Dr. Porter, creator of the Manchester DA, was living only two states away in North Carolina. Dr. Lawrence and a few of her students visited Dr. Porter and returned with their final, and most important, dose of inspiration for the project. In 2006, Lizzie, named after the Ford Model T - Tin Lizzie, was the first machine completed by the Marshall DA Team. Lizzie consists of two integrators and is easily portable. In May of 2009, construction was completed on a four-integrator DA capable of running nonlinear differential equations. This machine, Art, is named after Dr. Arthur Porter.

Dr. Lawrence, and her colleague, Dr. Clayton Brooks (also her husband), began traveling to universities giving presentations with Lizzie. Molly Peterson, a former graduate student, met Dr. Lawrence at a conference at the University of Wyoming in 2009. Molly was able to visit Marshall University and build her own DA, Miles-Diffy, over an 11-day period. Miles-Diffy, a two-integrator machine, was taken back to Simpson College, Molly's home college in Iowa, which is now the second location in the United States to have a publicly accessible DA [7].

The final differential analyzer built at Marshall is similar in design to Miles-Diffy and is known as DA Vinci. DA Vinci has two integrators and a large avenue of interconnect, which contains all of the rods and gears used in setting up a problem. Both are capable of running more complex equations because of their vast avenues of interconnect. Over the span of 8 years, the Marshall DA Team, under the guidance of Dr. Lawrence, has produced four successful and unique DAs.

2.1.3 Current Events

Dr. Lawrence, current research students, and DA Vinci continue to travel to universities and conferences to talk about differential analyzers and hopefully inspire other schools to build their own. At Marshall, professors of calculus and differential equations courses bring their students into the DA lab to help them understand integration in a mechanical fashion. Art has been incorporated into six masters theses, including this one, and numerous undergraduate capstone projects since 2009.

I became interested in the DAs at Marshall after Molly began telling me about her thesis work in the fall of 2013. I began reading the appropriate material and understanding differential analyzers in general, but particularly, studying the mechanics of Art. By the following spring, I was helping Molly perform runs on Art for her thesis. Since becoming a part of the Marshall DA Team, I have been fortunate enough to travel to Emmanuel College in Boston, MA and Indiana State University to run DA Vinci. We also had the opportunity this March to run Art during the Mathematical Assosiaction of America (MAA) Ohio sectional meeting hosted at Marshall.

Currently, students from Lindenwood University in St. Charles, Missouri are making moves toward the construction of a DA. The DAs at Marshall continue to attract students locally and nationally. They may be outdated by modern technologies, but differential analyzers will remain a learning tool and continue to hold their educational value.

2.2 Art: Marshall University's DA

The purpose of the differential analyzer is to model a variety of differential equations. The design and size of a particular machine determine the maximum order of the differential equation it is capable of solving. Differential equations are programmed into a DA through a specific arrangement of gears and shafts known as the *avenue of interconnect* [4]. The setup for the avenue of interconnect is based on a Bush Schematic Diagram, named after Dr. Vannevar Bush, which details each component and connection for a particular differential equation.

2.2.1 Mechanical Makeup

At the core of a differential analyzer is a series of mechanical integrators, one of which is pictured in Figure 2.1. Each integrator consists of a movable carriage which houses a horizontal disk, attached by a vertical axis. The vertical axis allows the disk to spin while the carriage translates the disk along a set of rails. Atop the disk rests a vertical wheel which is attached to a shaft that spins but is fixed in the horizontal plane. So, as the disk rotates and is translated, the wheel rotates, powered by the frictional force created at their point of contact. Note that the wheel may be positioned at various locations along a diameter of the disk since the disk is attached to a movable carriage by a lead screw.

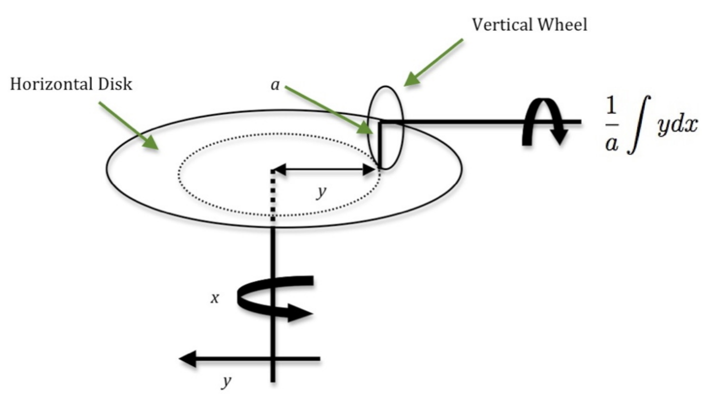

Figure 2.1: Setup of the wheel and disk for each integrator [7].

The motion created by the turning of the wheel, and in turn the rod connected to it, must make its way to the avenue of interconnect to feed other components. The torque produced from this motion is not strong enough to power all of the necessary components. Instead of increasing the friction between the wheel and disk, a torque amplifier is used to increase torque. Throughout the DA's time of use, many different types of torque amplifiers were used. Art, the four-integrator DA used in this work, amplifies torque using a servo-type motor [4]. It is important to mention that the torque amplifier does not affect the mathematical output of the integrator.

Another important component is the adder. A mechanical adder connects three rods where two rods send motion in and the third rod outputs the sum of those two motions [5]. The two rods send in motion using bevel gears attached to independent rods which create a sum (or difference) that is transferred to a single rod. This allows us to combine the motions produced from two or more integrators.

A beneficial accessory is the output table which requires two motions to be fed into it to produce a graphical representation of what one wants to model. One motion moves the physical table horizontally along a set of rails while the second motion moves the pen vertically. Depending on which rods are connected, the output table has the potential to graph the solution, the derivative, or other aspects of a particular DE. Art has two output tables, but an output table is not a necessity for the DA to run. The DA is designed to model DEs without sending motion into an output table.

2.2.2 Mathematics Behind Art

Mechanical integration requires two inputs to create the desired output motion. An independent variable is needed to spin the horizontal disk about the vertical axis. An input motion is necessary to move the disk along the carriage by means of a threaded lead screw. Finally, an output is produced and sent to other components along the avenue of interconnect via the fixed rod attached to the wheel and the torque amplifier. As the carriage moves the disk along the track, the position of the wheel on the disk is changing. The position of the wheel on the disk, measured in inches relative to the center of the disk, determines the number of times the wheel turns. This means if the wheel is positioned at the center of the disk, where the distance is considered zero, the wheel will not rotate. Another way to look at this is, if the wheel is positioned a inches from the center of the disk and the radius of the wheel is a inches, one turn of the disk will create one turn of the wheel.

Notice that the position of the wheel is measured in inches but needs to be represented in terms of shaft rotations. Focusing on Art's setup, there is a geardown of the output respresented by an integrator constant. This constant comes from the pitch of the lead screw, the embedded gear ratios, and the radius of the wheel. The motion coming out of the integrator is only $\frac{p \cdot k}{a}$ of the actual integral where p is the pitch of the lead screw, k is the ratio of gear reductions, and a is the radius of the wheel in inches.

On Art, the lead screw has 32 threads per inch, so p is $\frac{1}{32}$ inches per rotation. The gear near the clutch that moves the lead screw has a gear ratio of $\frac{50}{57}$ while the helical gears under the disk have a gear ratio of $\frac{2}{5}$. This makes $k = \frac{20}{57}$ rotations squared. Finally, the radius of the wheel on each of Art's integrators measures $\frac{15}{16}$ of an inch. So, the output of an integrator on Art with respect to the input is calculated by

$$\frac{p \cdot k}{a} = \frac{\frac{1}{32} \text{ inches/rotation} \cdot \frac{20}{57} \text{ rotations}^2}{\frac{15}{16} \text{ inches}} \approx 0.0117 \text{ rotations.}$$

Thus, one turn of an input rod leads to a factor of 0.0117 turns of the output rod. Excluding the counters, this means that our unit of measure becomes

$$\frac{1}{\frac{p \cdot k}{a}} = \frac{1}{0.0117} \approx 85.5 \text{ rotations.}$$

So, we get 85.5 turns of output for each turn of input.

In order to set the position of the wheel on the disk on Art, we use a counter mechanism. The counter has a gear ratio of $\frac{19}{57} = \frac{1}{3}$ and has a multiplier of 10 built into it. So the ratio of the counter becomes $\frac{10}{3}$. We also have to take into consideration the lead screw that connects the carriage to the counter. From above we know this gear ratio is $\frac{50}{57}$. If we multiply these ratios along with our unit of measure, we get

$$\frac{10}{3} \cdot \frac{50}{57} \cdot 85.5 = 250 \text{ rotations.}$$

Thus, 250 turns on the counter represent one unit from the center of the disk to the position of the wheel. We use the counters and this measure of unity to set initial conditions on the integrators. For example, if we want to set initial conditions $y(x_0) = 1/2$ and $y'(x_0) = 5/4$, we would set the counter on the first integrator to 125 and the counter on the second integrator to 312.5.

2.2.3 Riemann Sums

To begin the process of integration on the differential analyzer, we begin with the wheel remaining a fixed distance, y, from the center of the disk. This means that the carriage holding the disk is stationary. So as the disk rotates, the wheel creates a circle outlined on the disk. The circumference of this "outlined" circle will be $C_d = 2\pi y$. Also, if we denote the radius of the wheel as a, we know the circumference of the wheel is $C_w = 2\pi a$. So, the disk to wheel rotation ratio is 1 to y/a. That is, one turn of the disk results in y/a turns of the wheel.

Keeping this in mind, consider the case when an input is rotating the lead screw attached to the carriage so that the disk is moving, creating a change in position of the wheel on the disk. The change in radius, y, of the outlined circle on the disk will affect the arc lengths created by the rotations of the wheel. The formula for arc length, s, is $s = r\theta$. If we let $r = y$ and $\theta = 2\pi\triangle x$, where $\triangle x$ is the portion of a circle traced by the wheel at radius y, then we have $s = y \cdot 2\pi\triangle x$.

If we let n represent the number of portions of a rotation of the disk, then we can relate the arc lengths of the wheel to the turns of the disk. The total distance, for n portions of a turn, that the wheel covers on the disk is the sum of the arc lengths at the different radii, r_i, of the circles outlined by the wheel such that

$$\sum_{i=1}^{n} y(x_i) 2\pi \triangle x_i$$

where $y(x_i) = r_i$ and $2\pi\triangle x_i = \theta_i$ [4].

Since we know the distance the wheel travels on the disk and the circumference of the wheel, to find the total number of rotations of the wheel, we just need to divide:

$$\begin{aligned} \text{Rotations of the wheel} &= \frac{\text{Distance wheel travels}}{\text{Circumference of wheel}} \\ &= \sum_{i=1}^{n} \frac{y(x_i) 2\pi \triangle x_i}{2\pi a} \\ &= \frac{1}{a} \sum_{i=1}^{n} y(x_i) \triangle x_i. \end{aligned}$$

Notice that the distance traveled by the wheel is measured in length while the circumference of the wheel can be thought of as length per rotation. When we divide these two, we are left with a sum of rotations.

Now, if we let n, the number of portions of a rotation of the disk, decrease, the formula for the rotations of the wheel becomes

$$\frac{1}{a}\int y(x)\,dx.$$

These attributes resemble that of a Riemann sum [4]. Below is a formal definition of Riemann sum.

Definition 1 ([1]). *If P is the tagged partition, which is a partition of a given interval together with a finite sequence of numbers $i = 1, 2, ..., n$, we define the **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ corresponding to P to be the number*

$$S(f; P) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

If the function f is positive on $[a, b]$, then the Riemann sum is the sum of the areas of n rectangles whose bases are subintervals $I_i = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$.

The mechanical integrators provide a visual representation of integration. The position of the wheel on the disk, relative to the center, determines the direction in which the wheel spins. One side of the disk represents positive values while the other side represents negative values. This is due to the fact that the wheel comes to a stop at zero (the center of the disk) and changes direction when it crosses to the other side [4]. The motion created by the wheel is carried out through the rod connected to it and sent to the desired locations.

Now, consider the wheel leaving an outline of its movements on the disk. If the DA runs through one period of a periodic differentiable function, like $y = \sin x$, then the wheel passes through zero on two occasions. This causes the wheel to outline four spirals on the disk, defined by the particular equation set up on Art. The path of the wheel on the disk is modeled by these spirals, and they display the motion (definite integral) that was sent out through the rod attached to the wheel. If we plot these spirals on an $xy-$plane with the number of rotations of the wheel along the $x-$axis and the distance from the center of the disk to the wheel along the $y-$axis, then we can think about definite integrals as being the areas under the curves [7].

To generalize, consider the value of a function's nth derivative determining the position of the wheel on the disk of a specific integrator. Then the rotation of the wheel represents the value of the $(n-1)$st derivative of our function [4]. Suppose this motion is sent into another integrator determining the position of the wheel on the disk of this integrator. Then, the motion of this wheel represents the $(n-2)$nd derivative of our function. If this continues through n integrators, then the motion being outputed from the nth integrator will represent the desired function.

CHAPTER 3

TIME SCALE CALCULUS

Stefan Hilger had a vision to unite discrete and continuous analysis into one study. This vision came to life in his PhD thesis in 1988 and is known as Time Scale Calculus. In the past, dynamic equations have been classified by the structure of their domain being an interval subset of $\mathbb{R}$. The equations were then studied accordingly with the use of differential equations in the case of an interval subset of $\mathbb{R}$ and difference equations in the case of a (totally) discrete domain set. With the bond created between the two by Time Scale Calculus, dynamic equations can now be studied over a hybrid of these two domain types. Time Scale Calculus closes the gap between discrete and continuous analysis.

3.1 Basic Terms

A **time scale**, denoted $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers. Examples of time scales include the real numbers, $\mathbb{R}$, the integers, $\mathbb{Z}$, the natural numbers, $\mathbb{N}$, and the nonnegative integers, $\mathbb{N}_0$. Other examples include $[0,2] \cup [4,5] \cup \{6\}$ and $[0,1] \cup \mathbb{N}$. Nonexamples include the rational numbers, $\mathbb{Q}$, the irrational numbers, $\mathbb{R}\backslash\mathbb{Q}$, the complex numbers, $\mathbb{C}$, and the open interval $(0,2)$.

Notice that it is not necessary for a time scale to strictly be an interval subset of $\mathbb{R}$ or strictly be a (totally) discrete set. This fact requires us to define operators which allow us to move from one element to another in the time scale. So, we have the following two definitions.

Definition 2 ([2]). *(i) For $t \in \mathbb{T}$ we define the **forward-jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

*(ii) For $t \in \mathbb{T}$ we define the **backward-jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Note: If $t = \sup \mathbb{T}$, then $\sigma(t) = t$, and if $t = \inf \mathbb{T}$, then $\rho(t) = t$.

There are two cases for each of the jump operators. The forward-jump operator may result in $\sigma(t) > t$, in which case, there is a non-zero distance between t and the next element in $\mathbb{T}$. On the other hand, the forward-jump operator may result in $\sigma(t) = t$. This implies that there does not exist a "next" element that is greater than t. Similarly, the backward-jump operator may result in $\rho(t) < t$ implying a non-zero distance between t and the previous element in $\mathbb{T}$ or $\rho(t) = t$, in which case, there does not exist a "previous" element that is less than t.

Using these definitions, we are able to classify elements in the time scale as right-scattered or left-scattered and right-dense or left-dense. When there is a gap between elements in $\mathbb{T}$, we say an element is *right-scattered* if $\sigma(t) > t$ and *left-scattered* if $\rho(t) < t$. When there does not exist a "next" element greater than t or a "previous" element less than t, the jump operators tell us to remain at the current location, and we say an element is *right-dense* if $\sigma(t) = t$ and *left-dense* if $\rho(t) = t$. If an element is both left-scattered and right-scattered at the same time $(\rho(t) < t < \sigma(t))$ we call the element *isolated*. Alternately, if an element is both left-dense and right-dense at the same time $(\rho(t) = t = \sigma(t))$ we call the element *dense*. Finally, we need a way to calculate the distance between elements in $\mathbb{T}$, so we define the *graininess function*.

Definition 3 ([2])**.** *The change in position of consecutive elements, $\mu : \mathbb{T} \to [0, \infty)$, is definied by $\mu(t) := \sigma(t) - t$ and we call μ the* ***graininess function****.*

It is important to mention that the function $\mu(t)$ measures the distance between an element t and the *next* element $\sigma(t)$ in $\mathbb{T}$ which means it will always have a nonnegative value.

In order to define and use differentiation and integration in Time Scale Calculus, we must first define the set denoted by $\mathbb{T}^\kappa$.

Definition 4 ([2])**.** *If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. So,*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases}.$$

Lastly, we want to define composition of f with σ notation. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$. It is important to mention that these notations may be used interchangeably.

Now, we will look at a few examples to better understand the jump operators and graininess function. For each of the following examples, we will calculate $\sigma(t)$, $\rho(t)$, and $\mu(t)$ as well as classify the elements $t \in \mathbb{T}$.

Example 1. *We consider three examples.*

(i) Let $\mathbb{T} = \mathbb{R}$. Then for any $t \in \mathbb{R}$, we have

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf\{(t, \infty)\} = t.$$

Similarly, we have $\rho(t) = t$. Since $\rho(t) = t = \sigma(t)$ for every element $t \in \mathbb{R}$, all elements in $\mathbb{T} = \mathbb{R}$ are dense. The graininess function yields

$$\mu(t) = \sigma(t) - t = t - t = 0 \quad \forall \quad t \in \mathbb{T}.$$

(ii) Let $\mathbb{T} = \mathbb{Z}$. Then for any $t \in \mathbb{Z}$, we have

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t + 1, t + 2, t + 3, ...\} = t + 1.$$

Similarly, we have $\rho(t) = t - 1$. Since $\rho(t) < t < \sigma(t)$ for every element $t \in \mathbb{Z}$, all elements in $\mathbb{T} = \mathbb{Z}$ are isolated. The graininess function yields

$$\mu(t) = \sigma(t) - t = (t + 1) - t = 1 \quad \forall \quad t \in \mathbb{T}.$$

Figure 3.1: Visual representation of Example 1(iii), $\mathbb{T} = [0, 2] \cup \{6\}$.

(iii) Refer to Figure 3.1. Let $\mathbb{T} = [0, 2] \cup \{6\}$. Then for $t \in (0, 2)$, we have

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = t.$$

Similarly, $\rho(t) = t$ *for the above values of* t*. The graininess function for the above* t *values becomes*

$$\mu(t) = \sigma(t) - t = t - t = 0.$$

Thus, the elements $t \in (0, 2)$ *are dense. For* $t = 0$*, we have*

$$\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = 0$$

and

$$\rho(0) = \sup\{s \in \mathbb{T} : s < 0\} = \sup \emptyset = \inf \mathbb{T} = 0.$$

Note that the element $t = 0 = \inf \mathbb{T}$ *is a right-dense minimum which behaves as left-dense. The graininess function yields*

$$\mu(0) = \sigma(0) - 0 = 0.$$

For $t = 2$*, we have*

$$\sigma(2) = \inf\{s \in \mathbb{T} : s > 2\} = 6$$

and

$$\rho(2) = \sup\{s \in \mathbb{T} : s < 2\} = 2.$$

The graininess function yields

$$\mu(2) = \sigma(2) - 2 = 4.$$

Thus, the element $t = 2$ *is a left-dense and right-scattered element. For* $t = 6$*, we have*

$$\sigma(6) = \inf\{s \in \mathbb{T} : s > 6\} = \inf \emptyset = \sup \mathbb{T} = 6.$$

Note that the element $t = 6 = \sup \mathbb{T}$ *is a left-scattered maximum which behaves as right-dense. Also,*

$$\rho(6) = \sup\{s \in \mathbb{T} : s < 6\} = 2$$

and the graininess function yields

$$\mu(6) = \sigma(6) - 6 = 0.$$

The above examples provide insight into how the forward-jump operator, backward-jump operator, and graininess function behave on diverse time scales. These are the building blocks we will use in the next section to describe differentiation on a time scale.

3.2 Differentiation

In order to solve a logistic dynamic equation on a time scale, we must first understand differentiation on a time scale. In Time Scale Calculus, differentiation merges the rules from traditional calculus and from discrete analysis into one uniform concept. We begin with the definition of the delta derivative and follow with useful properties and applications. Also, from this point forward when we say *derivative*, we are referring to the *delta derivative*.

Definition 5 ([2])**.** *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\triangle}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (for example, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\Big|[f^{\sigma}(t) - f(s)] - f^{\triangle}(t)[\sigma(t) - s]\Big| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

*We read $f^{\triangle}(t)$ as the **delta derivative** of f at t.*

Now we introduce a theorem which relates differentiability with continuity for elements characterized by the jump operators. The reader may find proofs (not provided) for this theorem and all future theorems in the referenced texts.

Theorem 1 ([2])**.** *Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\triangle}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists and is a finite number. In this case

$$f^{\triangle}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\triangle}(t).$$

This equation is known as the "simple useful formula."

The following example provides a connection to traditional calculus and discrete analysis along with a combination of the two.

Example 2. *Let $f : \mathbb{T} \to \mathbb{R}$ be continuous.*

(i) Let $\mathbb{T} = \mathbb{R}$. Then $f^{\triangle}(t) = f'(t)$.

(ii) Let $\mathbb{T} = \mathbb{Z}$. Then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$.

(iii) Let $\mathbb{T} = [0, 2] \cup \{3\} \cup [5, 6]$. Then computing the derivative at the points $t = 1, 2, 3,$ and 5 using the definition of delta derivative, we obtain

(a) The point $t = 1$ is dense, so we have

$$f^{\triangle}(1) = \lim_{s \to 1} \frac{f(1) - f(s)}{1 - s}.$$

(b) The point $t = 2$ is right-scattered, so we have

$$f^{\triangle}(2) = \frac{f^{\sigma}(2) - f(2)}{\mu(2)} = \frac{f(3) - f(2)}{1} = f(3) - f(2).$$

(c) The point $t = 3$ is isolated, so we have

$$f^{\triangle}(3) = \frac{f^{\sigma}(3) - f(3)}{\mu(3)} = \frac{f(5) - f(3)}{2}.$$

(d) The point $t = 5$ is right-dense, so we have

$$f^{\triangle}(5) = \lim_{s \to 5} \frac{f(5) - f(s)}{5 - s}.$$

These examples show that the method used to find the derivative at a point in a time scale changes based on the classification of the element. As one would hope, the delta derivative at a dense element is precisely that of a traditional derivative and the delta derivative at a right-scattered element with graininess 1 is precisely that of a difference operator.

The next theorem states properties of the delta derivative that are also similar to those in traditional calculus.

Theorem 2 ([2]). *Suppose $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(t) + f^{\sigma}(t)g^{\triangle}(t) = f(t)g^{\triangle}(t) + f^{\triangle}(t)g^{\sigma}(t).$$

(iv) If $f(t)f^{\sigma}(t) \neq 0$*, then* $\frac{1}{f}$ *is differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\triangle}(t) = -\frac{f^{\triangle}(t)}{f(t)f^{\sigma}(t)}.$$

(v) If $g(t)g^{\sigma}(t) \neq 0$*, then* $\frac{f}{g}$ *is differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g^{\sigma}(t)}.$$

As stated previously, the delta derivative is all-inclusive. It provides a way to differentiate, for example, on an interval, or on a union of intervals, or on a union of intervals and isolated elements. Just as it is important to understand differentiation on a time scale, we must have a solid foundation for integrating on a time scale in order to solve a logistic dynamic equation. The next section will build this foundation.

3.3 Integration

Before discussing the antiderivative with respect to time scales, we must describe what it means for a function to be regulated and rd-continuous since these are properties that our pre-antiderivative must possess. Also, any mention of the term *limit* refers to *finite limit*.

Definition 6 ([2])**.** *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called* ***regulated*** *provided its right-sided limits exist at all right-dense points in* $\mathbb{T}$ *and its left-sided limits exist at all left-dense points in* $\mathbb{T}$*.*

Definition 7 ([2])**.** *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called* ***rd-continuous*** *provided it is continuous at right-dense points in* $\mathbb{T}$ *and its left-sided limits exist at left-dense points in* $\mathbb{T}$*. The set of rd-continuous functions* $f : \mathbb{T} \to \mathbb{R}$ *will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The following theorem describes a few results related to continuity and regulated and rd-continuous functions.

Theorem 3 ([2])**.** *Assume* $f : \mathbb{T} \to \mathbb{R}$*.*

(i) If f *is continuous, then* f *is rd-continuous.*

(ii) If f *is rd-continuous, then* f *is regulated.*

(iii) The jump operator σ *is rd-continuous.*

(iv) If f *is regulated or rd-continuous, then so is* f^σ*.*

(v) Assume f *is continuous. If* $g : \mathbb{T} \to \mathbb{R}$ *is regulated or rd-continuous, then* $f \circ g$ *has that property too.*

Combining these ideas of regulated and rd-continuous functions with the concept of pre-differentiability and the existence of pre-antiderivatives will allow us to define the delta antiderivative. Also, note a region of differentiability, D, is a region such that $D \subset \mathbb{T}^\kappa$ and $\mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$.

Theorem 4 ([2])**.** *Let* f *be regulated. Then there exists a function* F *which is pre-differentiable with region of differentiation* D *such that*

$$F^{\triangle}(t) = f(t) \qquad \textit{holds for all} \qquad t \in D.$$

Definition 8 ([2])**.** *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a regulated function. Any function* F *as in Theorem 4 is called a* ***pre-antiderivative*** *of* f*. We define the* ***indefinite integral*** *of a regulated function* f *by*

$$\int f(t) \triangle t = F(t) + C,$$

where C *is an arbitrary constant and* F *is a pre-antiderivative of* f*. We define the* ***Cauchy integral*** *by*

$$\int_r^s f(t) \triangle t = F(s) - F(r) \qquad \textit{for all} \qquad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ *is called an* ***antiderivative*** *of* $f : \mathbb{T} \to \mathbb{R}$ *provided*

$$F^{\triangle}(t) = f(t) \qquad \textit{holds for all} \qquad t \in \mathbb{T}^\kappa.$$

3.4 The Hilger Complex Plane

In order to define the exponential function on a time scale, we must first understand the structure of the Hilger complex plane. In the Hilger complex plane, the reciprocal of the graininess function, h, is the radius of the Hilger imaginary circle which is tangent to the imaginary axis.

Definition 9 ([2])**.** *For $h > 0$ we define the **Hilger complex numbers**, the **Hilger real axis**, the **Hilger alternating axis**, and the **Hilger imaginary circle** as*

$$\begin{aligned}
\mathbb{C}_h &:= \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\} \\
\mathbb{R}_h &:= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\} \\
\mathbb{A}_h &:= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\} \\
\mathbb{I}_h &:= \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},
\end{aligned}$$

respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 := i\mathbb{R}$, and $\mathbb{A}_0 := \emptyset$. A visualization of the Hilger complex plane can be seen in Figure 3.2.

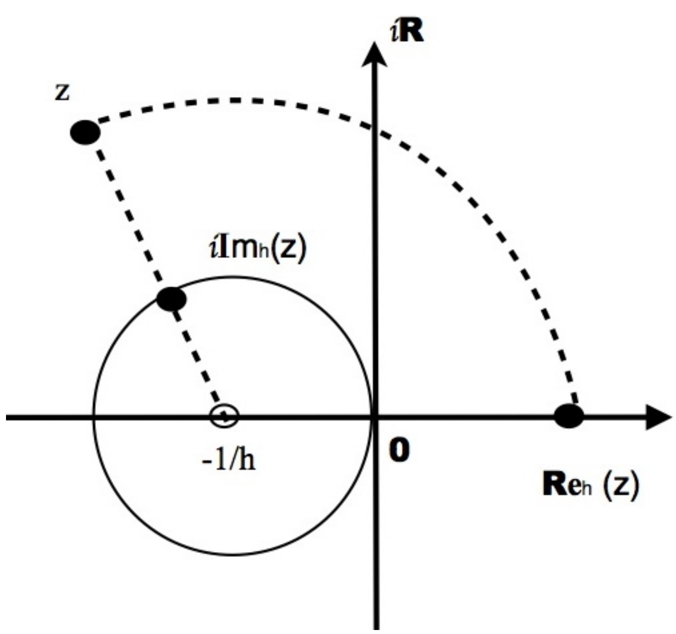

Figure 3.2: The Hilger Complex Plane [7]

Refering to Figure 3.2, notice that the radius of the Hilger imaginary circle is $\frac{1}{h}$ and that the circle is centered to the left of the origin $\frac{1}{h}$ units. So, as the graininess becomes smaller, the Hilger imaginary circle grows larger.

Additionally, we define the Hilger real and imaginary parts of a complex number z, along with the Hilger purely imaginary number $\mathring{\iota}\omega$.

Definition 10 ([2]). *Let $h > 0$ and $z \in \mathbb{C}_h$. We define the Hilger real part of z by*

$$\mathbb{R}e_h(z) := \frac{|zh+1|-1}{h}$$

and the Hilger imaginary part of z by

$$\mathbb{I}m_h(z) := \frac{Arg(zh+1)}{h},$$

where $-\pi < Arg(z) \leq \pi$.

Definition 11 ([2]). *Let $-\frac{\pi}{h} < \omega \leq \frac{\pi}{h}$. We define the **Hilger purely imaginary number** $\mathring{\iota}\omega$ by*

$$\mathring{\iota}\omega = \frac{e^{\mathring{\iota}\omega h}-1}{h}.$$

To define our logistic dynamic equation on a time scale, we must introduce the operation "circle plus" and its connection to the Hilger real and imaginary parts, and for $z \in \mathbb{C}_h$, introduce operation "circle minus." Both circle plus and circle minus are defined on $\mathbb{C}_h$.

Definition 12 ([2]). *We define the **circle plus addition** $\oplus$ on $\mathbb{C}_h$ by*

$$z \oplus w := z + w + zwh.$$

It is important to mention that $(\mathbb{C}_h, \oplus)$ is an Abelian group.

Theorem 5 ([2]). *For $z \in \mathbb{C}_h$ we have*

$$z = \mathbb{R}e_h z \oplus i\mathbb{I}m_h z.$$

The operation circle minus appears in the properties of the exponential function and in our logistic dynamic equation.

Definition 13 ([2]). *We define the* ***cirlce minus subtraction*** *on* $\mathbb{C}_h$ *by*

$$z \ominus w := z \oplus (\ominus w)$$

where

$$\ominus z := \frac{-z}{1+zh}.$$

Also, if $z, w \in \mathbb{C}_h$ with $h \geq 0$, then we have the following properties

(i) $z \ominus z = 0$;

(ii) $z \ominus w = \dfrac{z-w}{1+wh}$;

(iii) $z \ominus w = z - w$ if $h = 0$.

The cylinder transformation is at the heart of the exponential function. In order to define it, however, we need to define its codomain, $\mathbb{Z}_h$.

Definition 14 ([2]). *For* $h > 0$, *let* $\mathbb{Z}_h$ *be the strip*

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\},$$

and for $h = 0$, *let* $\mathbb{Z}_0 := \mathbb{C}$.

Definition 15 ([2]). *For* $h > 0$, *we define the* ***cylinder transformation*** $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h} Log(1 + zh),$$

where Log is the principal logarithm function. For $h = 0$, *we define* $\xi_0(z) = z$ *for all* $z \in \mathbb{C}$.

Definition 16 ([2]). *For* $h > 0$, *we define the* ***inverse transformation*** $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h$ *of the cylinder transformation by*

$$\xi_h^{-1}(z) = \frac{1}{h}\left(e^{zh} - 1\right).$$

The cylinder transformation, ξ_h, gets its name from $\mathbb{Z}_h$. For $h > 0$, if the boundary lines, $\text{Im}(z) = -\frac{\pi}{h}$ and $\text{Im}(z) = \frac{\pi}{h}$, of $\mathbb{Z}_h$ are joined, $\mathbb{Z}_h$ can be viewed as a cylinder.

3.5 Exponential Function

The exponential function is essential for solving the logistic dynamic equation. However, we must first describe what it means for a linear function to be regressive before formally stating the definition of the exponential function.

Definition 17 ([2])**.** *We say that a function $p : \mathbb{T} \to \mathbb{R}$ is **regressive** provided*

$$1 + \mu(t)p(t) \neq 0 \quad \textit{for all} \quad t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Now, using the cylinder transformation, we can define the exponential function on an arbitrary time scale.

Definition 18 ([2])**.** *If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}\left(p(\tau)\right) \triangle\tau\right) \quad \textit{for all} \quad s, t \in \mathbb{T}.$$

Here is a look at a few useful properties of the exponential function. In particular, properties *iii*, *iv*, and *v*, are essential in solving our logistic dynamic equation on a time scale.

Theorem 6 ([2])**.** *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*;*

(iii) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$*;*

(iv) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$*;*

(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

(viii) $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot,s)}$;

(ix) $e_p^{\triangle}(t,t_0) = p(t)\cdot e_p(t,t_0)$.

3.6 Logistic Dynamic Equations

Let us consider the linear equation

$$v^{\triangle} = -p(t)v^{\sigma} + f(t) \tag{3.1}$$

where $p \in \mathcal{R}$ and $f \in C_{rd}$. If $v(t)$ is a solution of (3.1) and $x(t) = 1/v(t)$, then

$$\begin{aligned} x^{\triangle} &= \left(\frac{1}{v}\right)^{\triangle} \\ &= \frac{-v^{\triangle}}{vv^{\sigma}} \\ &= \frac{p(t)v^{\sigma} - f(t)}{vv^{\sigma}} \\ &= \left(p(t) - f(t)x^{\sigma}\right)x \\ &= p(t)x - f(t)x\left(x + \mu(t)x^{\triangle}\right) \\ &= \left(p(t) - f(t)x\right)x - f(t)x\mu(t)x^{\triangle} \\ &= \frac{p(t) - f(t)x}{1 + f(t)x\mu(t)}\cdot x \\ &= \left[p(t) \ominus f(t)x\right]x. \end{aligned}$$

We call

$$x^{\triangle} = \left[p(t) \ominus f(t)x\right]x \tag{3.2}$$

a logistic dynamic equation.

Definition 19 ([2])**.** *The equation*

$$x^{\triangle} = p(t)x + f(t)$$

is called ***regressive*** *provided* $p \in \mathcal{R}$ *and* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous.*

In order to obtain the solution to (3.2), we must use the variation of constants approach explained in the following theorem.

Theorem 7 ([2])**.** *Suppose*

$$v^{\triangle} = p(t)v + f(t)$$

is regressive. Let $t_0 \in \mathbb{T}$ *and* $v_0 \in \mathbb{R}$*. Then the unique solution to the initial value problem*

$$v^{\triangle} = -p(t)v^{\sigma} + f(t), \qquad v(t_0) = v_0 \tag{3.3}$$

is given by

$$v(t) = e_{\ominus p}(t, t_0)v_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\triangle\tau.$$

Using this theorem, we prove the solution of the logistic dynamic equation (3.2).

Theorem 8 ([3])**.** *Suppose* $p \in \mathcal{R}$ *and* $f \in C_{rd}$*. Let* $x_0 \neq 0$*. If*

$$v(t) = e_{\ominus p}(t, t_0)v_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\triangle\tau \qquad \text{for all } t \in \mathbb{T},$$

then

$$x(t) = \frac{1}{v(t)} = \frac{e_p(t, t_0)}{\frac{1}{x_0} + \int_{t_0}^{t} e_p(\tau, t_0)f(\tau)\triangle\tau}$$

solves (3.2) and satisfies $x(t_0) = x_0$*.*

Proof. Let $x_0 \neq 0$ and suppose

$$v(t) = \frac{e_{\ominus p}(t, t_0)}{x_0} + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\triangle\tau \neq 0$$

for all $t \in \mathbb{T}$. Let $v_0 = 1/x_0$. Then

$$v(t) = e_{\ominus p}(t, t_0)v_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\triangle\tau,$$

from Theorem 7, is the unique solution to the initial value problem (3.3). Now, let

$$x(t) = \frac{1}{v(t)} = \frac{e_p(t, t_0)}{\dfrac{1}{x_0} + \displaystyle\int_{t_0}^{t} e_p(\tau, t_0) f(\tau) \triangle\tau}.$$

From the way we derived (3.2), we know that $x(t)$ solves

$$x^{\triangle} = [p(t) \ominus f(t)x]\, x.$$

Also, since we let $v_0 = 1/x_0$, $x(t_0) = x_0$ is satisfied. □

So, we will use this general equation

$$x(t) = \frac{1}{v(t)} = \frac{e_p(t, t_0)}{\dfrac{1}{x_0} + \displaystyle\int_{t_0}^{t} e_p(\tau, t_0) f(\tau) \triangle\tau},$$

$t \in \mathbb{T}$ and $x(t_0) = x_0$, to solve the logistic dynamic equation

$$x^{\triangle} = [p(t) \ominus f(t)x]\, x$$

on various time scales.

CHAPTER 4

A LOGISTIC DYNAMIC EQUATION: ANALYTICAL SOLUTIONS

This chapter focuses on the solution of a particular logistic dynamic equation. Analytical solutions will be provided on various time scales. Theorem 8 along with many properties of the exponential function detailed in Chapter 3 will be utilized throughout these analyses.

4.1 The Initial Value Problem

Our focus is on the solution of a particular logistic dynamic equation. In differential equations ($\mathbb{T} = \mathbb{R}$), our equation of interest has the form $x' = \left(1 - \frac{1}{2}x\right) x$ with initial condition $x(0) = \frac{1}{5}$. From the method of separation of variables, the general solution to this initial value problem (IVP) is given by

$$x(t) = \frac{2x_0 e^t}{e^{t_0}(2 - x_0) + x_0 e^t}.$$

So, the particular solution of this IVP is

$$x(t) = \frac{2e^t}{e^t + 9}.$$

Our IVP, with general form,

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x, \quad x(0) = \frac{1}{5},$$

involves the circle minus operation and has the unique solution defined in Theorem 8. We will begin by finding the general solution on an arbitrary time scale, $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$ and initial condition $x(t_0) = x_0$. Using the general solution, we will find a particular solution on a time scale with a large gap, $\mathbb{T}_1 = [0, 1] \cup [3, 5]$. We will continue to find particular solutions on time scales with smaller gaps between intervals to show that the solution converges to the solution on $\mathbb{T} = \mathbb{R}$, a time scale with no gap. It is impoartant to note that our IVP has only one initial condition. Even though we are able to have disjoint sets for our domains in Time Scale Calculus, we only need one initial condition for a first order problem. We are able to obtain the value of the solution at the beginning of the second interval by using the information obtained at the end of the first interval.

4.2 The General Solution

First, we will look at our IVP

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right)x, \quad x(t_0) = x_0,$$

on an arbitrary time scale $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$. Using concepts from Chapter 3, we know the dynamic equation

$$x^{\triangle} = [p(t) \ominus f(t)x]\, x, \quad x(t_0) = x_0,$$

has the general solution

$$x(t) = \frac{e_p(t, t_0)}{\frac{1}{x_0} + \int_{t_0}^{t} e_p(\tau, t_0) f(\tau) \triangle\tau}.$$

For our IVP, we have $p(t) = 1$ and $f(t) = \frac{1}{2}$. Also recall that our circle minus definition gives us

$$x^{\triangle} = \left(\frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}\mu(t)x}\right)x,$$

the differential equation divided by the factor $\left(1 + \frac{1}{2}\mu(t)x\right)$ where $\mu(t)$ is the size of our gap at t. Notice that when $\mu(t) = 0$, ($\mathbb{T} = \mathbb{R}$), we simply have the differential equation $x' = \left(1 - \frac{1}{2}x\right)x$.

Now, we will calculate the solution for $t \in [t_0, t_1]$, $t = t_2$, and $t \in (t_2, t_3]$ analytically.

- For $t \in [t_0, t_1]$ with $\mu(t) = 0$, we have

$$\begin{aligned} x(t) &= \frac{e_p(t, t_0)}{\frac{1}{x_0} + \int_{t_0}^{t} e_p(\tau, t_0) f(\tau) \triangle\tau} \\ &= \frac{e_1(t, t_0)}{\frac{1}{x_0} + \frac{1}{2}\int_{t_0}^{t} e_1(\tau, t_0) \triangle\tau} \\ &= \frac{e^{t - t_0}}{\frac{1}{x_0} + \frac{1}{2}\int_{t_0}^{t} e^{\tau - t_0} \triangle\tau} \\ &= \frac{2x_0 e^{t - t_0}}{2 + x_0 e^{-t_0}\int_{t_0}^{t} e^{\tau} \triangle\tau} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2x_0e^t}{2e^{t_0} + x_0\int_{t_0}^t e^\tau \triangle\tau} \\
&= \frac{2x_0e^t}{2e^{t_0} + x_0(e^t - e^{t_0})} \\
&= \frac{2x_0e^t}{e^{t_0}(2 - x_0) + x_0e^t}.
\end{aligned}
$$

- For $t = t_2$ with $\mu(t_1) = t_2 - t_1$, we have

$$
\begin{aligned}
x(t) &= \frac{e_p(t, t_0)}{\frac{1}{x_0} + \int_{t_0}^t e_p(\tau, t_0)f(\tau)\triangle\tau} \\
&= \frac{e_1(t, t_0)}{\frac{1}{x_0} + \frac{1}{2}\int_{t_0}^t e_1(\tau, t_0)\triangle\tau} \\
&= \frac{2x_0e_1(t_2, t_1)e_1(t_1, t_0)}{2 + x_0\left(\int_{t_1}^{t_2} e_1(\tau, t_0)\triangle\tau + \int_{t_0}^{t_1} e_1(\tau, t_0)\triangle\tau\right)} \\
&= \frac{2x_0(1 + t_2 - t_1)e^{t_1 - t_0}}{2 + x_0\left(\int_{t_1}^{t_2} e_1(\tau, t_1)e_1(t_1, t_0)\triangle\tau + e^{t_1 - t_0} - 1\right)} \\
&= \frac{2x_0(1 + t_2 - t_1)e^{t_1 - t_0}}{2 + x_0\left(e^{t_1 - t_0}\int_{t_1}^{t_2} e_1(\tau, t_1)\triangle\tau + e^{t_1 - t_0} - 1\right)} \\
&= \frac{2x_0(1 + t_2 - t_1)e^{t_1 - t_0}}{2 + x_0\left(e^{t_1 - t_0}(t_2 - t_1) + e^{t_1 - t_0} - 1\right)} \\
&= \frac{2x_0(1 + t_2 - t_1)e^{t_1}}{2e^{t_0} + x_0e^{t_1}(1 + t_2 - t_1) - x_0e^{t_0}} \\
&= \frac{2x_0e^{t_1}(1 + t_2 - t_1)}{e^{t_0}(2 - x_0) + x_0e^{t_1}(1 + t_2 - t_1)}.
\end{aligned}
$$

- For $t \in (t_2, t_3]$ with $\mu(t) = 0$, we have

$$
\begin{aligned}
x(t) &= \frac{e_p(t, t_0)}{\frac{1}{x_0} + \int_{t_0}^t e_p(\tau, t_0)f(\tau)\triangle\tau} \\
&= \frac{e_1(t, t_0)}{\frac{1}{x_0} + \frac{1}{2}\int_{t_0}^t e_1(\tau, t_0)\triangle\tau} \\
&= \frac{2x_0e_1(t, t_2)e_1(t_2, t_1)e_1(t_1, t_0)}{2 + x_0\left(\int_{t_2}^t e_1(\tau, t_0)\triangle\tau + \int_{t_1}^{t_2} e_1(\tau, t_0)\triangle\tau + \int_{t_0}^{t_1} e_1(\tau, t_0)\triangle\tau\right)} \\
&= \frac{2x_0e^{t - t_2}(1 + t_2 - t_1)e^{t_1 - t_0}}{2 + x_0\left(e_1(t_2, t_1)e_1(t_1, t_0)\int_{t_2}^t e_1(\tau, t_2)\triangle\tau + e_1(t_1, t_0)\int_{t_1}^{t_2} e_1(\tau, t_1)\triangle\tau + e^{t_1 - t_0} - 1\right)}
\end{aligned}
$$

$$= \frac{2x_0 e^{t-t_2+t_1-t_0}(1+t_2-t_1)}{2+x_0\left(e^{t_1-t_0}(1+t_2-t_1)(e^{t-t_2}-1)+e^{t_1-t_0}(t_2-t_1)+e^{t_1-t_0}-1\right)}$$
$$= \frac{2x_0 e^{t-t_2+t_1-t_0}(1+t_2-t_1)}{2+x_0 e^{t_1-t_0}\left((e^{t-t_2}-1)(1+t_2-t_1)+t_2-t_1+1\right)-x_0}$$
$$= \frac{2x_0 e^{t-t_2+t_1}(1+t_2-t_1)}{2e^{t_0}+x_0 e^{t-t_2+t_1}(1+t_2-t_1)-x_0 e^{t_0}}$$
$$= \frac{2x_0 e^{t-t_2+t_1}(1+t_2-t_1)}{e^{t_0}(2-x_0)+x_0 e^{t-t_2+t_1}(1+t_2-t_1)}.$$

Therefore, the general solution for $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$ is

$$x(t) = \begin{cases} \dfrac{2x_0 e^t}{e^{t_0}(2-x_0)+x_0 e^t}, & \text{for } t \in [t_0, t_1] \\ \\ \dfrac{2x_0 e^{t_1}(1+t_2-t_1)}{e^{t_0}(2-x_0)+x_0 e^{t_1}(1+t_2-t_1)}, & \text{for } t = t_2 \\ \\ \dfrac{2x_0 e^{t-t_2+t_1}(1+t_2-t_1)}{e^{t_0}(2-x_0)+x_0 e^{t-t_2+t_1}(1+t_2-t_1)}, & \text{for } t \in (t_2, t_3] \end{cases} .$$

Note that all of the above terms, excluding e^t, are constants because they do not depend on t.

4.3 Particular Solutions

We can now calculate particular solutions of our IVP ,

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right)x, \quad x(0) = \frac{1}{5},$$

on three different time scales using the general solution we just found. With each successive time scale we will decrease the gap between the intervals, with the last gap measuring only 0.5 units wide.

The first time scale we are interested in is $\mathbb{T}_1 = [0, 1] \cup [3, 5]$. Using our general solution, we obtain the following calculations:

- For $t \in [0, 1]$, we have

$$x(t) = \frac{2x_0 e^t}{e^{t_0}(2-x_0)+x_0 e^t}$$

$$
\begin{aligned}
&= \frac{2\left(\frac{1}{5}\right)e^t}{e^0\left(2-\frac{1}{5}\right)+\frac{1}{5}e^t} \\
&= \frac{2e^t}{(10-1)+e^t} \\
&= \frac{2e^t}{e^t+9}.
\end{aligned}
$$

- For $t = 3$, we have

$$
\begin{aligned}
x(3) &= \frac{2x_0e^{t_1}(1+t_2-t_1)}{e^{t_0}(2-x_0)+x_0e^{t_1}(1+t_2-t_1)} \\
&= \frac{2\left(\frac{1}{5}\right)e^1(1+3-1)}{e^0\left(2-\frac{1}{5}\right)+\frac{1}{5}e^1(1+3-1)} \\
&= \frac{2e(3)}{(10-1)+e(3)} \\
&= \frac{6e}{9+3e} \\
&= \frac{2e}{e+3}.
\end{aligned}
$$

We can also find the value of $x(3)$ using the simple useful formula as follows

$$
\begin{aligned}
x(3) &= x(1)+\mu(1)x^{\triangle}(1) \\
&= \frac{2e}{e+9}+2\left(\frac{\left(1-\frac{1}{2}x(1)\right)x(1)}{1+\frac{1}{2}x(1)}\right) \\
&= \frac{2e}{e+9}+2\left(\frac{6e}{(e+3)(e+9)}\right) \\
&= \frac{2e(e+3)+12e}{(e+3)(e+9)} \\
&= \frac{2e(e+9)}{(e+3)(e+9)} \\
&= \frac{2e}{e+3}.
\end{aligned}
$$

Notice that we arrive at the same value for our solution using the simple useful formula as we do using the general solution.

- For $t \in (3,5]$, we have

$$\begin{aligned}
x(t) &= \frac{2x_0 e^{t-t_2+t_1}(1+t_2-t_1)}{e^{t_0}(2-x_0)+x_0 e^{t-t_2+t_1}(1+t_2-t_1)} \\
&= \frac{2\left(\frac{1}{5}\right)e^{t-3+1}(1+3-1)}{e^0\left(2-\frac{1}{5}\right)+\frac{1}{5}e^{t-3+1}(1+3-1)} \\
&= \frac{2e^{t-2}(3)}{(10-1)+e^{t-2}(3)} \\
&= \frac{6e^{t-2}}{9+3e^{t-2}} \\
&= \frac{2e^{t-2}}{e^{t-2}+3}.
\end{aligned}$$

Therefore, the particular solution for $\mathbb{T}_1 = [0,1] \cup [3,5]$ is

$$x(t) = \begin{cases} \dfrac{2e^t}{e^t+9}, & \text{for } t \in [0,1] \\ \dfrac{2e}{e+3}, & \text{for } t = 3 \\ \dfrac{2e^{t-2}}{e^{t-2}+3}, & \text{for } t \in (3,5] \end{cases}.$$

Now, for the second time scale, we have $\mathbb{T}_2 = [0,1] \cup [2,5]$. Using the general solution, we find this particular solution to be

$$x(t) = \begin{cases} \dfrac{2e^t}{e^t+9}, & \text{for } t \in [0,1] \\ \dfrac{4e}{2e+9}, & \text{for } t = 2 \\ \dfrac{4e^{t-1}}{2e^{t-1}+9}, & \text{for } t \in (2,5] \end{cases}.$$

Finally, for our third time scale, we have $\mathbb{T}_3 = [0,1] \cup \left[\frac{3}{2},5\right]$. Again, using the general solution, we arrive at the following particular solution:

$$x(t) = \begin{cases} \dfrac{2e^t}{e^t+9}, & \text{for } t \in [0,1] \\ \dfrac{2e}{e+6}, & \text{for } t = \frac{3}{2} \\ \dfrac{2e^{t-1/2}}{e^{t-1/2}+6}, & \text{for } t \in \left(\frac{3}{2},5\right] \end{cases}.$$

Consider our IVP as a differential equation where $\mathbb{T} = \mathbb{R}$ with the same initial condition $x(0) = \frac{1}{5}$. Then, to calculate the solution at any value of $t \in \mathbb{R}$, we would use

$$x(t) = \frac{2e^t}{e^t + 9}.$$

For each of the above time scales, we found particular solution values depending on where t lived in the time scale. In Table 4.1 below, we compare the solutions for the left-scattered elements from each of the time scales with their respective solution in $\mathbb{T} = \mathbb{R}$.

	$\mathbb{T} = \mathbb{R}$	$\mathbb{T} = \mathbb{T}_i$		
$t_{2,i} = t_2 \in \mathbb{T}_i$	$x_{\mathbb{R}}(t_{2,i})$	$x_{\mathbb{T}}(t_{2,i})$	$x_{\mathbb{R}}(t_{2,i}) - x_{\mathbb{T}}(t_{2,i})$	$\mu(\rho(t_{2,i}))$
$t_{2,1} = 3$	1.38	0.951	0.429	2
$t_{2,2} = 2$	0.902	0.753	0.149	1
$t_{2,3} = \frac{3}{2}$	0.665	0.624	0.041	$\frac{1}{2}$

Table 4.1: Solution values for left-scattered elements in $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$ versus their solution values in $\mathbb{T} = \mathbb{R}$.

Notice as the size of the gap $\mu(\rho(t_{2,i}))$ decreases, the difference in the solution values for each $t_{2,i}$ also decreases. This gave us the idea that as the gap between the intervals of a time scale narrows, the solution on the time scale may approach the solution on $\mathbb{R}$. In fact, this is the case. Before we formally state this result, let us look at the general solution on a sequence of time scales.

4.4 A Sequence of Time Scales

Rather than working with individual time scales, we will now focus on a sequence of time scales. We are still considering our IVP

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x, \qquad x(m) = x_0.$$

Note that x_0 is a constant. We want to analyze the general solution on the time scale $\mathbb{T}_n = [m, a] \cup [a + \delta_n, b]$ where m, a, and b are constants and $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we define $\delta_n = \frac{1}{n}$ where $0 \leq \delta_n \leq b - \left(a + \frac{1}{n}\right)$ and $t_n = a + \frac{1}{n}$. From Theorem 8, we know our general solution has the form

$$\begin{aligned} x(t) &= \frac{e_p(t,t_0)}{\frac{1}{x_0} + \int_{t_0}^t e_p(\tau,t_0) f(\tau) \triangle\tau} \\ &= \frac{e_1(t,m)}{\frac{1}{x_0} + \frac{1}{2}\int_m^t e_1(\tau,m)\triangle\tau}. \end{aligned}$$

Similar to our process in the previous section, we want to analytically describe the solution on each interval and at the point after the jump. We will use the same general solution and initial condition each time.

On $\mathbb{T}_n = [m,a] \cup \left[a+\frac{1}{n}, b\right]$, $n \in \mathbb{N}$, we have the following calculations:

- For $t \in [m,a]$ with $\mu(t) = 0$, we have

$$\begin{aligned} x(t) &= \frac{e_1(t,m)}{\frac{1}{x_0} + \frac{1}{2}\int_m^t e_1(\tau,m)\triangle\tau} \\ &= \frac{e^{t-m}}{\frac{1}{x_0} + \frac{1}{2}\int_m^t e^{\tau-m}\triangle\tau} \\ &= \frac{e^{t-m}}{\frac{1}{x_0} + \frac{1}{2}e^{-m}\int_m^t e^{\tau}\triangle\tau} \\ &= \frac{2x_0 e^t}{2e^m + x_0(e^t - e^m)} \\ &= \frac{2x_0 e^t}{e^m(2-x_0) + x_0 e^t}. \end{aligned}$$

- For $t_n = a + \frac{1}{n}$ with $\mu(a) = a + \frac{1}{n} - a = \frac{1}{n}$, we have

$$\begin{aligned} x_n(t) &= \frac{e_1(t,m)}{\frac{1}{x_0} + \frac{1}{2}\int_m^t e_1(\tau,m)\triangle\tau} \\ &= \frac{2x_0 e_1\left(a+\frac{1}{n},a\right) e_1(a,m)}{2 + x_0\left(\int_a^{a+1/n} e_1(\tau,m)\triangle\tau + \int_m^a e_1(\tau,m)\triangle\tau\right)} \\ &= \frac{2x_0\left(1+a+\frac{1}{n}-a\right)e^{a-m}}{2 + x_0\left(\int_a^{a+1/n} e_1(\tau,a)e_1(a,m)\triangle\tau + e^{a-m} - 1\right)} \\ &= \frac{2x_0\left(1+\frac{1}{n}\right)e^{a-m}}{2 + x_0\left(e^{a-m}\int_a^{a+1/n} e_1(\tau,a)\triangle\tau + e^{a-m} - 1\right)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2x_0\left(1+\frac{1}{n}\right)e^{a-m}}{2+x_0\left(e^{a-m}\left(a+\frac{1}{n}-a\right)+e^{a-m}-1\right)}\\
&= \frac{2x_0\left(1+\frac{1}{n}\right)e^{a}}{2e^m+x_0e^a\left(1+\frac{1}{n}\right)-x_0e^m}\\
&= \frac{2x_0e^a\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^a\left(1+\frac{1}{n}\right)}.
\end{aligned}
$$

- For $t \in \left(a+\frac{1}{n}, b\right]$ with $\mu(t)=0$, we have

$$
\begin{aligned}
x_n(t) &= \frac{e_1(t,m)}{\frac{1}{x_0}+\frac{1}{2}\int_m^t e_1(\tau,m)\triangle\tau}\\
&= \frac{2x_0e_1\left(t,a+\frac{1}{n}\right)e_1\left(a+\frac{1}{n},a\right)e_1(a,m)}{2+x_0\left(\int_{a+1/n}^t e_1(\tau,m)\triangle\tau+\int_a^{a+1/n}e_1(\tau,m)\triangle\tau+\int_m^a e_1(\tau,m)\triangle\tau\right)}\\
&= \frac{2x_0e^{t-a-\frac{1}{n}}\left(1+a+\frac{1}{n}-a\right)e^{a-m}}{2+x_0\left(e_1\left(a+\frac{1}{n},a\right)e_1(a,m)\int_{a+1/n}^t e_1\left(\tau,a+\frac{1}{n}\right)\triangle\tau+e_1(a,m)\int_a^{a+1/n}e_1(\tau,a)\triangle\tau+e^{a-m}-1\right)}\\
&= \frac{2x_0e^{t-1/n-m}\left(1+\frac{1}{n}\right)}{2+x_0\left(e^{a-m}\left(1+a+\frac{1}{n}-a\right)\left(e^{t-a-1/n}-1\right)+e^{a-m}\left(a+\frac{1}{n}-a\right)+e^{a-m}-1\right)}\\
&= \frac{2x_0e^{t-1/n-m}\left(1+\frac{1}{n}\right)}{2+x_0e^{a-m}\left(\left(e^{t-a-1/n}-1\right)\left(1+\frac{1}{n}\right)+\frac{1}{n}+1\right)-x_0}\\
&= \frac{2x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}{2e^m+x_0e^{t-1/n}\left(1+\frac{1}{n}\right)-x_0e^m}\\
&= \frac{2x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}.
\end{aligned}
$$

Therefore, the general solution for $\mathbb{T}_n=[m,a]\cup\left[a+\frac{1}{n},b\right]$ is

$$
x_n(t)=\begin{cases}
\dfrac{2x_0e^t}{e^m(2-x_0)+x_0e^t}, & \text{for } t\in[m,a]\\[2em]
\dfrac{2x_0e^a\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^a\left(1+\frac{1}{n}\right)}, & \text{for } t_n=a+\frac{1}{n}\\[2em]
\dfrac{2x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}, & \text{for } t\in\left(a+\frac{1}{n},b\right]
\end{cases}.
$$

4.5 Convergence of Solutions

Now, we focus on a sequence of time scales, $\mathbb{T}_n$, that converges to a time scale $\mathbb{T}$ in the Hausdorff metric. That is, we want to show, for a sequence $\mathbb{T}_n$, there exists $\mathbb{T}$ such that

$$\lim_{n\to\infty} \mathbb{T}_n = \mathbb{T}.$$

The Hausdorff metric measures a distance between two sets, so this limit implies that this distance between $\mathbb{T}_n$ and $\mathbb{T}$ goes to 0. Let the space of closed nonempty subsets of $\mathbb{R}$ be denoted by $CL(\mathbb{R})$, and define the metric

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad d(x, y) = \min \{|x - y|, 1\}.$$

Using the above metric, we now have the following:

Definition 20 ([6])**.** *Let $H(\mathbb{T}, \mathbb{T}_n)$ represent the Hausdorff metric, then*

$$H(\mathbb{T}, \mathbb{T}_n) = \max \left\{ \sup_{s\in\mathbb{T}} d'(s, \mathbb{T}_n), \sup_{t\in\mathbb{T}_n} d'(\mathbb{T}, t) \right\}$$

where

$$d'(s, \mathbb{T}_n) = \inf_{t\in\mathbb{T}_n} d(s, t)$$

and

$$d'(\mathbb{T}, t) = \inf_{s\in\mathbb{T}} d(s, t).$$

Since $\mathbb{T}_n$ and $\mathbb{T}$ are closed,

$$\inf_{t\in\mathbb{T}_n} d(s, t) = \min_{t\in\mathbb{T}_n} d(s, t)$$

and

$$\inf_{s\in\mathbb{T}} d(s, t) = \min_{s\in\mathbb{T}} d(s, t).$$

Also, $\sup_{s\in\mathbb{T}} d'(s,\mathbb{T}_n)$ and $\sup_{t\in\mathbb{T}_n} d'(\mathbb{T},t)$ are bounded above. Thus,

$$\sup_{s\in\mathbb{T}} d'(s,\mathbb{T}_n) = \max_{s\in\mathbb{T}} d'(s,\mathbb{T}_n)$$

and

$$\sup_{t\in\mathbb{T}_n} d'(\mathbb{T},t) = \max_{t\in\mathbb{T}_n} d'(\mathbb{T},t)$$

and the Hausdorff metric becomes

$$H(\mathbb{T},\mathbb{T}_n) = \max\left\{\max_{s\in\mathbb{T}} d'(s,\mathbb{T}_n), \max_{t\in\mathbb{T}_n} d'(\mathbb{T},t)\right\}.$$

Using the Hausdorff metric, we can define the distance between two time scales $\mathbb{T}_n$ and $\mathbb{T}$. When we allow this distance to approach zero, the solution $x_n(t)$ defined on $\mathbb{T}_n$, approaches the solution $x(t)$, defined on $\mathbb{T}$. This result is formally stated in the following proposition.

Proposition 1. *Let $\mathbb{T}_n$ denote the time scale $\mathbb{T}_n = [m,a] \cup \left[a+\frac{1}{n}, b\right]$ $\left(a+\frac{1}{n} < b, \forall\, n \in \mathbb{N}\right)$ and $\mathbb{T} = [m,b]$. Consider the logistic dynamic equation*

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right)x \quad \text{with} \quad x(m) = x_0. \tag{4.1}$$

Let $x_n = x_n(t)$ be a solution of (4.1) on $\mathbb{T}_n$. Then for all $t \in [m,b]$,

$$\lim_{n\to\infty} x_n(t) = \frac{2x_0e^t}{e^m(2-x_0)+x_0e^t}.$$

That is, as $\mathbb{T}_n \to \mathbb{T}$, $x_n(t) \to \dfrac{2x_0e^t}{e^m(2-x_0)+x_0e^t}$ for all $t \in [m,b]$.

Proof. Let $\mathbb{T}_n$ denote the time scale $\mathbb{T}_n = [m,a] \cup \left[a+\frac{1}{n}, b\right]$ with $n \in \mathbb{N}$ and $\mathbb{T} = [m,b]$. For all

$n \in \mathbb{N}$ the solution of (4.1) on $\mathbb{T}_n$ is

$$x_n(t) = \begin{cases} \dfrac{2x_0 e^t}{e^m(2-x_0)+x_0e^t}, & \text{for } t \in [m,a] \\ \\ \dfrac{2x_0 e^a\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^a\left(1+\frac{1}{n}\right)}, & \text{for } t_n = a+\frac{1}{n} \\ \\ \dfrac{2x_0 e^{t-1/n}\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}, & \text{for } t \in \left(a+\frac{1}{n}, b\right] \end{cases}.$$

Using the Hausdorff metric, we have

$$\begin{aligned} H(\mathbb{T},\mathbb{T}_n) &= \max\left\{\max_{s\in\mathbb{T}} d'(s,\mathbb{T}_n), \max_{t\in\mathbb{T}_n} d'(\mathbb{T},t)\right\} \\ &= \max\left\{\max_{s\in[m,b]} d'\left(s,[m,a]\cup\left[a+\frac{1}{n},b\right]\right), \max_{t\in[m,a]\cup\left[a+\frac{1}{n},b\right]} d'\left([m,b],t\right)\right\} \\ &= \max\left\{\max_{s\in[m,b]}\left\{\min_{t\in[m,a]\cup\left[a+\frac{1}{n},b\right]}|s-t|\right\}, \max_{t\in[m,a]\cup\left[a+\frac{1}{n},b\right]}\left\{\min_{s\in[m,b]}|s-t|\right\}\right\} \\ &= \max\left\{\frac{1}{n},0\right\} \\ &= \frac{1}{n}. \end{aligned}$$

If $t \in [m,a]$, then

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \frac{2x_0e^t}{e^m(2-x_0)+x_0e^t} = \frac{2x_0e^t}{e^m(2-x_0)+x_0e^t}.$$

Now, consider when $t \in \left(a+\frac{1}{n}, b\right]$. Since $\mathbb{T}_n \to \mathbb{T}$, there exists $n_1 \in \mathbb{N}$ such that $t \in \mathbb{T}_{n_1}$. Since $\mathbb{T}_n \subset \mathbb{T}_{n+1}$, $t \in \mathbb{T}_n$ for all $n \geq n_1$. Then

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \frac{2x_0e^{t-1/n}\left(1+\frac{1}{n}\right)}{e^m(2-x_0)+x_0e^{t-1/n}\left(1+\frac{1}{n}\right)} = \frac{2x_0e^t}{e^m(2-x_0)+x_0e^t}.$$

Therefore, $x_n(t) \to x(t) = \dfrac{2x_0e^t}{e^m(2-x_0)+x_0e^t}$ for all $t \in [m,b]$. □

Notice that $x(t) = \dfrac{2x_0e^t}{e^m(2-x_0)+x_0e^t}$ is the solution of our DE on the interval $[m, b]$. Thus, using the dynamic logistic equation (4.1) and Hausdorff metric, we were able to show that the solution to (4.1) converges as $\mathbb{T}_n$ converges to $\mathbb{T}$. In other words,

$$x_n(t) \to x(t) = \frac{2x_0e^t}{e^m(2-x_0)+x_0e^t}.$$

This convergence is noticable in graphical representations of the solutions to (4.1) on various time scales. Chapter 5 presents these visual representations.

CHAPTER 5

A LOGISTIC DYNAMIC EQUATION: GRAPHICAL SOLUTIONS ON THE DIFFERENTIAL ANALYZER

Differential equations may be solved graphically using a number of computer programs even if no closed form solutions exist. Although these programs are readily available to us, our work utilizes Art, the Marshall University Differential Analyzer, to evaluate our IVP $x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x$ with initial condition $x(0) = \frac{1}{5}$. Art has the same capabilities as computer programs with a few advantages. The DA allows access to the behavior of each of the derivatives along with the solution at any time in the domain. These behaviors are represented numerically, using the counters, and mechanically, using the integrators. The convergence proposition in Chapter 4 was inspired by the graphical solutions produced by the DA.

5.1 Programming the IVP

It is important to mention that a differential analyzer is designed to model the derivative from traditional calculus and not necessarily the delta derivative defined in Time Scale Calculus. For this reason, Art is set up to run the equivalent differential equation $x' = \left(1 - \frac{1}{2}x\right) x$ and a process is put into place to solve the logistic dynamic equation $x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x$. Recall from Chapter 4, when $\mu(t) = 0$ ($\mathbb{T} = \mathbb{R}$), we have

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x$$

equivalent to

$$x' = \left(1 - \frac{1}{2}x\right) x.$$

Also, recall what our equation looks like using the definition of circle minus:

$$x^{\triangle} = \left(\frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}\mu(t)x}\right) x.$$

So, when $\mu(t) \neq 0$, the differential analyzer is modeling the dynamic equation using only the numerator on the right-hand side of the equation.

We will graph the solution on four different time scales beginning with $[0,1] \cup [3,5]$, the time scale with the largest gap. Each time scale after that will have a gap reduced 0.5 units relative to the previous time scale, with the last time scale being $[0,1] \cup [1.5,5]$. To decide how to set up Art's avenue of interconnect to run our IVP, we drew a Bush Schematic Diagram (see Figure 5.1) as mentioned in Chapter 2.

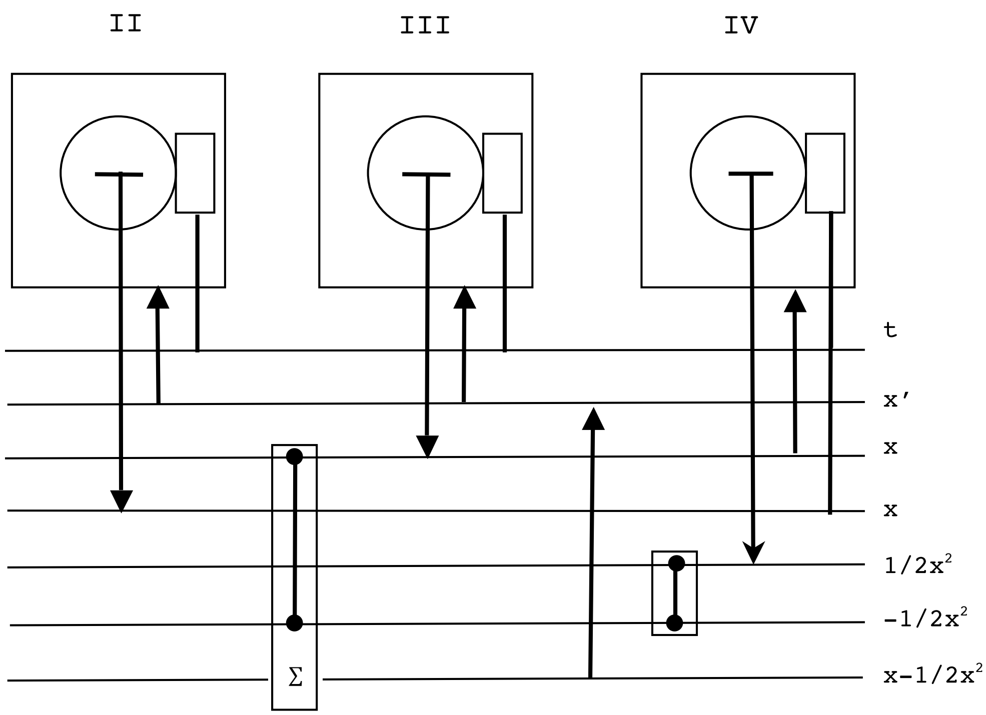

Figure 5.1: Bush Schematic used to set up $x' = \left(1 - \frac{1}{2}x\right)x$ on the differential analyzer.

The motion of x, created by integrating x', has to be sent to multiple locations; it has to turn the horizontal disk and move the carriage of Integrator IV. When we tried using one integrator to create the motion for both of these tasks, we were met with resistance as the machine groaned and rods bent. As a result, we set up two integrators (II and III) to perform the same task of integrating x'. This allowed us to send the output of Integrator II to turn the horizontal disk of Integrator IV and send the output of Integrator III to move the carriage of Integrator IV.

Art will be set up to run

$$x' = \left(1 - \frac{1}{2}x\right)x, \quad x(0) = \frac{1}{5}, \quad x'(0) = \frac{9}{50}.$$

We will use a process outlined in the next section to incorporate the circle minus operation into the mechanics. Notice that now we have two initial conditions $x(0)$ and $x'(0)$. For a first order dynamic equation, we only need one initial condition, perhaps $x(t_0)$. The DA requires $x(t_0)$ and $x'(t_0)$ for a first-order differential equation because we need to set the initial value for the output table as well as the initial value for x' on an integrator. We obtain $x'(0) = \frac{9}{50}$ by substituting $x(0) = \frac{1}{5}$ into our equation and evaluating $x'(0)$.

These initial conditions are entered using counters on Art. One unit in the domain of the solution of our DE is equivalent to a reading of 250 on a counter. We start with these initial conditions and obtain all other necessary values along the way. The two integrators that will be integrating x' will have initial values set at 45 (on the counters) while the integrator that will be integrating x will have its initial value set at 50 (on the counter).

Following the initial set up, we are ready to describe the process used to graphically solve our IVP on Art.

5.2 The Process

The steps we use to solve our particular logistic dynamic equation on various time scales incorporates the mechanics of Art and the theorry of Time Scale Calculus, particularly the definition of circle minus. We use the following steps to solve the IVP

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right)x, \quad x(0) = \frac{1}{5}, \quad x^{\triangle}(0) = \frac{9}{50}$$

over each closed interval of a time scale and jump the gap:

1. Run to the first gap ($t = t_1$) in the time scale $\mathbb{T} = [0, t_1] \cup [t_2, t_3]$ and stop the machine.

2. Record $x(t_1)$ and $x'(t_1^-)$ placement on the disk using the counters.

3. Use $x(t_1)$ and $x'(t_1^-)$ to calculate $x^{\triangle}(t_1)$ using

$$x^{\triangle}(t_1) = \frac{x'(t_1^-)}{1 + \frac{1}{2}\mu(t_1)x(t_1)}.$$

   Record this value.

4. Set the value of the x' integrators using the counters to $x^{\triangle}(t_1)$, calculated in Step 3.

5. Disconnect the lead screws to x' and lift the pen so we do not plot anything across the gap. Note that we will use the simple useful formula, Theorem 1(iv), to find $x(t_2)$:

$$x(t_2) = x^{\sigma}(t_1) = x(t_1) + \mu(t_1)x^{\triangle}(t_1).$$

6. Run to the end of the time gap ($t = t_2$) and stop the machine.

7. Record $x(t_2)$ position using the counter and reset the values of x and x' on the counters back to $x(t_1)$ and $x'(t_1^-)$, respectively.

8. Disconnect the output table (the pen is where we want it to be at the end of the gap, $x(t_2)$) and reconnect the lead screws to the x' integrators.

9. Run until the x counter reads the value of $x(t_2)$ and stop the machine. This gives us the $x^{\triangle}(t_2)$ value that we use at the beginning of the second interval.

10. Reconnect the output table, drop the pen, and run until the machine shuts off.

This process ensures that we have the correct values for $x(t_2)$ and $x^{\triangle}(t_2)$ on the counters and that the pen is located at the correct $x(t_2)$ position to begin graphing the solution on $[t_2, t_3]$.

5.3 Graphical Solutions on Four Time Scales

The four time scales and their respective graphs, after the gap, in Figure 5.2 are as follows:

$\mathbb{T}_1 = [0, 1] \cup [3, 5]$ where $[3, 5]$ is the domain for the graph that begins farthest right;

$\mathbb{T}_2 = [0, 1] \cup [2.5, 5]$ where $[2.5, 5]$ is the domain for the graph that begins second from the right;

$\mathbb{T}_3 = [0, 1] \cup [2, 5]$ where $[2, 5]$ is the graph that begins second from the left; and

$\mathbb{T}_4 = [0, 1] \cup [1.5, 5]$ where $[1.5, 5]$ is the graph that begins farthest left.

Note that all of the time scales contain the closed interval $[0, 1]$, so each of the solutions also contain the graph the appears before the gaps.

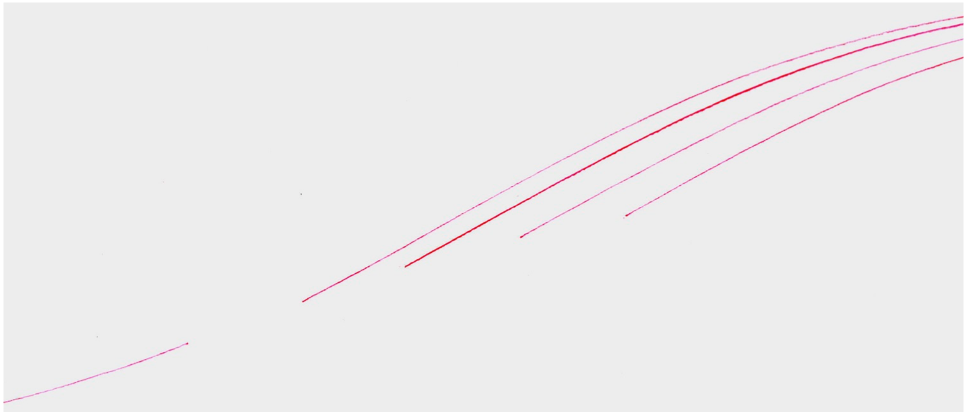

Figure 5.2: Plot of $x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x$ on four time scales. Illustration of the convergence of the graph as the gap closes.

For each run, we had to record the counters on steps 2, 3, and 7. Table 5.1 is a chart of the recordings for the four runs. Notice that the readings for $x(t_1)$ and $x'(t_1^-)$ are the same in all four runs. This is because each of our four time scales begins with the same closed interval $[0, 1]$. Also, notice the difference in the $x'(t_1^-)$ and $x^{\triangle}(t_1)$ values. For the graphs to converge as the gap closes, $x^{\triangle}(t_1)$ must converge to the value as $x'(t_1^-)$. So, not only do the plots represent convergence, the counters on Art do as well. Notice that as the gap shortens, the values of $x^{\triangle}(t_1)$ approach the value of $x'(t_1^-)$.

$\mu(t_1)$	$x(t_1)$	$x'(t_1^-)$	$x^{\triangle}(t_1)$	$x(t_2)$
2	122	93	62.5	257
1.5	122	93	68	233
1	122	93	75	200
0.5	122	93	83	167

Table 5.1: Readings from the counters on Art at different values in the time scale. Notice that as the gap, $\mu(t)$, decreases, the $x^{\triangle}(t_1)$ value is approaching the $x'(t_1^-)$ value and the $x(t_2)$ value is approaching the $x(t_1)$ value.

Therefore, the graphical solutions of $x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x$ with initial condition $x(0) = \frac{1}{5}$ show that as the gap closes for consecutive time scales, the graph of the solutions converge to the solution on the limiting interval case, when $\mathbb{T} = [0, t_2]$. This is the same result we saw with the analytical solutions. Art's ability to produce graphical solutions of our IVP on a number of time scales adds a visual aspect to our analytical solutions. Most importantly, we are able to see the difference in the values of a logistic dynamic equation's solution for a closed interval versus the value at a point after the gap. Thus, as the gap in the domain closes for the logistic dynamic equation

$$x^{\triangle} = \left(1 \ominus \frac{1}{2}x\right) x, \quad x(t_0) = x_0,$$

the solution converges to

$$x(t) = \frac{2x_0e^t}{e^{t_0}(2 - x_0) + x_0e^t},$$

which is the solution of this equation with $\mathbb{T} = \mathbb{R}$ on the limiting interval.

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APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

MARSHALL UNIVERSITY®
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Office of Research Integrity
Institutional Review Board

April 6, 2015

Alexandria Amorim
Mathematics
1 John Marshall Dr.
Huntington, WV 25755

Dear Ms. Amorim:

This letter is in response to the submitted thesis abstract entitled *"Solutions of a Logistic Equation on Varying Time Scales: Quantitative and Qualitative Analysis."* After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director

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Education

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Thesis Advisor: Dr. Bonita Lawrence

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Marshall University, Huntington, WV. May 2013

Major: Applied Mathematics

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Major: Math Education 5-Adult

Research

Graduate Research

Marshall University, 2013-2015. *Advisor Dr. Bonita Lawrence.*

Solutions of a Logistic Dynamic Equation on Various Time Scales.

Undergraduate Research

Marshall University, 2012. *Advisor Dr. Anna Mummert.*

Modeling Predator-Prey Population Cycles of the Canadian Lynx and Snowshoe Hare.

Undergraduate Research

Marshall University, 2011. *COMAP Mathematical Contest in Modeling*

Can you hear me now?

Teaching Experience

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Primary Instructor. Marshall University. Huntington, WV. 2013-2015.

Spring 2015: MTH 099 - Mathematics Skills II, 3 hrs

Fall 2014: MTH 099 - Mathematics Skills II, 3 hrs

Spring 2014: MTH 099 - Mathematics Skills II, 3 hrs

Fall 2013: MTH 122 - Plane Trigonometry, 3 hrs

Presentations

Co-presenter with Dr. Bonita Lawrence and Chad Lott.

Lindenwood University, St. Charles, MO. April 2015.

Demonstrated the use of the Differential Analyzer programmed to solve simple harmonic motion.

MAA Sectional Meeting Invited Co-presenter with Dr. Bonita Lawrence and Marshall University's DA Team.

Marshall University, Huntington, WV. March 2015.

Demonstrated the use of the Differential Analyzer programmed to a logistic dynamic equation.

Co-presenter with Dr. Bonita Lawrence and Chad Lott.

Indiana State University, Terre Haute, IN. November 2014.

Demonstrated the use of the Differential Analyzer programmed to solve simple harmonic motion.

Co-presenter with Dr. Bonita Lawrence and Molly Peterson.

Emmanuel College, Boston, MA. March 2014.

Demonstrated the use of the Differential Analyzer programmed to solve simple harmonic motion.

NIMBioS Undergraduate Research Conference Poster Presentation

University of Tennessee, Knoxville, TN. November 2012.

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Training

Critical Thinking Faculty Development Workshop (Marshall University) – Feb 2014

iPED Teaching Workshop (Marshall University) – Aug 2013

Graduate Coursework

Abstract Algebra II
Advanced Calculus I/II
Advanced Differential Equations
Coding Theory
Complex Variables I
Game Theory
Number Theory
Partial Differential Equations
Time Scale Calculus
Topology

Awards and Scholarships

Pi Mu Epsilon - Mathematics Honor Society 2013-2014

Phi Kappa Phi - Collegiate Honor Society 2013

Who's Who Among Students in American Universities and Colleges 2012

Promise Scholarship 2009-2013

Presidential Scholarship 2009-2013

Employment

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Marshall University, 2013–2015.

Borders, 2009–2011.

Last updated: May 1, 2015