2016

A Mechanical Investigation of Second Order Homogeneous Dynamic Equations on a Time Scale

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A MECHANICAL INVESTIGATION OF SECOND ORDER HOMOGENEOUS DYNAMIC EQUATIONS ON A TIME SCALE

A thesis submitted to
the Graduate College of
Marshall University
In partial fulfillment of
the requirements for the degree of
Master of Arts
in
Mathematics
by
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Marshall University
May 2016

APPROVAL OF THESIS/DISSERTATION

We, the faculty supervising the work of Jacob Emerson Fischer, affirm that the thesis, *A Mechanical Visualization of a Second Order Dynamic Equation on a Time Scale*, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the editorial standards of our discipline and the Graduate College of Marshall University. With our signatures, we approve the manuscript for publication.

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May 5, 2016

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ACKNOWLEDGEMENTS

I first off have to give a huge thank you to Dr. Bonita Lawrence. This last year has been beyond stressful for the both of us, for a variety of reasons, and I very much so appreciate the kick to make sure I got everything done, and not giving up until the final bell tolled.

Thank you to Chad Lott for helping me learn the ins and outs of the DA, and to Paige Yankey for helping get my run finished on the DA.

Finally, thank you to Dr. Clayton Brooks and Dr. Ralph Oberste-Vorth for volunteering their time to serve on my committee.

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ABSTRACT

This thesis covers the basic aspects of time scale calculus, a branch of mathematics combining the theories of differential equations and difference equations. Using the properties of time scale calculus we analyze a second order homogeneous dynamic equation with constant coefficients, in particular, $y^{\Delta\Delta} - \frac{1}{6}y^{\Delta} + \frac{1}{8}y = 0$. Following the analysis, this problem will be graphically evaluated using Marshall University's Differential Analyzer, affectionately named Art. A differential analyzer is a machine that mechanically integrates by way of related rates of rotating rods. The process for making the jump between intervals on a time scale will be discussed, and the behavior of the solution as the gaps decrease will be evaluated.

CHAPTER 1

Introduction

Time scales calculus is the marriage of discrete and continuous analysis that was formally conceived in a PhD dissertation by Stefan Hilger in 1988. This new branch of mathematics allowed for differential equations and difference equations to be defined simultaneously. For instance, if one wanted to work on a combination of intervals and discrete points, they would have to use differential equation theorems and definitions on the intervals, and difference equation theorems and definitions on the discrete points. Time scales calculus allows us to work on these combination of intervals and points with a newly defined dynamic equation. The cornerstone of time scales calculus is the forward jump operator which allows us to travel between consecutive points, regardless of whether these points are dense or there is a gap between them. Dr. Martin Bohner and Dr. Allan Peterson formalized the material presented in Hilger's dissertation, as well as contributions from a variety of sources, to author the book, *Dynamic Equations on Time Scales - An Introduction with Applications.* In the end, this book reads similar to an introductory analysis book, starting with defining the basic definitions and functions, and building all the way to differentiation and integration, then transitioning into a "differential equation-esque" text. These "differential equations" on a time scale, referred to as dynamic equations, prevents us from needing to prove both continuous and discrete results, and in the end give results for a domain called a time scale.

This thesis is the study of second order dynamic equations with constant coefficients. The interest revolves around the behavior of the solutions of dynamic equations of this sort on time scale composed of a sequence of two compact intervals. Primarily, we will look at the relationship of the behavior of solutions to these dynamic equations as the gap between the two compact intervals decreases.

CHAPTER 2

Fundamentals of Time Scales Calculus

2.1 Basic Terms

Time scales calculus is a branch of mathematics that gained notoriety through a PhD dissertation by Stefan Hilger in 1988. Time scales calculus fills the gap between differential equations and difference equations. Differential equations are evaluated over a compact set or interval whereas difference equations are evaluated on a discrete set. Prior to the formation of time scales calculus, if one wanted to evaluate a problem involving both dense and discrete sets, the problem would have to be evaluated in parts using both methods, differential and difference equations. The development of time scales allows for the problems to be solved in one step, and in some instances, avoids proving a result multiple times. At its core, time scales calculus operates on sets called *time scales*, denoted $\mathbb{T}$, which are arbitrary nonempty closed subsets of real numbers. Common sets such as $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$, as well as closed interval sets like $[0, 1]$ are time scales, while $\mathbb{Q}$ and $(0, 1)$ are not time scales. Once differentiation and integration are introduced, a second type of time scale must be defined. If a time scale $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^\kappa$ is defined as follows:

$$\mathbb{T}^\kappa = \mathbb{T} - \{m\}$$

If the time scale $\mathbb{T}$ does not have a left scattered maximum them we simply define $\mathbb{T}^\kappa \equiv \mathbb{T}$. The use for this subset of the time scale will become more evident once the derivative and integral have been defined on a time scale. In time scales calculus, we define a forward and a backward jump operator. These operators help define subsequent terms on a time scale, provided they exist.

Definition 1. *Forward Jump Operator [2]*
For every $t \in \mathbb{T}$ the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}.$$

Definition 2. *Backward Jump Operator [2]*

For every $t \in \mathbb{T}$ the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) := sup\{s \in \mathbb{T} : s < t\}.$$

Notice that if t is a maximum of $\mathbb{T}$, then we define $\sigma(t) = t$, and if t is a minimum of $\mathbb{T}$, then we defined $\rho(t) = t$.

Based on the forward and backward jump operators, elements within time scales can be classified in several ways. An element, t, is said to be *right-scattered* if $\sigma(t) > t$, while t is said to be *left-scattered* if $\rho(t) < t$. If an element is simultaneously left-scattered and right-scattered, the element is said to be *isolated*. An element, t, is said to be *right-dense* if $t < sup\ \mathbb{T}$ and $\sigma(t) = t$, while t is said to be *left-dense* if $t > inf\ \mathbb{T}$ and $\rho(t) = t$. If an element is simultaneously left-dense and right-dense, then the element is referred to as *dense*. Finally the graininess function is defined. The graininess of a time scale gives one the idea of how sparse or dense the elements of the time scale are relative to each other.

Definition 3. *Graininess Function [2]*

For every $t \in \mathbb{T}$ the graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ is defined as

$$\mu(t) = \sigma(t) - t$$

2.2 Differentiation

Similar to traditional calculus, differentiation is possible within a time scale. However, the definition must be altered so that it will fit within constraints of a time scale. In time scale calculus, the derivative is referred to as the *delta (or Hilger) derivative* and is defined as follows:

Definition 4. *[2] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t ($U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U. \tag{2.1}$$

Because of the way the delta derivative is defined, we can see the need for $\mathbb{T}^\kappa$. If the derivative was defined on just the regular time scale, $\mathbb{T}$, then if one tried to evaluate the derivative at a left scattered maximum point, then the derivative would essentially jump out of the time scale, so we must stop it one element short so that the derivative can be defined for all elements in the time scale.

Just as we can define what it means to take a derivative, we can also give a classification to functions which are able to be differentiated. This class of functions are said to be *pre-differentiable*.

Definition 5. *[2] A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided:*

1. $D \subset \mathbb{T}^\kappa$
2. $\mathbb{T} \setminus D$ *is countable*
3. $\mathbb{T} \setminus D$ *contains no right scattered elements of* $\mathbb{T}$.
4. f *is differentiable for every* $t \in D$.

Notice here that the region with which we can take a derivative is not necessarily the entirety of the time scale. Also it is important to be aware, and in most cases, it is easier to say what is not in the region of differentiation, D, than what the region will contain, and that is because $\mathbb{T} \setminus D$ must be countable.

Now we must introduce the following theorem to establish some important properties of the delta derivative.

Theorem 1. *[2] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

1. *If f is differentiable at t, then f is continuous at t.*
2. *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{2.2}$$

3. If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.3}$$

exists as a finite number. In this case

$$f^{\Delta} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.4}$$

4. If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.5}$$

Part 4 of Theorem 1 is referred to as the *Simple Useful Formula*, and is used to find appropriate values of a function on a time scale while jumping a gap, among several other uses. Using the parts in this theorem as the basic properties of the delta derivative, we can begin to prove properties about the delta derivative by highlighting the differences between the derivative on a time scale and the derivative not on a time scale.

Just like a derivative on a compact interval, the delta derivative on a time scale has properties concerning the derivative of sums and products of functions. Thus we have the following theorem detailing those derivatives:

Theorem 2. *[2] Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}$. Then:*

1. The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

4. If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

5. If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) + f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Finally, we need to define what it means to take a second derivative on a time scale. Suppose we wish to evaluate the derivative of $f^{\Delta}(t)$ on $\mathbb{T}^{\kappa}$ which happens to have a left-scattered maximum. We know we cannot evaluate a derivate at a left-scattered maximum. Therefore we need to redefine $\mathbb{T}^{\kappa}$ for when we take higher ordered derivatives.

Definition 6. *[2] For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^{Δ} is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{\Delta\Delta} = \left(f^{\Delta}\right)^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$ and $\rho^2(t) = \rho(\rho(t))$.*

2.3 Integration

Integration is also possible on a time scale, however, similar to the delta derivative there are some adjustments that must be made to integration on the real line to make it well defined on a time scale. So, before we can define what it means to integrate on a time scale, we must first define two function characteristics.

Definition 7. *[2] A function $f : \mathbb{T} \to \mathbb{R}$ is called* ***regulated*** *provided its right-sided limits exisits (they are finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 8. *[2] A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exists (they are finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted as*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$$

We also establish some properties about regulated and rd-continuous functions in an attempt to better understand them. This theorem will take properties of functions that we are familiar with, and expand on their meaning in terms of regulated and rd-continuous.

Theorem 3. *[2] Assume $f : \mathbb{T} \to \mathbb{R}$*

1. *If f is continuous, then f is rd-continuous.*
2. *If f is rd-continuous, then f is regulated.*
3. *The forward jump operator σ is rd-continuous.*
4. *If f is regulated or rd-continuous, then so is $f(\sigma(t))$.*
5. *Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property as well.*

So now that we have established the properties of regulated and rd-continuous functions, we can begin discussing what it means to be able to integrate a function on a time scale. Traditionally, when thinking about integration, we know that if we integrate some function $f(t)$ we obtain some function $F(t)$, and at the same time, we can take the derivative of our new function, $F(t)$, and we will arrive back to our original function, $f(t)$. Because of this relationship, integration is sometimes referred to as *anti-differentiation.*

Theorem 4. *Existence of Pre-Antiderivative [2]*

Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in D \tag{2.6}$$

Definition 9. *[2] Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function as defined in Theorem 4 is called a* ***pre-antiderivative*** *of f. We define the indefinite integral of a regulated function f by*

$$\int f(t)\Delta t = F(t) + C, \tag{2.7}$$

where C is an arbitrary constant and F is the pre-antiderivative of f. We define the Cauchy Integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

In traditional calculus, we are able to integrate a function, differentiate the result, and return to our original function. However in a time scale, we are only able to differentiate over the function's region of differentiation, a subset of the time scale. What this means is that if we take a regulated function and differentiate it, our region of differentiation may not be the entire time scale. So if we integrate the resulting function, in an attempt to return to the original function, the time scale on which it is defined will not be the same. So with a regulated function, we are not able to freely differentiate and integrate back and forth, as we might lose elements within our time scale. However, with an added constraint, we are able to achieve this desired result.

Theorem 5. *Existence of Antiderivatives [2]*

Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau$$

for all $t \in \mathbb{T}$ is an antiderivative of f.

So if we start with a function that is rd-continuous, we are able to freely integrate and differentiate, and the timescale will not change (i.e. the region of differentiation will be the same as the time scale).

2.4 Hilger's Complex Plane

As we progress towards constructing a differential equation on a time scale, we must continue to fill in the gaps between time scale calculus and traditional calculus. One of the differences between the two branches is the need for the construction of subsets of the complex plane. In time scale calculus, we call this Hilger Complex Plane.

Definition 10. *[2] For $h > 0$ we define the following terms:*
The Hilger Complex Numbers

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}$$

The Hilger Real Axis

$$\mathbb{R}_h := \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\},$$

The Hilger Alternating Axis

$$\mathbb{A}_h := \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\},$$

and The Hilger Imaginary Circle

$$\mathbb{I}_h := \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},$$

Note that when $h = 0$, $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 := i\mathbb{R}$, and $\mathbb{A}_0 := \emptyset$.

As in the traditional complex plane, numbers in the Hilger Complex Plane also have a real and an imaginary part.

Definition 11. *[2] Let $h > 0$ and $z \in \mathbb{C}_h$. We define the Hilger real part of z by*

$$Re_h(z) := \frac{|zh + 1| - 1}{h}$$

and the Hilger imaginary part of z by

$$Im_h(z) := \frac{Arg(zh + 1)}{h},$$

where Arg(z) denotes the principal argument of z (i.e., $-\pi < Arg z < \pi$)

As we continue to define components within the Hilger Complex Plane, a goal is to define a general form of the exponential function. To do this, we must also define the cylinder transformation, and what it means for a function to be regressive.

Definition 12. *[2] For $h > 0$, we define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by*

$$\xi_h(z) = \frac{1}{h} Log(1 + zh), \tag{2.8}$$

where Log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Also note that $\mathbb{Z}_h$ is a strip of the complex plane. The strip is defined as follows:

Definition 13. *[2] For $h > 0$, let $\mathbb{Z}_h$ be the strip*

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\},$$

note that when $h = 0$, we let $\mathbb{Z}_0 := \mathbb{C}$

We must now define the regressive characteristic of a function.

Definition 14. *[2] A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^{\kappa} \tag{2.9}$$

The set of all regressive and rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted as

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

This condition of regressivity stems from our definition of the cylinder transformation, and the use of that transformation to define the exponential function.

Now that we have established all that we need within and about the Hilger Complex Plane, we define the exponential function on a time scale.

Definition 15. *[2] If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(t)}(p(t)) \Delta\tau \right) \text{ for all } s, t \in \mathbb{T} \tag{2.10}$$

Looking at this definition, we can see the rationale for the definition of a regressive function. If $1 + \mu(t)p(t) = 0$ then the Log function will not be defined at that point.

It is valuable to discuss some of the properties of the exponential function on a time scale to gain additional insight on its behavior. However before we do so, we will define the operators circle plus, $\oplus$, and circle minus, $\ominus$.

Definition 16. *[2] Suppose that $p, q \in \mathcal{R}$, then circle plus $\oplus$ is defined as*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$$

for all $t \in \mathbb{T}^\kappa$.

Definition 17. *[2] Suppose that $p, q \in \mathcal{R}$, then circle minus $\ominus$ is defined as*

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t)$$

for all $t \in \mathbb{T}^\kappa$ where $\ominus q$ is defined as

$$(\ominus q)(t) := -\frac{q(t)}{1 + \mu(t)q(t)}$$

for all $t \in \mathbb{T}^\kappa$.

Theorem 6. *[2] If $p, q \in \mathcal{R}$, then*

1. $e_0(t, s) \equiv 1$

2. $e_p(t, t) \equiv 1$

3. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$

4. $e_p(t, s)e_p(s, r) = e_p(t, r)$

5. $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$

These properties are important conceptually, allowing us to study the generalized exponential function and see how it compares to the properties of the exponential function on the real line, something with which we are very familiar. Also, many of the properties are used in the derivation of solutions of dynamic equations.

2.5 Dynamic Equations on a Time Scale

Differential equations on a time scale are regularly referred to as dynamic equations. Dynamic equations share the properties of both a differential equation, equations whose domain is some subset of the real line, and a difference equation, equations whose domain is discrete points rather than intervals. Dynamic equations utilize the delta derivative which allows for the combination of the two theories. A dynamic equation, in its simplest form as defined as follows.

Definition 18. *[2] Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation*

$$y^{\Delta} = f(t, y, y^{\sigma}) \tag{2.11}$$

is called a first order dynamic equation. A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of the dynamic equation if

$$y^{\Delta}(t) = f(t, y(t), y(\sigma(t))) \tag{2.12}$$

is satisfied for all $t \in \mathbb{T}^{\kappa}$.

The following theorem gives the solution to a first order dynamic initial value problem. This result is important to note as it gives an additional property of the exponential function on a time scale.

Theorem 7. *[2] Suppose $y^{\Delta} = p(t)y$ is regressive and $t_0 \in \mathbb{T}$. Then $y = e_p(\cdot, t_0)$ is a solution (on $\mathbb{T}$) of the initial value problem*

$$y^{\Delta} = p(t)y, \;\; y(t_0) = 1$$

While this theorem is important because it gives us a basic result for a general dynamic equation, what it does, more importantly, is offer us the derivative of the exponential function on a time scale. As shown, the derivative of the exponential function on a time scale is quite similar to the derivative of an exponential function on the real line. That is, the derivative is equal to the exponential function multiplied by the derivative of its exponent.

CHAPTER 3

Second Order Dynamic Equations on a Time Scale

Second order dynamic equations are dynamic equations that contain, at least a second degree delta derivative. In this investigation we will look at homogeneous equations of the form:

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = 0 \tag{3.1}$$

where we assume that $p, q \in C_{rd}$.

Definition 19. *[2] The operator $L_2 : C_{rd}^2 \to C_{rd}$ defined as*

$$L_2y(t) = y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y, \tag{3.2}$$

for all $t \in \mathbb{T}^{\kappa}$ is a linear operator, meaning

$$L_2(\alpha y_1 + \beta y_2) = \alpha L_2(y_1) + \beta L_2(y_2), \tag{3.3}$$

for all $\alpha, \beta \in \mathbb{R}$ and $y_1, y_2 \in C_{rd}^2$. If y_1 and y_2 solve the homogeneous equation $L_2y = 0$ then so does $y = \alpha y_1 + \beta y_2$, where α and β are any real constants.

This is an important consequence when deriving the solutions to the dynamic equations. Many of the methods will result in two linearly independent solutions, and this result allows us to make a linear combination of those solutions to arrive at the general solution.

These dynamic equations can be solved using a variety of methods. However this study only focuses on the method of solution through a characteristic equation. Before discussing the derivation of solution using this method, we first will establish some properties of second order dynamic equations.

The following theorem establishes that for any given dynamic equation, the solution that is derived is unique. This allows us to say that there is only one solution for a given dynamic equation and a certain set of initial conditions.

First, however, let's first define what it means for a second order dynamic equation to be regressive.

Definition 20. *[2] Equation (3.1) is called regressive provided $p, q, f \in C_{rd}$ such that the regressivity condition*

$$1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0$$

for all $t \in \mathbb{T}^\kappa$.

Theorem 8. *[2] Assume that the dynamic equation $y^{\Delta\Delta} + p(t)y^\Delta + q(t)y = 0$ is regressive. If $t_0 \in \mathbb{T}^\kappa$ then the initial value problem*

$$L_2 y = f(t), y(t_0) = y_0, y^\Delta(t_0) = y_0^\Delta \tag{3.4}$$

where y_0 and y_0^Δ are given constants, has a unique solution, and the solution is defined for all $t \in \mathbb{T}$.

The crux of this proof states that the solution, $y(t)$, can be written as a linear combination as stated in Definition 17. The issue is determining the values for α and β once the linearly independent solutions, y_1 and y_2, have been found. Once these constants have been found, they should satisfy the following matrix equation:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1^\Delta(t_0) & y_2^\Delta(t_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0^\Delta \end{pmatrix}$$

Due to this fact we can state the following definition concerning the 2×2 matrix.

Definition 21. *[2] For two differentiable functions y_1 and y_2 we define the Wronskian $W = W(y_1, y_2)$ by*

$$W(t) = det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1^\Delta(t) & y_2^\Delta \end{pmatrix} \tag{3.5}$$

We say that two solutions y_1 and y_2 of $L_2 y = 0$ form a ***fundamental set of solutions*** *for $L_2 y = 0$ provided $W(y_1, y_2)(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$.*

So we can solve the matrix equation to find the values for α and β as long as the Wronskian of our two solutions, y_1 and y_2, does not equal zero. It is evident that $W(t_0) = W(y_1(t_0), y_2(t_0))$ may not equal zero for the matrix equation as it would produce a trivial solution. The following theorem gives the general solution to an initial value dynamic equation given assuming that a fundamental systems of solutions has been found.

Theorem 9. *[2] If the pair of functions y_1, y_2 forms a fundamental system of solutions for $L_2y = 0$, then*

$$y(t) = \alpha y_1(t) + \beta y_2(t), \text{ where } \alpha, \beta \in \mathbb{R} \tag{3.6}$$

is a general solution of $L_2y = 0$. By general solution we mean every function of this form is a solution and every solution is in this form. In particular the solution of the initial value problem

$$L_2y = 0, \ y(t_0) = y_0, \ y^{\Delta}(t_0) = y_0^{\Delta} \tag{3.7}$$

is given by

$$y(t) = \frac{y_2^{\Delta}(t_0)y_0 - y_2(t_0)y_0^{\Delta}}{W(y_1, y_2)(t_0)} y_1(t) + \frac{y_1(t_0)y_0^{\Delta} - y_1^{\Delta}(t_0)y_0}{W(y_1, y_2)(t_0)} y_2(t) \tag{3.8}$$

Theorem 9 gives a method for finding the constants α and β from (3.6). We can see from the result that $\alpha = \frac{y_2^{\Delta}(t_0)y_0 - y_2(t_0)y_0^{\Delta}}{W(y_1, y_2)(t_0)}$ and $\beta = \frac{y_1(t_0)y_0^{\Delta} - y_1^{\Delta}(t_0)y_0}{W(y_1, y_2)(t_0)}$. This shell of a general solution will be utilized once we start looking at a particular type of dynamic equation, specifically those that have constant coefficients. Notice that this theorem gives a second reason, beyond producing a trivial solution, as to why (2.5) must be non-zero. If the Wronskian was equal to zero, and those solutions were attempted to be used in this theorem, the constants would be undefined. Thus, trivial solution aside, the Wronskian must never equal zero.

3.1 General Solution to a Second Order Homogeneous Dynamic Equation with Arbitrary Coefficients

For this study, the solution to a second order dynamic equation will be derived through the method of characteristic equations. This method is similar to its counterpart on the real line, with some adjustments made to the final solution to fulfill the requirements of a time scale. Let's first consider a general second order homogeneous dynamic equation with constant coefficients,

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0 \text{ with } \alpha, \beta \in \mathbb{R}. \tag{3.9}$$

We will assume that $y(t) = e_\lambda(t, t_0)$ is a solution to the dynamic equation for some λ. When this solution is substituted into the equation, the result is:

$$(\lambda^2 + \alpha\lambda + \beta)e_\lambda(t, t_0) = 0.$$

Because we know that an exponential function can never equal zero, we must find the roots of the quadratic. These roots, denoted λ_1 and λ_2, equal:

$$\lambda_1 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \text{ and } \lambda_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \tag{3.10}$$

Now consider the following theorem for the solution to the general second order homogeneous dynamic equation.

Theorem 10. *[2] Suppose $\alpha^2 - 4\beta \neq 0$. If $\mu\beta - \alpha \in \mathcal{R}$, then a fundamental system of the dynamic equation $y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0$ is given by*

$$y_1 = e_{\lambda_1}(\cdot, t_0) \textit{ and } y_2 = e_{\lambda_2}(\cdot, t_0)$$

where $t_0 \in \mathbb{T}^\kappa$ and λ_1, λ_2 are given as above. The solution of the initial value problem (3.7) *is given by*

$$y(\cdot, t_0) = y_0 \frac{e_{\lambda_1}(\cdot, t_0) + e_{\lambda_2}(\cdot, t_0)}{2} + \frac{\alpha y_0 + 2y_0^{\Delta}}{\sqrt{\alpha^2 - 4\beta}} \frac{e_{\lambda_2}(\cdot, t_0) - e_{\lambda_1}(\cdot, t_0)}{2} \tag{3.11}$$

Because this theorem is so instrumental in the overall study, the proof will be given in detail.

Proof. Let us first remind ourselves of the values of $\lambda_1(t, t_0)$ and $\lambda_2(t, t_0)$ which are given in (2.10). Since they are both solutions to the characteristic equation, then we know that $e_{\lambda_1}(t, t_0)$ and $e_{\lambda_2}(t, t_0)$ are solutions to the dynamic equation in question. Let us first confirm that the Wronskian of these two solutions is in fact not equal to zero. Using the definition of $W(y_1, y_2)(t)$, we obtain the following:

$$
\begin{aligned}
W(y_1,y_2)(t) &= det\begin{pmatrix} e_{\lambda_1}(t,t_0) & e_{\lambda_2}(t,t_0) \\ \lambda_1 e_{\lambda_1}(t,t_0) & \lambda_2 e_{\lambda_2}(t,t_0)\end{pmatrix} \\
&= \lambda_2 e_{\lambda_1}(t,t_0)e_{\lambda_2}(t,t_0) - \lambda_1 e_{\lambda_1}(t,t_0)e_{\lambda_2}(t,t_0) \\
&= (\lambda_2 - \lambda_1)e_{\lambda_1}(t,t_0)e_{\lambda_2}(t,t_0) \\
&= \sqrt{\alpha^2 - 4\beta}\; e_{\lambda_1 \oplus \lambda_2}(t,t_0)
\end{aligned}
$$

So notice that $e_{\lambda_1 \oplus \lambda_2}(t,t_0)$ can never be zero, and stated in the theorem, $\alpha^2 - 4\beta \neq 0$. Thus the Wronskian is not equal to zero, and the two solutions must form a fundamental set of solutions, $y_1 = e_{\lambda_1}(t,t_0)$ and $y_2 = e_{\lambda_2}(t,t_0)$, for the dynamic equation in question. This means that the general solution of the initial value problem is $y(t) = c_1 e_{\lambda_1}(t,t_0) + c_2 e_{\lambda_2}(t,t_0)$, and by Theorem 9, we know that the solution must be in the following form.

$$
y(t) = \frac{y_2^\Delta(t_0)y_0 - y_2(t_0)y_0^\Delta}{W(y_1,y_2)(t_0)} y_1(t) + \frac{y_1(t_0)y_0^\Delta - y_1^\Delta(t_0)y_0}{W(y_1,y_2)(t_0)} y_2(t)
$$

This means that we can find c_1 and c_2 for this problem as follows:

$$
\begin{aligned}
c_1 &= \frac{y_2^\Delta(t_0)y_0 - y_2(t_0)y_0^\Delta}{W(y_1,y_2)(t_0)} \\
&= \frac{\lambda_2 e_{\lambda_2}(t_0,t_0)y_0 - e_{\lambda_2}(t_0,t_0)y_0^\Delta}{\lambda_1 - \lambda_2} \\
&= \frac{\lambda_2 y_0 - y_0^\Delta}{\sqrt{\alpha^2 - 4\beta}} \\
&= \frac{\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right) y_0 - 2y_0^\Delta}{2\sqrt{\alpha^2 - 4\beta}} \\
&= \frac{y_0}{2} - \frac{\alpha y_0 - 2y_0^\Delta}{2\sqrt{\alpha^2 - 4\beta}}
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= \frac{y_1(t_0)y_0^\Delta - y_1^\Delta(t_0)y_0}{W(y_1,y_2)(t_0)} \\
&= \frac{e_{\lambda_1}(t_0,t_0)y_0^\Delta - \lambda_1 e_{\lambda_1}(t_0,t_0)y_0}{\lambda_1 - \lambda_2} \\
&= \frac{y_0^\Delta - \lambda_1 y_0}{\sqrt{\alpha^2 - 4\beta}} \\
&= \frac{2y_0^\Delta - \left(-\alpha - \sqrt{\alpha^2 - 4\beta}\right) y_0}{2\sqrt{\alpha^2 - 4\beta}} \\
&= \frac{y_0}{2} + \frac{\alpha y_0 + 2y_0^\Delta}{2\sqrt{\alpha^2 - 4\beta}}
\end{aligned}
$$

Therefore we can write the solution as

$$
\begin{aligned}
y(t) &= \left(\frac{y_0}{2} - \frac{\alpha y_0 - 2y_0^\Delta}{2\sqrt{\alpha^2 - 4\beta}}\right) e_{\lambda_1}(\cdot,t_0) + \left(\frac{y_0}{2} + \frac{\alpha y_0 + 2y_0^\Delta}{2\sqrt{\alpha^2 - 4\beta}}\right) e_{\lambda_2}(\cdot,t_0) \\
&= \frac{y_0}{2}\left(e_{\lambda_1}(\cdot,t_0) + e_{\lambda_2}(\cdot,t_0)\right) + \frac{\alpha y_0 + 2y_0^\Delta}{2\sqrt{\alpha^2 - 4\beta}}\left(e_{\lambda_2}(\cdot,t_0) - e_{\lambda_1}(\cdot,t_0)\right) \\
&= y_0 \frac{e_{\lambda_1}(\cdot,t_0) + e_{\lambda_2}(\cdot,t_0)}{2} + \frac{\alpha y_0 + 2y_0^\Delta}{\sqrt{\alpha^2 - 4\beta}} \frac{e_{\lambda_2}(\cdot,t_0) - e_{\lambda_1}(\cdot,t_0)}{2}
\end{aligned}
$$

So now we have confirmed the form of the solution to a particular class of second order homogeneous dynamic equations with constant coefficients. Before going on to discuss a specific dynamic equation, let's first define a few trigonometric functions on a time scale. Trigonometric functions arise very frequently in solutions of differential equations, and the case is no different when working on a time scale. In particular, cosine and sine will be define on a time scale.

Definition 22. *[2]Trigonometric Functions*
If $p \in C_{rd}$ and $\mu p^2 \in \mathcal{R}$, then we define the trigonometric functions cos_p and sin_p by

$$
cos_p = \frac{e_{ip} + e_{-ip}}{2} \text{ and } sin_p = \frac{e_{ip} - e_{-ip}}{2i} \tag{3.12}
$$

Note that μp^2 is regressive if and only if both ip and $-ip$ are regressive, so cos_p and sin_p are well defined.

Now let's look at a specific case of this problem where, $\alpha^2 - 4\beta < 0$.

Theorem 11. *[2] Suppose $\alpha^2 - 4\beta < 0$. Then define*

$$p = -\frac{\alpha}{2} \text{ and } q = \frac{\sqrt{4\beta - \alpha^2}}{2}$$

If p and $\mu\beta - \alpha$ are regressive, then a fundamental system of (3.7) is given by

$$cos_{\frac{q}{1+\mu p}}(\cdot, t_0)e_p(\cdot, t_0) \text{ and } sin_{\frac{q}{1+\mu p}}(\cdot, t_0)e_p(\cdot, t_0)$$

where $t_0 \in \mathbb{T}$, and the Wronskian of these two solutions is

$$q \cdot e_{\mu\beta-\alpha}(\cdot, t_0).$$

The solution of the initial value problem (3.7) is given by

$$\left[y_0 cos_{\frac{q}{1+\mu p}}(\cdot, t_0) + \frac{y_0^\Delta - py_0}{q} sin_{\frac{q}{1+\mu p}}(\cdot, t_0)\right] e_p(\cdot, t_0) \tag{3.13}$$

3.2 Deriving the Solution to a Second Order Homogeneous Dynamic Equation with Constant Coefficients

First consider the initial value dynamic equation.

$$y^{\Delta\Delta} - \frac{1}{6}y^\Delta + \frac{1}{8}y = 0, \quad y(0) = 1,\ y^\Delta(0) = 0 \tag{3.14}$$

This system can be solved through the methods defined in Section 3.1. Notice that in this example

$$\alpha = -\frac{1}{6},\ \beta = \frac{1}{8},\ y_0 = 1,\ y_0^\Delta = 0.$$

First let's evaluate $\alpha^2 - 4\beta$

$$\begin{aligned}\alpha^2 - 4\beta &= \left(-\frac{1}{6}\right)^2 - 4\left(\frac{1}{8}\right) \\ &= \frac{1}{36} - \frac{1}{2} \\ &= \frac{-17}{36} \\ &< 0\end{aligned}$$

Because $\alpha^2 - 4\beta < 0$ then we can use the general solution given in equation 3.12, where $p = -\frac{-\frac{1}{6}}{2} = \frac{1}{12}$ and $q = \frac{\sqrt{4(\frac{1}{8}) - (\frac{1}{6})^2}}{2} = \frac{\sqrt{17}}{12}$

$$\begin{aligned}y(t) &= \left[y_0 cos_{\frac{q}{1+\mu p}}(\cdot, t_0) + \frac{y_0^\Delta - py_0}{q} sin_{\frac{q}{1+\mu p}}(\cdot, t_0)\right] e_p(\cdot, t_0) \\ &= \left[(1) cos_{\frac{\sqrt{17}}{12+\mu}}(\cdot, t_0) + \frac{0 - \frac{1}{12}}{\frac{\sqrt{17}}{12}} sin_{\frac{\sqrt{17}}{12+\mu}}(\cdot, t_0)\right] e_{\frac{1}{12}}(\cdot, t_0) \\ &= \left[cos_{\frac{\sqrt{17}}{12+\mu}}(\cdot, t_0) + \frac{1}{\sqrt{17}} sin_{\frac{\sqrt{17}}{12+\mu}}(\cdot, t_0)\right] e_{\frac{1}{12}}(\cdot, t_0)\end{aligned}$$

Now before proceeding, let's compare the solution of the dynamic equation on a general time scale to the solution of the differential equation on the real line. Consider the following initial value differential equation:

$$y''(t) - \frac{1}{6}y'(t) + \frac{1}{8} = 0, \quad y(0) = 1,\ y'(0) = 0 \tag{3.15}$$

The solution for (2.14) is

$$y(t) = e^{\frac{1}{12}t}\left[cos\left(\frac{\sqrt{17}}{12}t\right) + \frac{1}{\sqrt{17}} sin\left(\frac{\sqrt{17}}{12}t\right)\right] \tag{3.16}$$

Comparing this solution to the solution for a general time scale, we can see they differ only by the definition of the exponential and trigonometric functions on the real line versus a general time scale. Notice that on the real line, $\mu(t) = 0$.

3.3 Deriving the Solution to a Second Order Homogeneous Dynamic Equation on a Time Scale

Using the general solution (3.12), we have presented the solution when $\mathbb{T} = \mathbb{R}$. In this section we will see what the solution looks like when our time scale is composed of two compact intervals.

First let's suppose that $\mathbb{T} = [0, \frac{12}{\sqrt{17}}\pi] \cup [\frac{24}{\sqrt{17}}\pi, 6\pi]$. The first interval will have the same result as the solution when $\mathbb{T} = \mathbb{R}$, because it is simply a portion of the solution on the real line.

So let $t \in [0, \frac{12}{\sqrt{17}}\pi]$, then

$$y(t) = e^{\frac{1}{12}t}\left[cos\left(\frac{\sqrt{17}}{12}t\right) + \frac{1}{\sqrt{17}}sin\left(\frac{\sqrt{17}}{12}t\right)\right]$$

Now we must make the jump from $\frac{12}{\sqrt{17}}\pi$ to $\frac{24}{\sqrt{17}}\pi$. To do this, the simple useful formula (1.5) will be utilized. In this case $\sigma(\frac{12}{\sqrt{17}}\pi) = \frac{24}{\sqrt{17}}\pi$.

Let $t = \frac{24}{\sqrt{17}}\pi$. Then the following results

$$y(t) = e_{\frac{1}{12}}(t, t_0)\left(cos_{\frac{\sqrt{17}}{12+\mu}}(t, t_0) + \frac{1}{\sqrt{17}}sin_{\frac{\sqrt{17}}{12+\mu}}(t, t_0)\right)$$

Due to the complexity of this solution, we will evaluate each part of the solution individually, then piece them back together once finished.

$$\begin{aligned}
e_{\frac{1}{12}}\left(\frac{24}{\sqrt{17}}\pi, 0\right) &= e_{\frac{1}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right)e_{\frac{1}{12}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right) \\
&= exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}}\xi_0\left(\frac{1}{12}\right)\Delta\tau\right)exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}}\xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{1}{12}\right)\Delta\tau\right) \\
&= exp\left(\frac{\pi}{\sqrt{17}}\right)exp\left(Log\left(1 + \frac{\pi}{\sqrt{17}}\right)\right) \\
&= e^{\frac{\pi}{\sqrt{17}}}\left(1 + \frac{\pi}{\sqrt{17}}\right)
\end{aligned}$$

$$
\begin{aligned}
cos_{\frac{\sqrt{17}}{12+\mu}}\left(\frac{24}{\sqrt{17}}\pi, 0\right) &= \frac{e_{\frac{i\sqrt{17}}{12+\mu}}(t,0) + e_{\frac{-i\sqrt{17}}{12+\mu}}(t,0)}{2} \\
&= \frac{e_{\frac{i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right)}{2} \\
&\quad + \frac{e_{\frac{-i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{-i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right)}{2} \\
&= \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{i\sqrt{17}}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right)}{2} \\
&\quad + \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{-i\sqrt{17}}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{-i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right)}{2} \\
&= \frac{e^{i\pi}\left(1 + \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) + e^{-i\pi}\left(1 - \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)}{2} \\
&= \frac{\left(-1 - \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) + \left(-1 + \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)}{2} \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
sin_{\frac{\sqrt{17}}{12+\mu}}\left(\frac{24}{\sqrt{17}}\pi, 0\right) &= \frac{e_{\frac{i\sqrt{17}}{12+\mu}}(t,0) - e_{\frac{-i\sqrt{17}}{12+\mu}}(t,0)}{2i} \\
&= \frac{e_{\frac{i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right)}{2i} \\
&\quad - \frac{e_{\frac{-i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{-i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right)}{2i} \\
&= \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{i\sqrt{17}}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right)}{2i} \\
&\quad - \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{-i\sqrt{17}}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{-i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right)}{2i} \\
&= \frac{e^{i\pi}\left(1 + \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) - e^{-i\pi}\left(1 - \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)}{2i} \\
&= \frac{\left(-1 - \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) - \left(-1 + \frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)}{2i} \\
&= \frac{\frac{-2i\pi}{1+\frac{\pi}{\sqrt{17}}}}{2i} \\
&= \frac{-\pi\sqrt{17}}{\sqrt{17}+\pi}
\end{aligned}
$$

Therefore the solution when $t = \dfrac{24}{\sqrt{17}}\pi$ is

$$
\begin{aligned}
y(t) &= e^{\frac{\pi}{\sqrt{17}}}\left(1 + \frac{\pi}{\sqrt{17}}\right)\left[-1 + \frac{1}{\sqrt{17}}\left(\frac{-\pi\sqrt{17}}{\sqrt{17}+\pi}\right)\right] \\
&= e^{\frac{\pi}{\sqrt{17}}}\left(1 + \frac{\pi}{\sqrt{17}}\right)\left(-1 + \frac{-\pi}{\sqrt{17}+\pi}\right) \\
&= -e^{\frac{\pi}{\sqrt{17}}}\left(1 + \frac{2\pi}{\sqrt{17}}\right) \qquad (3.17)
\end{aligned}
$$

Additionally, we can see that using the Simple Useful Formula (1.5) produces the same result.

$$\begin{aligned} y(\frac{24}{\sqrt{17}}\pi) &= y\left(\sigma(\frac{12}{\sqrt{17}}\pi)\right) \\ &= y\left(\frac{12}{\sqrt{17}}\pi\right) + \mu\left(\frac{12}{\sqrt{17}}\pi\right) y^{\Delta}\left(\frac{12}{\sqrt{17}}\pi\right) \\ &= \left(-e^{\frac{\pi}{\sqrt{17}}}\right) + \left(\frac{12\pi}{\sqrt{17}}\right)\left(-\frac{1}{6}e^{\frac{\pi}{\sqrt{17}}}\right) \\ &= -e^{\frac{\pi}{\sqrt{17}}}\left(1+\frac{2\pi}{\sqrt{17}}\right) \end{aligned} \tag{3.18}$$

Now let $t \in \left(\frac{24}{\sqrt{17}}\pi, 6\pi\right]$

$$y(t) = e_{\frac{1}{12}}(t,t_0)\left(cos_{\frac{\sqrt{17}}{12+\mu}}(t,t_0) + \frac{1}{\sqrt{17}}sin_{\frac{\sqrt{17}}{12+\mu}}(t,t_0)\right)$$

Again due to the complexity of the solution, we will derive each of the parts individually, then put them all together in the end.

$$\begin{aligned} e_{\frac{1}{12}}(t,0) &= e_{\frac{1}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{1}{12}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right) e_{\frac{1}{12}}\left(t, \frac{24}{\sqrt{17}}\pi\right) \\ &= exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{1}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{1}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{24\pi}{\sqrt{17}}}^{t} \xi_0\left(\frac{1}{12}\right)\Delta\tau\right) \\ &= exp\left(\frac{\pi}{\sqrt{17}}\right) exp\left(Log\left(1+\frac{\pi}{\sqrt{17}}\right)\right) exp\left(\left(\frac{t}{12}-\frac{2\pi}{\sqrt{17}}\right)\right) \\ &= \left(1+\frac{\pi}{\sqrt{17}}\right) exp\left(\frac{t}{12}-\frac{\pi}{\sqrt{17}}\right) \end{aligned}$$

$$
\begin{aligned}
cos_{\frac{\sqrt{17}}{12+\mu}}(t,0) &= \frac{e_{\frac{i\sqrt{17}}{12+\mu}}(t,0)+e_{\frac{-i\sqrt{17}}{12+\mu}}(t,0)}{2}\\
&= \frac{e_{\frac{i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi,0\right)e_{\frac{i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi,\frac{12}{\sqrt{17}}\pi\right)e_{\frac{i\sqrt{17}}{12}}\left(t,\frac{24}{\sqrt{17}}\pi\right)}{2}\\
&+\frac{e_{\frac{-i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi,0\right)e_{\frac{-i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi,\frac{12}{\sqrt{17}}\pi\right)e_{\frac{-i\sqrt{17}}{12}}\left(t,\frac{24}{\sqrt{17}}\pi\right)}{2}\\
&= \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}}\xi_0\left(\frac{i\sqrt{17}}{12}\right)\Delta\tau\right)exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}}\xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right)exp\left(\int_{\frac{24\pi}{\sqrt{17}}}^{t}\xi_0\left(\frac{i\sqrt{17}}{12}\right)\Delta\tau\right)}{2}\\
&+\frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}}\xi_0\left(\frac{-i\sqrt{17}}{12}\right)\Delta\tau\right)exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}}\xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{-i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right)exp\left(\int_{\frac{24\pi}{\sqrt{17}}}^{t}\xi_0\left(\frac{-i\sqrt{17}}{12}\right)\Delta\tau\right)}{2}\\
&= \frac{exp(i\pi)\ exp\left(Log\left(1+\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)\right)exp\left(\frac{it\sqrt{17}}{12}-2i\pi\right)}{2}\\
&+\frac{exp(-i\pi)\ exp\left(Log\left(1-\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)\right)exp\left(\frac{-it\sqrt{17}}{12}+2i\pi\right)}{2}\\
&= \frac{\left(1+\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)exp\left(\frac{it\sqrt{17}}{12}\right)e^{-i\pi}+\left(1-\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)exp\left(\frac{-it\sqrt{17}}{12}\right)e^{i\pi}}{2}\\
&= \frac{\left(-1-\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)exp\left(\frac{it\sqrt{17}}{12}\right)+\left(-1+\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)exp\left(\frac{-it\sqrt{17}}{12}\right)}{2}\\
&= cos\left(\frac{\sqrt{17}}{12}t\right)-\frac{\pi sin\left(\frac{\sqrt{17}}{12}t\right)}{1+\frac{\pi}{\sqrt{17}}}
\end{aligned}
$$

$$
\begin{aligned}
sin_{\frac{\sqrt{17}}{12+\mu}}(t,0) &= \frac{e_{\frac{i\sqrt{17}}{12+\mu}}(t,0) - e_{\frac{-i\sqrt{17}}{12+\mu}}(t,0)}{2i} \\
&= \frac{e_{\frac{i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right) e_{\frac{i\sqrt{17}}{12}}\left(t, \frac{24}{\sqrt{17}}\pi\right)}{2i} \\
&\quad - \frac{e_{\frac{-i\sqrt{17}}{12}}\left(\frac{12}{\sqrt{17}}\pi, 0\right) e_{\frac{-i\sqrt{17}}{12+\frac{12\pi}{\sqrt{17}}}}\left(\frac{24}{\sqrt{17}}\pi, \frac{12}{\sqrt{17}}\pi\right) e_{\frac{-i\sqrt{17}}{12}}\left(t, \frac{24}{\sqrt{17}}\pi\right)}{2i} \\
&= \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{i\sqrt{17}}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right) exp\left(\int_{\frac{24\pi}{\sqrt{17}}}^{t} \xi_0\left(\frac{i\sqrt{17}}{12}\right)\Delta\tau\right)}{2i} \\
&\quad - \frac{exp\left(\int_0^{\frac{12\pi}{\sqrt{17}}} \xi_0\left(\frac{-i\sqrt{17}}{12}\right)\Delta\tau\right) exp\left(\int_{\frac{12\pi}{\sqrt{17}}}^{\frac{24\pi}{\sqrt{17}}} \xi_{\frac{12\pi}{\sqrt{17}}}\left(\frac{-i\sqrt{17}}{12(1+\frac{\pi}{\sqrt{17}})}\right)\Delta\tau\right) exp\left(\int_{\frac{24\pi}{\sqrt{17}}}^{t} \xi_0\left(\frac{-i\sqrt{17}}{12}\right)\Delta\tau\right)}{2i} \\
&= \frac{exp(i\pi)\ exp\left(Log\left(1+\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)\right) exp\left(\frac{it\sqrt{17}}{12} - 2i\pi\right)}{2i} \\
&\quad - \frac{exp(-i\pi)\ exp\left(Log\left(1-\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right)\right) exp\left(\frac{-it\sqrt{17}}{12} + 2i\pi\right)}{2i} \\
&= \frac{\left(1+\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) exp\left(\frac{it\sqrt{17}}{12}\right) e^{-i\pi} - \left(1-\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) exp\left(\frac{-it\sqrt{17}}{12}\right) e^{i\pi}}{2i} \\
&= \frac{\left(-1-\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) exp\left(\frac{it\sqrt{17}}{12}\right) - \left(-1+\frac{i\pi}{1+\frac{\pi}{\sqrt{17}}}\right) exp\left(\frac{-it\sqrt{17}}{12}\right)}{2i} \\
&= -sin\left(\frac{\sqrt{17}}{12}t\right) - \frac{\pi cos\left(\frac{\sqrt{17}}{12}t\right)}{1+\frac{\pi}{\sqrt{17}}}
\end{aligned}
$$

Thus the solution when $t \in \left(\frac{24}{\sqrt{17}}\pi, 6\pi\right]$ is

$$y(t) = e^{\left(\frac{t}{12} - \frac{\pi}{\sqrt{17}}\right)}\left(1 + \frac{\pi}{2\sqrt{17}}\right)\left[cos\left(\frac{\sqrt{17}}{12}t\right) - \frac{\pi sin\left(\frac{\sqrt{17}}{12}t\right)}{1 + \frac{\pi}{\sqrt{17}}} - \frac{1}{\sqrt{17}}\left(sin\left(\frac{\sqrt{17}}{12}t\right) + \frac{\pi cos\left(\frac{\sqrt{17}}{12}t\right)}{1 + \frac{\pi}{\sqrt{17}}}\right)\right]$$
$$= e^{\left(\frac{t}{12} - \frac{\pi}{\sqrt{17}}\right)}\left(1 + \frac{\pi}{2\sqrt{17}}\right)\left[cos\left(\frac{\sqrt{17}}{12}t\right) - \frac{1}{\sqrt{17}}sin\left(\frac{\sqrt{17}}{12}t\right)\right]$$
$$- e^{\left(\frac{t}{12} - \frac{\pi}{\sqrt{17}}\right)}\left(1 + \frac{\pi}{2\sqrt{17}}\right)\left\{\frac{\pi\sqrt{17}}{\sqrt{17} + \pi}\left[sin\left(\frac{\sqrt{17}}{12}t\right) + \frac{1}{\sqrt{17}}cos\left(\frac{\sqrt{17}}{12}t\right)\right]\right\}$$

And the solution for the time scale $\mathbb{T} = [0, \frac{12}{\sqrt{17}}\pi] \cup [\frac{24}{\sqrt{17}}\pi, 6\pi]$ is

$$y(t) = \begin{cases} e^{\frac{1}{12}t}\left[cos\left(\frac{\sqrt{17}}{12}t\right) + \frac{1}{\sqrt{17}}sin\left(\frac{\sqrt{17}}{12}t\right)\right] & t \in [0, \frac{12}{\sqrt{17}}\pi] \\ -e^{\frac{\pi}{\sqrt{17}}}\left(1 + \frac{2\pi}{\sqrt{17}}\right) & t = \frac{24}{\sqrt{17}}\pi \\ e^{\left(\frac{t}{12} - \frac{\pi}{\sqrt{17}}\right)}\left(1 + \frac{\pi}{2\sqrt{17}}\right)\left[cos\left(\frac{\sqrt{17}}{12}t\right) - \frac{1}{\sqrt{17}}sin\left(\frac{\sqrt{17}}{12}t\right)\right] & \\ \quad -e^{\left(\frac{t}{12} - \frac{\pi}{\sqrt{17}}\right)}\left(1 + \frac{\pi}{2\sqrt{17}}\right)\left\{\frac{\pi\sqrt{17}}{\sqrt{17} + \pi}\left[sin\left(\frac{\sqrt{17}}{12}t\right) + \frac{1}{\sqrt{17}}cos\left(\frac{\sqrt{17}}{12}t\right)\right]\right\} & t \in \left(\frac{24}{\sqrt{17}}\pi, 6\pi\right] \end{cases}$$

CHAPTER 4

Overview of the Differential Analyzer

A differential analyzer (DA) is a machine designed to mechanically solve and plot the solutions of a differential equation. Before the advent of high powered computers and numerical methods, the differential analyzer was the only method for solving some differential equations; and even today still shows it benefits versus some of the more modern methods. [3]

The differential analyzer at Marshall University, like most of its brethren, is almost entirely composed of Meccano, or Meccano replica, parts. The only pieces on the Marshall DA that are not Meccano are the torque amplifiers, because there isn't a reliable method for torque amplification solely using Meccano parts. Marshall's four integrator differential analyzer was constructed in 2004.

4.1 Mechanics of the Differential Analyzer

A differential analyzer utilizes mechanical movement to solve differential equations. The main component of the differential analyzer is the *integrator*. There are several types of integrators that can be used, but in the case of Marshall University's DA, the wheel and disk combination is implemented. For this method, the independent variable is driven by a motor, which turns a disk. The speed with which the disk turns is determined by a dial connected to the motor. We can consider the speed with which the disk turns to be constant. Sitting on the disk is a wheel representing the dependent variable. Although the disk spins with constant velocity, because it is spinning, the angular velocity increases as distance from the center of the disk increases. Because of this, the wheel sitting on the disk will spin with a speed equal to the angular velocity of the disk.

Following the motion of the rod coming from the wheel, the next component of the DA we encounter is the *torque amplifier*. While the torque amplifier does not affect the mathematics represented, it is important in ensuring that the motion of the wheel's rod is strong enough to push all of the other components in the DA. The torque amplifiers used in the DA at Marshall University are operated using a computer program loaded into a microprocessor.

Moving away from the integrators, the next component is what is lovingly referred to at the "highway of interconnect". There are two rods leading into, and one coming out of, each integrator. The first rod is the independent variable, which as stated, drives the disk on the integrator. The middle rod offers the motion describing the integration. For instance, if an integrator in question is integrating y'', then the second rod represents $\int y''$ or y'. The last rod pushes the carriage that houses the disk. This rod, representing the integrand, is what causes the change in position of the wheel, which in turn affects the location of the wheel, varying the speed the middle rod turns. While the second rod is pushing motion out of the integrator, the first and third rods are pushing motion into the integrator.

Each of these rods runs into a gear box in the highway of interconnect. These gear boxes allow for the motion of the rod to be moved, at equal speed, to a perpendicular rod. This allows us to both solve higher ordered problems on the DA, as well as plot the solution. For instance, if we want to solve a second order differential equation, then the first integrator is associated with y'', and as stated, the middle rod on the integrator represents y'. So if we allow the middle rod to push motion through its gearbox and into the the second rod of another integrator, then the middle rod of the second integrator represents the solution y.

Between the integrators on the highway of interconnect we can connect a system of gears of varying sizes. These gears allow for the motion coming from one gearbox to be slowed, or geared down. This allows the DA to solve more complex differential equations, i.e. those that have terms with coefficients other than 1. Also in this system of gears are gear trains referred at *adders*. These adders can take the motion of two rods and output the sum of the motion of those two rods onto a third rod.

The final destination for motion on the DA is the *output table*. The output table, can plot the relationship between any two motions on the DA chosen by the operator. It can take the motion from two rods, and through gearboxes, turn a system of gears: one that moves a pen vertically and another that pushes the table under the pen horizontally. These two rods can be of thought as the knobs on an Etch-a-Sketch, as the output table operates in a similar fashion to the toy.

The final component of the DA, the *initial condition counter*, is something that needs to be set before starting the machine. Each of the integrators must be set up with the initial conditions for the particular initial value problem. On each integrator a counter is associated with the position of the wheel on the disk. For these counters on Marshall's four integrator machine, it has been determined that one unit on the real line is equivalent to 250 clicks on the counter. When the wheel is at the center of the disk, the counter reads 000. A positive initial condition reads on the counter as expected. However, a negative initial condition causes the counter to run backwards to 999 and down from there. For instance, an initial condition of -1 would result in a counter value of 750. For the initial conditions on the integrators, we need not worry about any overlapping (i.e. is 750 equivalent to -1 or 3?) because the limits of the disk prevent the counter from achieving an absolute value larger than 2.

The system of gears and rods used to set up a dynamic equation on a differential analyzer can often be quite complex. There is a circular chain of motion, originating from the independent variable motor, that connect all of the integrators being used to each other through the gearboxes on the highway of interconnect, and finally to the output table. The blueprint for a setup of the differential analyzer is called a Bush Diagram (seen below):

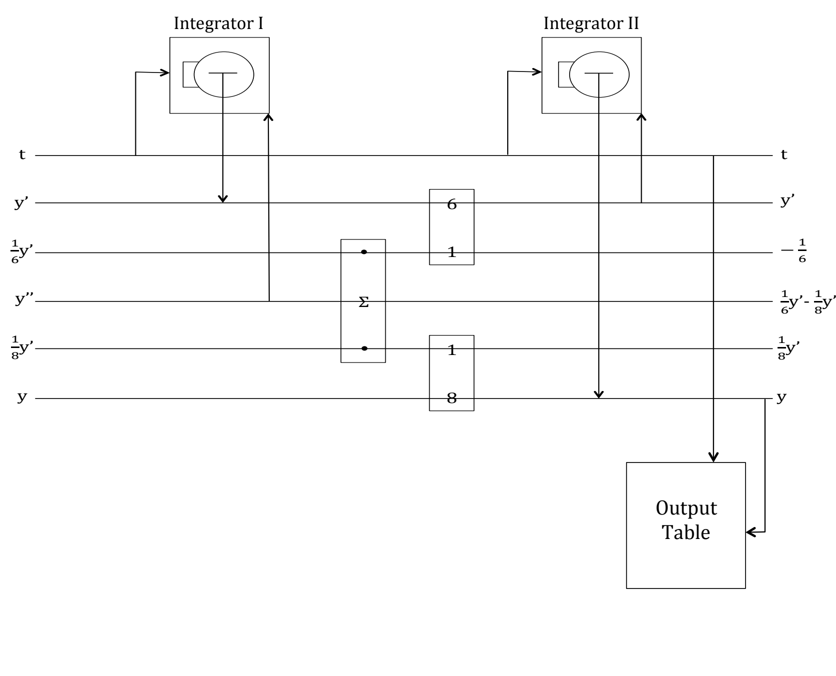

Figure 4.1: Bush Diagram of $y^{\Delta\Delta} - \frac{1}{6}y^{\Delta} + \frac{1}{8}y = 0$

It is important to note that there is a closed system of motion on the highway of interconnect. When setting up a differential equation on the DA, the differential equation is first solved in terms of the highest ordered term. For the example in Figure 4.1, we rewrite the differential equation as $y^{\Delta\Delta} = \frac{1}{6}y^{\Delta} - \frac{1}{8}y$. This tells us that the rod producing the summated motion of $\frac{1}{6}y^{\Delta} - \frac{1}{8}y$ must also be the rod associated with $y^{\Delta\Delta}$, and we can see that this is the case in the figure. If the DA did not have this closed system of motion, it would not produce the intended solution.

4.2 Graphing the Solution of $y^{\Delta\Delta} - \dfrac{1}{6}y^{\Delta} + \dfrac{1}{8}y = 0$ on a Differential Analyzer

When graphing the solution to the dynamic equation, we will assume that $t_0 = 0$ throughout. If the time scale is the real line, then there is no adjustments that need to be made to the plot. Figure 4.2 is the plot of the continuous solution to the dynamic equation.

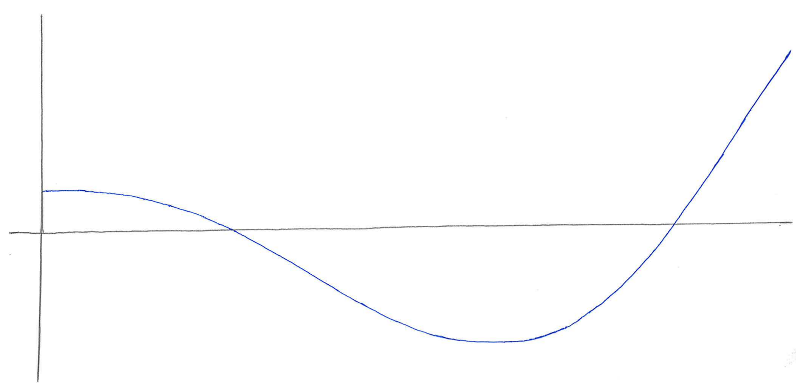

Figure 4.2: Differential Analyzer output of $y^{\Delta\Delta} - \frac{1}{6}y^{\Delta} + \frac{1}{8}y = 0$ for $\mathbb{T} = \mathbb{R}$

However suppose we would like to plot the solution on $\mathbb{T} = [0, t_1] \cup [t_2, t_3]$, then the process becomes more involved. When making a jump from the end of an interval t_1 to the beginning of another interval t_2, the solution doesn't start at t_2 on the continuous solution, but rather, according to the Simple Useful Formula, restarts at t_2 and is located on the line tangent to $y(t_1)$ from the first interval. Because of this, we must run the machine in a particular way to find the appropriate values for $y(t_2)$, $y^{\Delta}(t_2)$, and $y^{\Delta\Delta}(t_2)$. The following provides a list of steps, along with the rationale for taking those steps:

1. With the pen down on the output table, run the differential analyzer from t_0 to t_1.

   We do this to plot the solution of our first dense interval. The value for t_1 can be chosen arbitrarily, or it may be chosen because it results in a desired value for y, y^{Δ}, or $y^{\Delta\Delta}$

2. Disconnect Integrator I and Integrator II by engaging their respective clutches, lift the pen and run the differential analyzer to t_2, and mark this point on the paper with the pen. We disconnect the first and second integrator as this will keep the first and second derivatives constant so that we can jump from t_1 to t_2 in a straight line. Notice that the value for $y\left(\frac{24}{\sqrt{17}}\pi\right)$ is the same regardless of whether we evaluate the solution at a point (3.17), or use the Simple Useful Formula (3.18). Therefore we can use the Simple Useful Formula to find the value of t_2.

   $$\begin{aligned} y(t_2) &= y(\sigma(t_1)) \\ &= y(t_1) + (t_2 - t_1)y^{\Delta}(t_1^-) \end{aligned}$$

   Because we ran the DA to t_1 in step 1, we have the value of $y(t_1)$. By running from t_1 to t_2 we are simulating the calculation, $t_2 - t_1$, on the DA, and since we have engaged the clutch that is associated with y^{Δ}, we have kept it at its value $y^{\Delta}(t_1)$. Because the solution picks up at a point on the tangent line to $y(t_1)$, we are able to assume that $y^{\Delta}(t_1) = y^{\Delta}(t_1^-)$.

3. Engage the clutch for y and run the DA backwards from t_2 back to t_1.

   This begins the process of finding the value of $y^{\Delta}(t_2)$. At this point, the clutches for $y^{\Delta\Delta}, y^{\Delta}$, and y are all engaged so all of these values remain constant.

4. Disengage the clutch on Integrator II and again run the DA from t_1 to t_2.

   This step performs that same function as Step 2, but for y^{Δ} rather than y. Disengaging the clutch on Integrator II allows for y^{Δ} to change. Again we are using the Simple Useful

Formula in the following fashion.

$$\begin{aligned} y^{\Delta}(t_2) &= y^{\Delta}(\sigma(t_1)) \\ &= y^{\Delta}(t_1) + (t_2 - t_1)y^{\Delta\Delta}(t_1) \end{aligned}$$

We still have the clutch on integrator I engaged, so $y^{\Delta\Delta}$ remains constant across the gap.

5. Calculate $y^{\Delta\Delta}(t_2)$. Unfortunately, we cannot find the value of $y^{\Delta\Delta}(t_2)$ using the simple useful formula because we do not have a $y^{\Delta\Delta\Delta}$ defined. However what we can do is use the values found in Step 2, $y(t_2)$, and Step 4, $y^{\Delta}(t_2)$, in the dynamic equation to find the value for $y^{\Delta\Delta}(t_2)$,
$$y^{\Delta\Delta}(t_2) = \frac{1}{6}y^{\Delta}(t_2) - \frac{1}{8}y(t_2).$$

6. Reset the value on integrator I to the value found in step 5. Now disengage all clutches and allow the DA to run until the end of the plot.
   So Steps 2, 4, and 5 have provided the values for y, y^{Δ}, and $y^{\Delta\Delta}$ respectively. These values have as little manual manipulation on the DA as possible. So now that we have defined all applicable pieces at t_2 the solution can be plotted accurately moving forward.

When implementing this method, it is interesting to observe the differences in the solution as the distance of the gap decreases. To see this result, we looked at the sequence of time scales, $\mathbb{T}_n = [0, t_1] \cup [t_n, t_3]$, for $n \geq 2$ where t_n approaches t_1. To do this we repeat the steps stated above. It is important to note, that Step 1 only needs to be completed a single time. This is because this portion of the solution is still equivalent to the solution on the real line, so as the jump distance decreases, this portion of the solution will always remain the same. In previous studies of the solutions of dynamic equations with decreasing gap length, the gaps were chosen to achieve a specific value for the jump. However in this study, the first interval of solution was chosen, not based on the distance of the gap, but because of what those t values represented in the solution on $[0, \infty)$. The endpoint for the first interval of solution t_1 was chosen because $y(t_1) = 0$. While the value of t_2, the largest gap, was chosen since $y^{\Delta}(t_2) = 0$, then each subsequent gap distance was chosen such that the difference in distance between two consecutive gaps was constant.

For the first three time scales after the gap, the DA was allowed to run until completion, however the machine never reached the end of the paper because the value for y^{Δ} became so large that it exceeded the limits of the disk on the integrator, forcing the machine to shut off.

The Table 4.1 presents the counter values of $t, \mu(t), y^{\Delta}(t)$, and $y^{\Delta\Delta}(t)$ for each of the gaps that were run on the DA.

t	$\mu(t)$	$y(t)$	$y^{\Delta}(t)$	$y^{\Delta\Delta}(t)$
0		83	0	-31.13
320		0	-126	-21.5
500	180	-73	-170.5	-1.04
590	270	-123	-193	13.96
680	360	-180.5	-215	31.85
770	450	-227	-248	43.79
860	540	-273.5	-263	58.73

Table 4.1: Counter Values for Each Gap Run on DA

While this data is valuable, it is not very useful for someone not fully familiar with the DA, Table 4.2 provides the same data, but with the values converted into the real numbers, rather than counter values.

t	$\mu(t)$	$y(t)$	$y^{\Delta}(t)$	$y^{\Delta\Delta}(t)$
0		1	0	-0.12
3.84		0	-0.5	-0.09
6	2.16	-0.88	-0.68	0
7.08	3.24	-1.48	-0.77	0.06
8.16	4.32	-2.17	-0.86	0.13
9.24	5.4	-2.72	-0.99	0.18
10.32	6.48	-3.28	-1.05	0.23

Table 4.2: Real Values for Each Gap Run on DA

It is extremely important to note when analyzing the data in Table 4.2, a single unit on the real line is equivalent to 250 clicks on the counter on the DA. However, the y and t values have been scaled down by a factor of 3. This is because either time ran too fast, which didn't provide an output with enough distinguishable characteristics of the solution, or y ran too fast, which caused the graph to exceed the capabilities of the output table. The output of the differential analyzer for the decreasing gap sizes as described above is shown below:

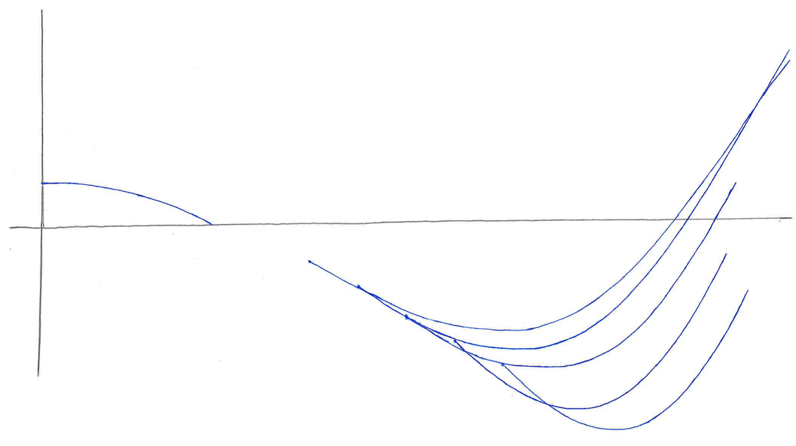

Figure 4.3: Differential Analyzer output of $y^{\Delta\Delta} - \frac{1}{6}y^{\Delta} + \frac{1}{8}y = 0$ with decreasing gap distances

The plots of the second interval, going from top to bottom, are associated with gap distances as shown in the first column of Table 4.2.

So based on the output from the DA, we can see that the solutions on the union of two intervals appears to converge to the solution on the real line as the gap closes between the intervals. This result qualitatively aligns with previous work on solutions to dynamic equations of decreasing gap length. [1] [4]

APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

MARSHALL UNIVERSITY.

www.marshall.edu

Office of Research Integrity

April 21, 2016

Jacob Fischer
319 Ferguson Court
Huntington, WV 25701

Dear Mr. Fischer:

This letter is in response to the submitted thesis abstract entitled *"Mechanical Visualization of a Second Order Dynamic Equation."* After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day

Bruce F. Day, ThD, CIP
Director

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A State University of West Virginia • An Affirmative Action/Equal Opportunity Employer

REFERENCES

[1] Alexandria Amorim, *Solutions of a logistic equation on varying time scales: A quantitative and qualitative analysis*, Master's thesis, Marshall University, 2015.

[2] Martin Bohner, *Dynamic equations on time scales - an introduction with applications*, Birkhauser Boston, 2001.

[3] J. Crank, *The differential analyzer*, Longmans, London, 1947.

[4] Molly Peterson, *Mechanical visualization of a second order dynamic equation on varying time scales*, Master's thesis, Marshall University, 2014.

JACOB E FISCHER
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EDUCATION

Aug 2014 to May 2016 **Marshall University**
Huntington, WV 25701
M.S. in Mathematics

- GPA – 3.66
- Anticipated Graduation – May 2016

Aug 2010 to May 2014 **Bethany College**
Bethany, WV 26032
B.S. in Chemistry and Mathematics

- GPA – 3.36 on 4.0 scale; Cum Laude Honors
- Chemistry GPA – 3.22
- Mathematics GPA – 3.50
- Departmental Fellow to the Department of Physical and Computational Sciences

WORK HISTORY

Aug 2015 to Feb 2016 **Mathematics Teacher**
St. Joseph Central Catholic High School
Huntington, WV 25701

Taught mathematics part time at St. Joseph Central Catholic High School. Instruct students of all grade levels over a variety of subjects.

Classes Taught

- *Algebra I*
- *Algebra 2*
- *Geometry*
- *Analytical Math*

Aug 2014 to May 2016 **Graduate Assistant**
Marshall University
Huntington, WV 25701

Served as tutor for general mathematics courses in tutoring lab on campus, taught introductory math classes, assisted with general maintenance and research relating to differential analyzer.

Classes Taught

- *Mathematics Skills I – Fall 2014*
- *Mathematics Skills II – Spring 2015*

Jan 2013 **Resident Assistant**
to *Bethany College*
May 2014 *Bethany, WV 26032*

Resident assistant in resident hall on campus. Job duties included, but were not limited to conducting rounds of the buildings on a regular basis, hosting various events and programs on and off campus, and assisting residents with a variety of academic, personal, and roommate issues.

Aug 2011 **Physical Science Academic Assistant**
to *Bethany College*
May 2014 *Bethany, WV 26032*

Tutored Classes approximately 10 hours per week

Classes Tutored

- *Calculus Sequence*
- *General, Organic, Physical, Analytical Chemistry Sequences*
- *Linear Algebra*
- *Differential Equations*
- *General Physics Sequence*

Classes Graded

- Calculus I
- Linear Algebra

Aug 2010 **Laboratory Proctor/Technician**
to *Bethany College*
May 2014 *Bethany, WV 26032*

Monitored and assisted with chemistry lab sections, prepped for labs weekly by organizing materials from stockroom, and creating solutions. Performed regular lab maintenance including cleaning glassware and recording stockroom contents for digital database.

Lab Sections Proctored

- *General Chemistry*
- *Physical Chemistry*
- *Analytical Chemistry*
- *Consumer Chemistry*
- *Forensics Chemistry*

May 2012 **Summer Fellow**
to *WV – INBRE Program*
Jul 2012 *Huntington, WV 25701*

As part of the WV-IDeA Networt of Biomedical Research Excellence Summer program, I worked in the Center for Diagnostic Nanosystems under Dr. Eric Blough at the Robert C. Byrd Biotechnology Science Center on Marshall University's campus

- Project Title – *Using FTIR in the Optimization of Immuniprecipitation of Magnetic Beads for us in Microfluidic Applications*

SPECIAL SKILLS

Familiarity with the following instruments

- Gas Chromatograph – Mass Spectrometer (GC-MS)
- Ultraviolet – Visible Spectrometer (UV-Vis)
- Fourier Transform Infrared Spectrometer (FTIR)
- Raman Spectrometer
- High Pressure Liquid Chromatograph (HPLC)
- Nuclear Magnetic Resonance Spectrometer (NMR)

Globally Harmonized System of Classification and Labeling of Chemicals (GHS) Safety Certified

Proficient in Mathematics 8

Proficient in LaTeX

Knowledge of JAVA programming

PRESENTATIONS

Synthesis of Polyaniline Nanofibers Sensors for use in Determining Rate of Carbon Dioxide Emission in Carbonated Beverages (Presentation and Poster)

- Bethany College
- April 2014

Preparation and Analysis of Conducting Polymers via Interfacial Polymerization (Poster)

- Bethany College
- April 2013

Getting off the Hill : Conducting a Summer Research Project away from Bethany College

- Bethany College
- Spring 2013

Using FTIR in the Optimization of Immuniprecipitation of Magnetic Beads for us in Microfluidic Applications (Poster)

- West Virginia University
- July 2012

GRANTS

Bethany College Gans Fund (2013)
NASA West Virginia Space Grant Consortium (2013)
NASA West Virginia Space Grant Consortium (2012)

CONFERENCES

Kappa Mu Epsilon National Conference; Jacksonville, AL; 2014
American Chemical Society National Convention; Dallas, TX; 2014
Gamma Sigma Epsilon National Conference, Frostburg, MD; 2013
Midstates Conference For Undergraduate Research in Computer Science and Mathematics (MCURCSM); Delaware, OH; 2013
PittCon Conference & Expo; Philadelphia PA; 2013
Midstates Conference For Undergraduate Research in Computer Science and Mathematics (MCURCSM); Delaware, OH; 2012

LEADERSHIP POSITIONS AND HONORS

Student Government Association of Bethany College
President – Fall 2013 – Spring 2014
American Chemical Society – Chapter at Bethany College
President – Spring 2013 – Spring 2014
Gamma Sigma Epsilon – National Chemistry Honors Society
President – Spring 2013 – Spring 2014
Kappa Mu Epsilon – National Mathematics Honors Society
Secretary – Spring 2013 – Spring 2014
Alpha Epsilon Delta – National Pre-Health Professions Honors Society
Vice President Spring 2013 – Fall 2014
Bethany College Kalon
Inducted Spring 2013
W.H Cramblet Prize in Mathematics, 2014