1-1-2010

The Beta Maxwell Distribution

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Thesis submitted to the Graduate College of Marshall University in partial fulfillment of the requirements for the degree of

Master of Arts
in
Mathematics

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Huntington, West Virginia
2010

Keywords: Beta distribution, Maxwell distribution, moments, parameter estimation

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ABSTRACT

In this work we considered a general class of distributions generated from the logit of the beta random variable. We looked at various works that have been done and discussed some of the results that were obtained. Special cases of this class include the beta-normal distribution, the beta-exponential distribution, the beta-Gumbell distribution, the beta-Weibull distribution, the beta-Pareto distribution and the beta-Rayleigh distribution. We looked at the probability distribution functions of each of these distributions and also looked at some of their properties. Another special case of this family, a three-parameter beta-Maxwell distribution was defined and studied. Various properties of the distribution were also discussed. The method of maximum likelihood was proposed to estimate the parameters of the distribution.

Acknowledgement

I would like to thank all those who helped and supported me. My sincere appreciation goes to my parents Prof. and Mrs O.O.G Amusan and family. I would not have made it this far without your tremendous support. I would like to thank Dr Akinsete, Dr Karna, and Dr Dementieva for their advice and patience with me throughout. I would also like to thank Luwin Witherspoon for his help and encourage. And most importantly I would like to thank God would made it possible for me to finish the work.

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1 Introduction

The Maxwell distribution also known as the Maxwell-Boltzmann distribution is a continuous probability distribution with applications in physics and chemistry. It is named after James Clark Maxwell and Ludwig Boltzmann. The distribution is commonly used in statistical mechanics to determine the speeds of molecules. The Maxwell distribution gives the distribution of the speeds of molecules as it is given by statistical mechanics in thermal equilibrium when the temperature is high enough and the density is low enough to render quantum effects negligible. Hence for this distribution, the temperature, the mass of the molecule, and the speed of the molecule are considered.

The probability density function (PDF) and the cumulative distribution function (CDF) of the Maxwell distribution with parameter $a > 0$ are respectively,

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{x^2 e^{-x^2/2a^2}}{a^3}, \quad 0 \leq x < \infty \tag{1}$$

and

$$F(x) = \frac{2\gamma(\frac{3}{2}, \frac{x^2}{2a^2})}{\sqrt{\pi}}, \quad 0 \leq x < \infty, \tag{2}$$

where the random variable x denotes the speed of a molecule, while the $a = \sqrt{kT/m}$, where k is the Boltzmann constant, T is the temperature and m is the mass of a molecule. Also note that $\gamma(a, b)$ is the lower incomplete gamma function, which is defined as

$$\gamma(a, b) = \int_0^b t^{a-1} e^{-t} dt.$$

The Maxwell distribution has various applications in physics and chemistry. Its most common application is in statistical mechanics.

Similarly, beta distribution is a continuous probability distribution with two positive shape parameters α and β. It is the conjugate prior of the binomial distribution. It is also a natural extension of the uniform distribution by dint of two parameters. The beta distribution models events which are constrained to take place within an interval defined by a minimum and maximum values. One of its main uses is to describe certainty or random variation. The other main use of the beta distribution is that one can rescale and shift it to create distributions with a wide range of shapes and over any finite range and as a result it has been used for a wide variety of other applications.

The PDF and the CDF of the beta distribution are respectively

$$f(x) = \frac{(1-x)^{\beta-1}x^{\alpha-1}}{B(\alpha,\beta)}, \quad 0 < x < 1, \alpha > 0, \beta > 0, \tag{3}$$

$$F(x) = \frac{1}{B(\alpha,\beta)}\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0 < t < 1, \alpha > 0, \beta > 0, \tag{4}$$

where $B(\alpha, \beta)$ is the beta function defined as

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The main use of the beta distribution as stated earlier has led to it being used in the generalization of different distributions, which has been studied and proposed by different authors in literature. This new class of distributions was made prominent by Jones [4], where the author talked about the families of distributions arising from distributions of order statistics. The author proposed an alternative derivation of this family of distributions as the result of applying the inverse probability integral transformation to the beta distribution. The properties including the location and scale parameters, the distribution function, the moments, the modality, the tail weight, and the limiting distributions were considered.

Suppose $F(x)$ is the cumulative distribution function of a random variable X. The cumulative distribution function for a generalized class of distributions for the random variable X can be defined as the logit of the beta random variable

$$G(x) = \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0 < \alpha, \beta < \infty. \tag{5}$$

Its probability density function is $g(x) = G'(x)$, which is

$$g(x) = \frac{1}{B(\alpha,\beta)} F(x)^{\alpha-1}(1-F(x))^{\beta-1}F'(x). \tag{6}$$

The above work by Jones [4] introduced a general family that triggered other authors to consider generalized distributions with some statistical applications. Thereafter, a few authors have defined and studied various forms of beta-compounded distributions. For example, Eugene et al. [9] introduced the beta-normal distribution and studied some properties of this distribution. Nadarajah and Gupta [7] later derived a more general expression for the nth moment of the beta-normal distribution which was introduced by Eugene et al [9]. The beta-exponential and the beta-Gumbel were both defined and studied by Nadarajah and Kotz [13], [12]. The beta-Weibull distribution was proposed by Lee et al [5]. The beta pareto distribution was defined and studied by Akinsete et al [2]. And most recently, the beta Rayleigh Distribution was proposed by Akinsete and Lowe [1].

In this thesis we define and study a three-parameter beta-Maxwell distribution. To achieve this, we will let $F(x)$ be the CDF of the Maxwell distribution in Equation (6). The resulting distribution is being referred to as the beta-Maxwell distribution. We will further verify that this is a legitimate distribution, in that $\int_0^\infty g(x) = 1$, where $g(x)$ refers to the PDF of the distribution. Some properties of the distribution will also be defined and discussed.

2 The Literature Review

We will now look at the various work that have been done and observe some of the results that were obtained in the literature.

2.1 The beta-Normal Distribution

Eugene et al [9] introduced the beta-normal distribution and its applications. In their work, they studied the case when $F(x)$ is the CDF of the normal distribution with the parameters μ and σ . This led to the random variable X having the beta-normal distribution denoted by $BN(\alpha, \beta, \mu, \sigma)$ with the PDF

$$\begin{aligned} BN(\alpha, \beta, \mu, \sigma) &= G[\Phi(\frac{x-\mu}{\sigma})] \\ &= \frac{1}{B(\alpha, \beta)}[\Phi(\frac{x-\mu}{\sigma})]^{\alpha-1}[1-\Phi(\frac{x-\mu}{\sigma})]^{\beta-1} \\ &\times \sigma^{-1}\phi(\frac{x-\mu}{\sigma}), \end{aligned} \tag{7}$$

where $\Phi(\frac{x-\mu}{\sigma})$ is the normal CDF with parameters μ and σ and $\phi(\frac{x-\mu}{\sigma})$ is the normal PDF.

Some properties of the beta-normal distributions were also discussed. One property discussed was the shape property. The beta-normal distribution was found to be symmetric about μ when $\alpha = \beta$. It was also noted that the class of beta-normal distributions uses the normal distribution $N(\mu, \sigma)$ as the distribution generator. Another property noted was that the beta-normal distribution can model both unimodal and bimodal data. The values of α and β were noted to affect the modality of the distribution since they are the shape parameters while μ is the location parameter and σ is the scale parameter. When α and β are less than 0.214, a region of α and β is identified where the beta-normal distribution is bimodal. However if α

and β are both larger than 0.214, a region of α and β is identified where the beta-normal distribution is always unimodal. These results were obtained by Eugene [8] earlier.

Eugene et al [9] further mention the shape properties of the unimodal beta-normal distribution. Included in these is that the distribution is right skewed when $\alpha > \beta$ and that the increase in the degree of right skewness is proportional to the increase in α. Another shape property is that the distribution is left skewed when $\alpha < \beta$ and that the increase in the degree of left skewness is proportional to the decrease in β. The distribution was noted to be symmetric and platykurtic when $\alpha, \beta < 1$ and $\alpha = \beta$, and that it is leptokurtic when $\alpha, \beta > 1$. The decrease in both α and β until bimodality occurs makes the tail heavier, and the increase in both α and β causes the peak of the distribution to be higher.

The relationships between the first two moments and the parameters of the beta-normal distribution were also noted. These include that the expected value μ_{BN} is an increasing function of α, while on the other hand, μ_{BN} is a decreasing function of β. Also, it was noted that for any given $\beta < \alpha$, the kurtosis is an increasing function of α. Likewise for a fixed $\alpha < \beta$, the kurtosis was noted to be a decreasing function of α.

The first moment of the beta-normal distribution was considered and it was concluded that the exact first moments cannot be evaluated. As a result the application of the differential equations technique used by Bose and Gupta [3] was employed to derive some exact first moments. The closed form solutions of $E(X)$ for some values of α and β were given and the proof of the result for $\alpha = 4$ and $\beta = 2$ was also provided. This approach expresses the first moment as

$$E(X) = \mu + \frac{10\sigma}{\sqrt{\pi}} - \frac{30\sigma}{\pi\sqrt{\pi}} \arctan\sqrt{2}. \tag{8}$$

The maximum likelihood method was used to estimate the parameters α, β, μ, and σ. The applications of the beta-normal distribution were also considered and it was noted that both the gamma and Lagrange-gamma distributions provided poor fits for some of the data sets fitted with the beta-normal distribution.

Nadarajah and Gupta [7] later derived a more general expression for the nth moment of the beta-normal distribution which was introduced by Eugene et al [9]. The proof of the results were much more simpler to follow in that the calculations made were based on the use of the complementary error function and some of its properties. Using these the nth moment was obtained as follows

$$E(X^n) = \mu^n + \frac{\mu^n}{B(\alpha, \beta)} \sum_{j=0}^{\beta-1} (-1)^j. \tag{9}$$

Several particular cases of different values of α and β were also considered, including the one considered by Eugene et al [9]. The case that coincided with that of Eugene et al [9] where $\alpha = 4$ and $\beta = 2$ gave the same result as the one obtained earlier by Eugene et al [9] refer to Equation(8).

2.2 The beta-Gumbel Distribution

The beta-Gumbel distribution was introduced by Nadarajah and Kotz [12] as a generalization of the Gumbel distribution. It was a result of the motivation by the work of Eugene et al [9]. It was also generated from the logit of a beta random variable. A generalization of the Gumbel distribution was proposed in the hope of attracting wider applicability in engineering.

The Gumbel distribution is most widely used in applied statistical distribution for problems encountered in engineering. It is also referred to as the extreme value distribution of type I. Some applications in engineering include flood frequency analysis, network engineering, nuclear engineering, and offshore engineering to name a few. Its PDF and CDF with parameters μ and σ, are given respectively as follows:

$$g(x) = \frac{1}{\sigma}\exp\left[\frac{x-\mu}{\sigma} - \exp\left(\frac{x-\mu}{\sigma}\right)\right], \;\; -\infty < x < \infty, \sigma > 0, -\infty < \mu < \infty,$$

$$G(x) = \exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \;\; -\infty < x < \infty, \sigma > 0, -\infty < \mu < \infty. \tag{10}$$

The generalization of the Gumbel distribution - referred to as the beta Gumbel (BG) is obtained by letting $F(x)$ in Equation (6) be the CDF of the Gumbel distribution. It follows that the PDF of the BG is given by

$$f(x) = \frac{u\exp(-\alpha u)(1-\exp(-u))^{\beta-1}}{\sigma B(\alpha,\beta)}, \tag{11}$$

where $u = \exp\{-(x-\mu)/\sigma\}$.

The generalization given above was noted to allow greater flexibility of its tail. This is of great importance because most problems studied using Gumbel distribution are concerned with the tail behavior of one or more variables. Thus, in capturing the tail behavior more accurately, one would obtain improved estimation and prediction, which is what the beta Gumbel distribution provides.

When $\alpha = 1$ and $\beta = 1$, it was noted that the beta Gumbel distribution is equivalent to the Gumbel distribution. Other particular cases were identified using the special properties of the incomplete beta function ratio. The possible shapes of the PDF of the beta-Gumbel were also considered. It was noted that the PDF exhibited a single mode.The hazard rate function which is an important quantity that characterizes life phenomena was also obtained. An expression for the nth moment of the distribution was also obtained and from this, the variance, skewness, and kurtosis measures were calculated. It was noted that the beta-Gumbel distribution is much more flexible than the Gumbel distribution. The estimation of the parameters by method of maximum likelihood was also discussed.

2.3 The beta-Exponential Distribution

The beta exponential distribution was introduced by Nadarajah and Kotz [13] as a generalization of the exponential distribution. Just like the previous distributions studied, it was generated from the logit of a beta random variable. The distribution was generalized with the hope that it will attract wider applications in reliability.

The exponential distribution with parameter λ is known to be the most widely applied statistical distribution used for problems in reliability. Its PDF and CDF are given respectively as follows:

$$\begin{aligned} g(x) &= \lambda \exp(-\lambda x), \quad x \geq 0, \\ G(x) &= 1 - \exp(-\lambda x), \quad x \geq 0. \end{aligned} \tag{12}$$

The generalization of the exponential distribution - referred to as the beta exponential (BE) is obtained by letting $F(x)$ in Equation (6) be the CDF of the exponential distribution with parameter λ. Such that the CDF of the BE is given by

$$F(x) = B(1 - \exp(-\lambda x); \alpha, \beta), \tag{13}$$

where $B(1 - \exp(-\lambda x); \alpha, \beta) = \int_0^{1-\exp(-\lambda x)} t^{\alpha-1}(1-t)^{\beta-1} dt$ is the incomplete beta function.

The PDF $f(x)$ and the hazard rate $h(x)$ of BE are respectively given as

$$
\begin{aligned}
f(x) &= \frac{\lambda}{B(\alpha,\beta)} \exp(-\beta\lambda x)\{1-\exp(-\lambda x)\}^{\alpha-1}, \\
h(x) &= \frac{\lambda \exp(-\beta\lambda x)\{1-\exp(-\lambda x)\}^{\alpha-1}}{B(\exp(-\lambda x);\beta,\alpha)},
\end{aligned}
\tag{14}
$$

where $B(\exp(-\lambda x);\alpha,\beta) = \int_0^{\exp(-\lambda x)} t^{\alpha-1}(1-t)^{\beta-1}dt$ is also the incomplete beta function.

The exponentiated exponential distribution is a particular case of BE when $\beta = 1$. The exponential distribution (with parameter $\beta\lambda$) is another particular case of BE when we have $\alpha = 1$. Other particular cases were also given. The beta exponential distribution can be used as an improved model for failure time data. This is because it displays both increasing and decreasing failure rates and the shape of the failure rate function depends only on the parameter α.

The shapes of the PDF and the hazard rate function of the BE distribution were considered and it was noted that their shapes depend only on the parameter α. The PDF was noted to monotonically decrease if $\alpha < 1$. Also noted was that if $\alpha > 1$ then the PDF had a unique mode at $x = x_0$ with the PDF increasing for $x < x_0$ and the PDF decreasing for $x > x_0$. It was noted that if $\alpha < 1$, then the hazard rate function monotonically decreases with x. Likewise, if $\alpha > 1$, then the hazard rate function monotonically increases with x.

The moment generating function and the characteristics function were derived for the BE distribution. The nth moment of the BE distribution was obtained from the moment generating function and from this, the first four moments were obtained. The first three central moments, skewness and the kurtosis of the distribution were also obtained. It was noted that the skewness and the kurtosis depend only on α and β.

The mean deviation about the mean $\delta_1(X)$ and the mean deviation about the median $\delta_2(X)$ were also found. These were verified for the case when $\alpha = 1$. It was noted that the mean deviation about the mean reduces to that of the exponential distribution. The Renyi entropy of the distribution was obtained to be

$$\vartheta_R(\gamma) = -\log\lambda + \frac{1}{1-\gamma}\log\left[\frac{B(\gamma\alpha-\gamma+1,\gamma\beta)}{B^{\gamma}(\alpha,\beta)}\right], \quad \alpha,\beta,\gamma>0,\gamma\neq 1.$$

The possibility of the sum of two random variables with a BE distribution with common parameters α, β, and λ having the gamma distribution, and their ratio has the F distribution was also exploded. It was noted that they were possible for the case when $\alpha = 1$.

2.4 The beta-Weibull Distribution

The beta-Weibull distribution with four parameters was proposed by Lee et al [5]. Like the previously mentioned generalized distributions, it was also generated from the logit of a beta random variable. It was shown that the beta-Weibull distribution is unimodal and some results on the non-central moments were obtained. The parameters were estimated using the maximum likelihood technique and a likelihood ratio test was derived for the beta-Weibull distribution.

The Weibull distribution is known to have wide applications in many disciplines. Different generalizations of the Weibull distribution have also been studied. The Weibull-exponential distribution was proposed by Zacks [14]. Also introduced was a generalized Weibull distribution with an additional shape parameter by Mudholkar and Kollia [10]. An application of this to a model survival data was done by Mudholkar et al [11]. They showed that the distribution has increasing, decreasing, bathtub, and unimodal hazard functions. Various properties of the exponentiated were studied. An application of the exponentiated Weibull distribution on extreme value data was also done. The exponentiated exponential distribution was also proposed to be a special case of the exponentiated Weibull family.

The beta-Weibull distribution with parameters α, β, c and γ, has a PDF expressed as

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{c}{\gamma} \left(\frac{x}{\gamma}\right)^{c-1} \left[1 - e^{-(x/\gamma)^c}\right]^{\alpha-1} e^{-\beta(x/\gamma)^c}, \tag{15}$$

for $x > 0$, $\alpha, \beta, c, \gamma > 0$. The hazard function, the Renyi entropy and the Shannon's entropy for the beta-Weibull were also obtained. The following were noted for the beta-Weibull distribution:

- It has a constant hazard rate when $\alpha = c = 1$.
- It has a decreasing hazard rate when $\alpha c \leq 1$ and $c \leq 1$.
- It has an increasing hazard rate when $\alpha c \geq 1$ and $c \geq 1$.
- It has a bathtub hazard rate when $\alpha c < 1$ and $c > 1$.
- It has an upside down bathtub (or unimodal) hazard rate when $\alpha c > 1$ and $c < 1$.

This is unique amongst distributions, since the beta-Weibull distribution shows all the kinds of hazard rate one could have in any distribution.

It was also noted that the biases of the maximum loglikelyhood estimates from beta-Weibull distribution were smaller than those from exponentiated Weibull model with comparable standard errors when $\beta < 1$.

2.5 The beta-Pareto Distribution

The Pareto distribution is mainly used in modeling the heavy-tailed distributions that are include data on income distribution, city population size and size of firms. Recent applications of the Pareto distribution include data sets of earthquakes, forest fire areas, fault lengths on Earth and Venus, and on oil and gas fields' sizes.

There are several generalizations of the Pareto distribution. Among these is the generalized Pareto distribution (GPD) which was used by Pickands [6] when making statistical inferences about the upper tail of a distribution function. The GPD was found to be useful in modeling extreme value data due to its long tail feature. Also noted was that the Pareto distribution is a special case of GPD.

The CDF of the Pareto distribution with parameters k and θ is given as

$$F(x) = 1 - \left(\frac{x}{\theta}\right)^{-k}, \tag{16}$$

where $\theta, k > 0$, and $x \geqslant \theta$.

A four-parameter beta-Pareto distribution (BPD) was defined and studied by Akinsete et al [2]. Just like the previous distributions studied, it was generated from the logit of a beta random variable. If we let $F(x)$ be the CDF of the Pareto distribution in Equation (6), the PDF for the beta-Pareto random variable is given by

$$g(x) = \frac{k}{\theta B(\alpha,\beta)}\left\{1 - \left(\frac{x}{\theta}\right)^{-k}\right\}^{\alpha-1}\left(\frac{x}{\theta}\right)^{-k\beta-1}, \tag{17}$$

where $x \geqslant \theta$ and $\alpha, \beta, \theta, k > 0$.

The corresponding CDF for the beta-Pareto was obtained to be

$$G(x) = 1 - \frac{z^{\beta}}{B(\alpha,\beta)}\left\{\frac{1}{\beta} + \frac{1-\alpha}{\beta+1}z + \ldots + \frac{(1-\alpha)(2-\alpha)\ldots(n-\alpha)}{n!(\beta+n)}z^{n} + \ldots\right\}, \tag{18}$$

where $z = (x/\theta)^{-k}$.

It was noted that when $\alpha = \beta = 1$ the BPD reduces to the Pareto distribution with parameters k and θ. Also noted was that when $\alpha = 1$, the BPD reduces to the Pareto distribution with parameters $k\beta = c$ and θ. It was shown that the BPD reduces to the acrsince distribution, the beta-Weibull distribution and the exponential distribution with appropriate transformations.

It was noted that the BPD is unimodal. The hazard rate of the BPD was derived, and it was noted that this reduces to the hazard rate of the Pareto distribution when $\alpha = \beta = 1$. The BPD was noted to have unimodal failure rate when $\alpha > 1$ and a decreasing failure rate when $0 < \alpha \leq 1$. The moments of the BPD were obtained using the beta integral. Using this, the mean, variance, skewness and kurtosis were also obtained. The mean and variance were noted to be increasing functions of α and θ, while the skewness and kurtosis were noted to be decreasing functions of α, β and k, but independent of parameter θ.

The mean deviation from the mean, the mean deviation from the median, the Renyi entropy and the Shannon entropy were obtained. The method of maximum likelihood was also used to estimate the parameters.

2.6 The beta-Rayleigh Distribution

The Rayleigh distribution is a right skewed distribution, characterized by a scale parameter σ, and usually arises when a two-dimensional vector has its two orthogonal components that are normally and independently distributed with equal variance. Its PDF is given as

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{1}{2}(x/\sigma)^2}; \quad x \geqslant 0, \sigma > 0.$$

It was observed to be a special case of the Weibull distribution with parameters α and β, where $\alpha = 2$. It is also a special case of the chi distribution with parameter $\nu = 2$. The CDF of the Rayleigh distribution is given as

$$F(x) = 1 - e^{-\frac{1}{2}(x/\sigma)^2}; \quad x \geqslant 0, \sigma > 0.$$

Akinsete and Lowe [1] defined the PDF for the beta-Rayleigh random variable to be

$$g(x) = \frac{x}{\sigma^2 B(\alpha,\beta)} e^{-\frac{1}{2}(\frac{x}{\sigma})^2 \beta} (1 - e^{-\frac{1}{2}(\frac{x}{\sigma})^2})^{\alpha-1}; \quad x \geqslant 0. \tag{19}$$

This was obtained by letting $F(x)$ in Equation (6) to be the CDF of the Rayleigh distribution. From the above PDF the distribution function of the BRD was obtained to be

$$G(x) = 1 - \frac{\exp\{-\frac{\alpha}{2}(x/\sigma)^2\}}{\alpha B(\alpha,\beta)} \times {}_2F_1(\alpha, 1-\beta; 1+\alpha; \exp\{-\frac{\alpha}{2}(x/\sigma)^2\}), \tag{20}$$

where ${}_2F_1(\alpha, 1-\beta; 1+\alpha; \exp\{-\frac{\alpha}{2}(x/\sigma)^2\})$ is the Gauss Hypergeometric function.

It was noted that if X is a random variable with a BRD with parameters α, β and σ, then $Y = 1 - e^{\frac{-1}{2}(\frac{X}{\sigma})^2}$ has the beta density with parameters α and β. Also if you set $\alpha = \beta = 1$, BRD reduces to the Rayleigh distribution with parameter σ. When $\alpha = 1$, the BRD is noted to reduce to the Rayleigh distribution with parameter $k = \frac{\sigma}{\sqrt{\beta}}$.

The nth raw moment of a random variable X with a BRD is given by

$$\mu_n' = \frac{(\sigma\sqrt{2})^n (n/2)!}{B(\alpha,\beta)} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} \frac{1}{(\beta+k)^{n/2+1}}.$$

The variance, skewness and kurtosis were obtained from this expression. The parameters of the BRD were estimated by the method of maximum likelihood estimation.

3 The beta-Maxwell Distribution

We will look at some results and discuss them.

3.1 Definition

Suppose $F(x)$ denote the cumulative distribution function (CDF) of a random variable X. The CDF for a generalized class of distribution for the random variable X, according to Eugene et al. [9], is generated by applying the inverse CDF to a beta distributed random variable to obtain,

$$G(x) = \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}dt; \quad 0 < \alpha, \beta < \infty. \tag{21}$$

The probability density function (PDF) corresponding to the above G(x) is given by

$$g(x) = \frac{1}{B(\alpha,\beta)}[F(x)]^{\alpha-1}[1-F(x)]^{\beta-1}F'(x). \tag{22}$$

Like we have discussed in the previous chapter, the density in Equation (22) has been studied by many authors by assuming various types of cumulative distribution functions $F(x)$. Now, suppose that $F(x)$ is the cumulative distribution function of the Maxwell distribution as given in Equation (2). We can show that the function

$$g(x) = \frac{1}{B(\alpha,\beta)}\left[\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x^2}{2a^2})\right]^{\alpha-1}\left[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x^2}{2a^2})\right]^{\beta-1}\sqrt{\frac{2}{\pi}}\frac{x^2e^{-\frac{x^2}{2a^2}}}{a^3}; \tag{23}$$

for $0 \leq x < \infty$ is a probability density function. Where $\gamma(a,b)$ is an incomplete gamma function.

Using the appropriate transformation, we will show that $\int_0^\infty g(x)dx = 1$.

Let $A = \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{x^2}{2a^2})$ such that $\frac{dA}{dx} = \sqrt{\frac{2}{\pi}}\frac{x^2\exp\{-\frac{x^2}{2a^2}\}}{a^3}$. So we have,

$$\begin{aligned}
g(x) &= \frac{1}{B(\alpha,\beta)}A^{\alpha-1}(1-A)^{\beta-1}\frac{dA}{dx} \\
\int_{-\infty}^{\infty} g(x)dx &= \int_{-\infty}^{\infty}\frac{1}{B(\alpha,\beta)}A^{\alpha-1}(1-A)^{\beta-1}dA \\
&= \int_{0}^{1}\frac{1}{B(\alpha,\beta)}A^{\alpha-1}(1-A)^{\beta-1}dA.
\end{aligned} \tag{24}$$

Since $0 \leq x < \infty$ then $0 \leq A < \infty$. Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &= \frac{1}{B(\alpha,\beta)}\int_{0}^{1} A^{\alpha-1}(1-A)^{\beta-1}dA \\
&= \frac{1}{B(\alpha,\beta)}B(\alpha,\beta) \\
&= 1.
\end{aligned} \tag{25}$$

This verifies that $g(x)$ is indeed a probability density function of a continuous distribution. We will refer to a random variable X with the probability density function $g(x)$ as the beta-Maxwell distribution. If a random variable X has the beta-Maxwell distribution with the density given in Equation (23), we shall write $X \sim BM(\alpha, \beta, a)$. The graphs of the probability density functions for various values of α, β and a are shown in Figure 1.

Given that $X \sim BM(\alpha, \beta, a)$ its distribution function may be expressed as

$$\begin{aligned}
G(x) = P(X \leq x) &= \int_{0}^{x}\frac{1}{B(\alpha,\beta)}\left[\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{t^2}{2a^2})\right]^{\alpha-1} \\
&\times \left[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{t^2}{2a^2})\right]^{\beta-1}\sqrt{\frac{2}{\pi}}\frac{t^2e^{-\frac{t^2}{2a^2}}}{a^3}dt.
\end{aligned} \tag{26}$$

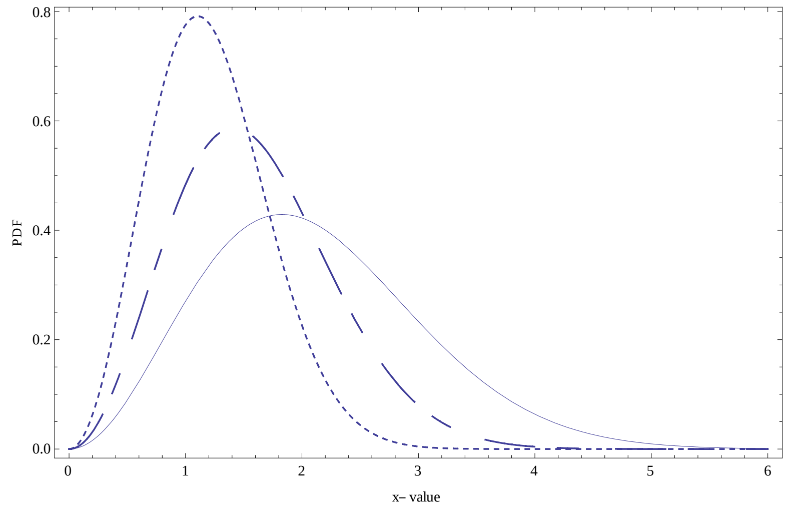

Figure 1: A graph of the pdf of Beta-Maxwell for different values of α, β and a (–: $\alpha = 1$,$\beta = 1$, $a = 1$; -: $\alpha = 1$, $\beta = 0.5$, $a = 1$; ...: $\alpha = 1$, $\beta = 2$, $a = 1$)

Let $z = \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{t^2}{2a^2})$ and $\frac{dz}{dt} = \sqrt{\frac{2}{\pi}}\frac{t^2 e^{-\frac{t^2}{2a^2}}}{a^3}$. We can simplify as follows

$$\begin{aligned} G(x) &= \int_0^A \frac{1}{B(\alpha,\beta)} z^{\alpha-1}(1-z)^{\beta-1} dz \\ &= \frac{1}{B(\alpha,\beta)} \int_0^A z^{\alpha-1}(1-z)^{\beta-1} dz \\ &= \frac{B(A;\alpha,\beta)}{B(\alpha,\beta)}, \end{aligned} \tag{27}$$

where $B(A;\alpha,\beta)$ is an incomplete beta function with $A = \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{x^2}{2a^2})$. The relationship between the incomplete beta function and series enable us to write Equation (27) alternatively as

$$G(x) = \frac{A^\alpha}{B(\alpha,\beta)}\left\{\frac{1}{\alpha} + \frac{1-\beta}{\alpha+1}A + \ldots + \frac{(1-\beta)(2-\beta)\ldots(n-\beta)A^n}{n!(\alpha+n)}\right\}.$$

The graph of the cumulative distribution functions for various values of α, β and a are shown in Figure 2.

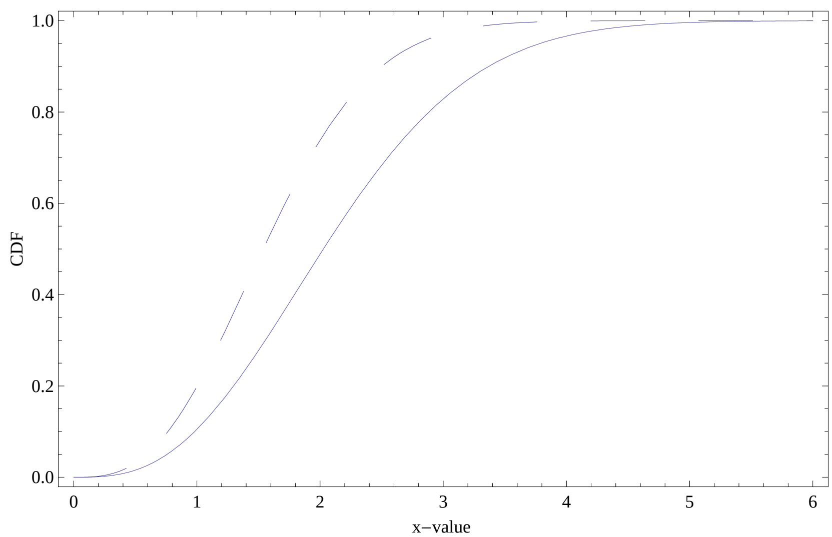

Figure 2: A graph of the cdf of Beta-Maxwell for different values of α, β and a

3.2 Some properties of the beta-Maxwell distribution

3.2.1 Limit behavior

LEMMA 1: The limit of beta-Maxwell density as $x \to \infty$ is 0 and the limit as $x \to 0$ is 0.

Proof: These can be shown as follows by taking the limit of the beta-Maxwell density in Equation (23).

For $x \to \infty$:

$$
\begin{aligned}
\lim_{x\to\infty} g(x) &= \lim_{x\to\infty} \frac{1}{B(\alpha,\beta)} \left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2a^2}\right)\right]^{\alpha-1} \left[1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2a^2}\right)\right]^{\beta-1} \\
&\times \sqrt{\frac{2}{\pi}} \frac{x^2 \exp\{-\frac{x^2}{2a^2}\}}{a^3} \\
&= \lim_{x\to\infty} \left\{ \frac{1}{B(\alpha,\beta)} \left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2a^2}\right)\right]^{\alpha-1} \right\} \\
&\times \lim_{x\to\infty} \left[1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2a^2}\right)\right]^{\beta-1} \times \lim_{x\to\infty} \sqrt{\frac{2}{\pi}} \frac{x^2 \exp\{-\frac{x^2}{2a^2}\}}{a^3} \\
&= 0.
\end{aligned}
$$

This is because,

$$
\lim_{x\to\infty} \sqrt{\frac{2}{\pi}} \frac{x^2 \exp\{-\frac{x^2}{2a^2}\}}{a^3} = 0.
$$

Likewise, as $x \to 0$, we immediately see that by replacing the limit as $x \to \infty$ with $x \to 0$, the above limit expression becomes zero. Since in this case,

$$
\lim_{x\to 0} \sqrt{\frac{2}{\pi}} \frac{x^2 \exp\{-\frac{x^2}{2a^2}\}}{a^3} = 0.
$$

The above indicates that the beta-Maxwell distribution has a mode.

3.2.2 Hazard Rate Function

The hazard rate function of a random variable X with the probability density function $g(x)$ and a cumulative distribution function $G(x)$ is given by

$$h(x) = \frac{g(x)}{1 - G(x)}.$$

For the beta-Maxwell distribution, the $g(x)$ and $G(x)$ are given respectively by equations (23) and (27). Using these expressions, the hazard rate function may be expressed as

$$h(x) = \frac{A^{\alpha-1}(1-A)^{\beta-1}A'}{B(\alpha,\beta) - B(A;\alpha,\beta)}, \tag{28}$$

where $A = \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{x^2}{2a^2})$.

LEMMA 2: The limit of the beta-Maxwell hazard rate function as $x \to \infty$ is 0 and the limit as $x \to 0$ is 0.

Proof: This can be shown straightforward by taking the limit of the beta-Maxwell hazard rate function in Equation (28) following the same steps used for the limit of the beta-Maxwell density function.

3.3 Parameter Estimations

The estimations of the parameters of the beta-Maxwell distribution are discussed by the method of maximum likelihood estimation. Using Equation (23), the maximum log-likelihood function of the beta-Maxwell distribution may be expressed as

$$
\begin{aligned}
L(x|\alpha,\beta,a) &= \left(\frac{2}{\pi}\right)^{n/2} \frac{\prod_{j=1}^{n} x_j^2 \exp\{-\frac{1}{2a^2}\sum_{j=1}^{n} x_j^2\}}{a^{3n}[B(\alpha,\beta)]^n} \\
&\times \prod_{j=1}^{n}\left[\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})\right]^{\alpha-1} \times \left[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})\right]^{\beta-1}, \\
lnL(x|\alpha,\beta,a) &= \frac{n}{2}\ln\left(\frac{2}{\pi}\right) - 3n\ln(a) - n\ln\left(B(\alpha,\beta)\right) + \sum_{j=1}^{n} 2\ln(x_j) \\
&- \frac{1}{2a^2}\sum_{j=1}^{n} x_j^2 + (\alpha-1)\sum_{j=1}^{n} ln\left[\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})\right] \\
&+ (\beta-1)\sum_{j=1}^{n} ln\left[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})\right].
\end{aligned}
\tag{29}
$$

Differentiating Equation (29) with respect to α, β and a respectively, and setting the results equal to zero we have,

$$
\begin{aligned}
\frac{\partial lnL(x|\alpha,\beta,a)}{\partial\alpha} &= -n\Psi(\alpha) + n\Psi(\alpha+\beta) + \sum_{j=1}^{n} ln\left[\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})\right] \\
&= 0 \\
\frac{\partial lnL(x|\alpha,\beta,a)}{\partial\beta} &= -n\Psi(\beta) + n\Psi(\alpha+\beta) + \sum_{j=1}^{n} ln\left[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})\right] \\
&= 0 \\
\frac{\partial lnL(x|\alpha,\beta,a)}{\partial a} &= -\frac{3n}{a} + \frac{1}{a^3}\sum_{j=1}^{n} x_j^2 - (\alpha-1)\sum_{j=1}^{n} \frac{x_j^3\exp\{-\frac{x_j^2}{2a^2}\}}{a^4\sqrt{2}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})} \\
&+ (\beta-1)\sum_{j=1}^{n}\sqrt{\frac{2}{\pi}}\frac{x_j^3\exp\{-\frac{x_j^2}{2a^2}\}}{a^4[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})]} \\
&= 0
\end{aligned}
\tag{30}
$$

where $\Psi(k)$ is the logarithmic derivative of the gamma function.

Solving the above equations simultaneously for α, β and a gives us their respective estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{a}$. Also finding the derivatives of these equations, gives us the diagonal elements of the Fisher's information matrix. They are as follows:

$$\begin{aligned}
\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\alpha^2} &= -n\Phi'(\alpha)+n\Phi'(\alpha+\beta) \\
\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\alpha\partial\beta} &= n\Phi'(\alpha+\beta)=\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\beta\partial\alpha} \\
\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\beta^2} &= -n\Phi'(\beta)+n\Phi'(\alpha+\beta) \\
\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\alpha\partial a} &= -\frac{1}{a^4\sqrt{2}}\sum_{j=1}^{n}\frac{x_j^3\exp\{-\frac{x_j^2}{2a^2}\}}{\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})} \\
\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\beta\partial a} &= \sqrt{\frac{2}{\pi}}\frac{x_j^3\exp\{-\frac{x_j^2}{2a^2}\}}{a^4}\sum_{j=1}^{n}\frac{1}{1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})} \\
\frac{\partial^2 lnL(x|\alpha,\beta,a)}{\partial\ a^2} &= \frac{3n}{a^2}-\frac{3}{a^4}\sum_{j=1}^{n}x_j^2-(\alpha-1)\sum_{j=1}^{n}\frac{x_j^6\exp\{-\frac{x_j^2}{a^2}\}}{2a^8(\gamma(\frac{3}{2},\frac{x_j^2}{2a^2}))^2} \\
&+ (\alpha-1)\sum_{j=1}^{n}\frac{2\sqrt{2}x_j^3\exp\{-\frac{x_j^2}{2a^2}\}}{a^5\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})} \\
&- (\alpha-1)\sum_{j=1}^{n}\frac{x_j^5\exp\{-\frac{x_j^2}{2a^2}\}}{\sqrt{2}a^7\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})} \\
&+ (\beta-1)\sum_{j=1}^{n}\frac{x_j^5\exp\{-\frac{x_j^2}{2a^2}\}}{a^7[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})]} \\
&- (\beta-1)\frac{4x_j^3\exp\{-\frac{x_j^2}{2a^2}\}}{a^5[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})]} \\
&+ (\beta-1)\sum_{j=1}^{n}\frac{\sqrt{\frac{2}{\pi}}x_j^7\exp\{-\frac{x_j^2}{2a^2}\}}{a^8[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2},\frac{x_j^2}{2a^2})]^2}.
\end{aligned} \tag{31}$$

The Fisher's information matrix is useful for the purpose of calculating the interval estimates, asymptotic variances and covariances, and tests of hypothesis of α, β and a.

4 Conclusion

We studied the use of beta distribution in the obtaining beta-compounded distributions. We noted some of the results which were obtained in doing this. Following the pattern of these distributions, we have defined the beta-Maxwell distribution and studied some of its properties. The Maxwell distribution is said to be used in physics and chemistry. It is therefore expected that the beta-Maxwell distribution would better describe situations where the Maxwell distribution has been found useful. Further work concerns finding real life data that would demonstrate the capabilities of the beta-Maxwell distribution. We also attempted to find the moments of the beta-Maxwell distribution which would have lead to its mean, variance, skewness and kurtosis, however this was not possible analytically. Perhaps, a numerical approximation could be obtained.

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