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Olusegun Michael Otunuga
Marshall University, otunuga@marshall.edu

Gangaram S. Ladde

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Stochastic Modeling of Energy Commodity Spot Price Processes with Delay in Volatility

Olusegun M. Otunuga
Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Avenue, Tampa, FL 33620, USA.

Gangaram S. Ladde
Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Avenue, Tampa, FL 33620, USA.

Abstract

Employing basic economic principles, we systematically develop both deterministic and stochastic dynamic models for the log-spot price process of energy commodity. Furthermore, treating a diffusion coefficient parameter in the non-seasonal log-spot price dynamic system as a stochastic volatility functional of log-spot price, an interconnected system of stochastic model for log-spot price, expected log-spot price and hereditary volatility process is developed. By outlining the risk-neutral dynamics and pricing, sufficient conditions are given to guarantee that the risk-neutral dynamic model is equivalent to the developed model. Furthermore, it is shown that the expectation of the square of volatility under the risk-neutral measure is a deterministic continuous-time delay differential equation. The presented oscillatory and non-oscillatory results exhibit the hereditary effects on the mean-square volatility process. Using a numerical scheme, a time-series model is developed to estimate the system parameters by applying the Least Square optimization and Maximum Likelihood techniques. In fact, the developed time-series model includes the extended GARCH model as a special case.

Keywords: Delayed Volatility, Stochastic Interconnected Model, GARCH model, Non-seasonal Log-Spot Price Process Dynamic, Risk-Neutral Model, Oscillatory, Non-Oscillatory

1. Introduction

In a real world situation, the expected spot price of energy commodities and its measure of variation are not constant. This is because of the fact that a spot price is subject to random environmental perturbations. Moreover, some statistical studies of stock price (Bernard & Thomas, 1989) raised the issue of market's delayed response. This indeed causes the price to drift significantly away from the market quoted price. It is well recognized that time-delay models in economics (Frisch R. & Holmes , 1935; Kalecki, 1935; Tinbergen, 1935) are more realistic than the models without time-delay. Discrete-time stochastic volatility models (Bollerslev, 1986; Engle, 1982) have been developed in economics. Recently, a survey paper by Hansen and Lunde (2001) has estimated these types of models and concluded that the performance of the GARCH (1,1) model is better than any other model. Furthermore, Cox-Ingersoll-Ross (CIR) developed a mean reverting interest rate model that was based on the mean-level interest rate with exponentially weighted integral of past history of interest rate, the relationship between level dependent volatility and the square root of the interest rate (Cox, Ingersoll & Ross, 1985). Employing the Ornstein Uhlenbeck (1930) and Cox-Ingersoll-Ross (CIR) (1985) processes, Heston developed a stochastic model for the volatility of stock spot asset. Recently (Hobson & Rogers, 1998), a continuous time stochastic volatility models have been generalized.

In this work, using basic economic principles, we systematically develop both deterministic and stochastic dynamic models for the log-spot price process. In addition, by treating a diffusion coefficient parameter in the non-seasonal log- spot price dynamic system as a stochastic volatility functional of log-spot price, a stochastic interconnected model for system of log-spot price, expected log-spot price and hereditary volatility processes is developed.

Introducing a numerical scheme, a time-series model is developed, and it is utilized to estimate the system parameters. The organization of the paper is as follows:

In Section 2, we develop a stochastic interconnected model for energy commodity spot price and volatility processes. An example is outlined in the context of Henry Hub Natural gas daily Spot price from 1997 to 2011. In Section 3, we obtain closed form solutions of the log-spot and the expected log of spot prices. In Section 4, by outlining the risk-neutral dynamics of price process, sufficient conditions are given to ensure that the presented risk-neutral dynamic model is equivalent to the developed model in Section 2. Furthermore, it is shown that the risk-neutral dynamic model is equivalent to the developed model in Section 2. Furthermore, it is shown that the mean of the square of volatility under the risk-neutral measure is a deterministic continuous-time delay differential equation. In addition, sufficient conditions are also given to investigate both the oscillatory and non-oscillatory behavior of the expected value of square of volatility (Ladde, 1979; Ladde 1977). In Section 5, numerical scheme is used to develop a time-series model. Using the Least Squares optimization and Maximum Likelihood techniques, we outline the parameter estimations for our model.

2. Model Derivation

We denote $S(t)$ to be the spot price for a given energy commodity at a time *t*. Since the price of energy commodity is non-negative. To minimize ambiguity and for the sake of simplicity, it is expressed as an exponential function of the following form;

$$S(t) = \exp(x_2(t) + f(t)), \tag{1}$$

where $x_2(t)$ stands for the non-seasonal log of the spot price at time *t*, $f(t)$ is the price at *t* influenced by the seasonality, and it is considered as a Fourier series comprising of linear combinations of sine and cosine functions;

$$f(t) = A_0 + \sum_{k=1}^{N} \left(A_k \cos\left[\frac{2\pi kt}{P}\right] + B_k \sin\left[\frac{2\pi kt}{P}\right] \right), \tag{2}$$

where P, A_0, A_k, B_k, $k = 1 \ldots N$ are all constant parameters. P is the period which represents the number of trading days in a year. Without loss in generality, we choose $N = 2$. By modeling the seasonal term this way, we are able to account for the peak season high price and off peak season low price of gas.

We present the dynamics for the spot price process.

2.1. Deterministic non-seasonal log-Spot price dynamic model

Under the basic economic principles of demand and supply processes, the price of a energy commodity will remain within a given finite upper bound. Let $\kappa > 0$ be the expected upper limit of x_2. In a real world situation, the non-seasonal log of spot price is governed by the spot price dynamic process. This leads to a development of dynamic model for the non-seasonal price process x_2. In this case, $\kappa - x_2(t)$ characterizes the market potential of non-seasonal log-spot price process per unit of time at a time *t*. This market potential is influenced by the underlying market forces on the non-seasonal log of spot price process. This leads to the following principle regarding the dynamic of non-seasonal log-spot price process of energy goods. The change in non-seasonal log of spot price of the energy commodity $\Delta x_2(t) = x_2(t + \Delta t) - x_2(t)$ over the interval of length $|\Delta t|$ is directly proportional to the market potential price, that is:

$$\Delta x_2(t) \propto (\kappa - x_2(t))\Delta t . \tag{3}$$

This implies

$$dx_2(t) = \gamma(\kappa - x_2(t))dt, \tag{4}$$

where γ is a positive constant of proportionality, $dx_2(t)$ and dt are differentials of $x_2(t)$ and t, respectively. From this mathematical model, we note that as the non-seasonal log price, $x_2(t)$ falls below the expected price κ, the market potential, $\kappa - x_2(t)$ is positive. Hence $x_2(t)$ is increasing at the constant rate γ per unit size of $\kappa - x_2(t)$ per unit time at a time t. On the other hand, if the non-seasonal log price $x_2(t)$ is above the expected price κ, then $\kappa - x_2(t)$ is negative and hence $x_2(t)$ decreases at the rate γ per unit size per unit time at a time t.

From (3), we note that the steady-state or equilibrium state non-seasonal log of spot price is given by

$$\kappa - x_2^* = 0. \tag{5}$$

In the real world situation, the expected price of the non-seasonal log spot price κ is not a constant parameter. Therefore, we consider the expected non-seasonal log of spot price to be the mean of non-seasonal log spot price, $x_2(t)$ at time t, and it is denoted by $x_1(t)$. Under this assumption, (4) reduces to

$$dx_2(t) = \gamma(x_1(t) - x_2(t))\mathrm{d}t. \tag{6}$$

Moreover, in order to preserve the equilibrium of non-seasonal log spot price ($\kappa = x_2^*$), we further assume that the mean of non-seasonal spot price process is operated under the principle described by (3).

$$\Delta x_1(t) \propto \big(\kappa - x_1(t)\big)\Delta t \tag{7}$$

and hence

$$dx_1(t) \;=\; \mu\big(\kappa - x_1(t)\big)dt, \tag{8}$$

where μ is a positive constant of proportionality.

From (6) and (7), the mathematical model for the deterministic non-seasonal spot price process is described by the following system of differential equations:

$$\begin{aligned} dx_1(t) &= \mu\big(\kappa - x_1(t)\big)dt \\ dx_2(t) &= \gamma\Big(x_1(t) - x_2(t)\Big)dt. \end{aligned} \tag{9}$$

2.2. Stochastic non-seasonal log-spot price dynamic model

We note that in (3), κ is not just the time-varying deterministic log of spot price, instead it is a stochastic process describing random environmental perturbations as follows:

$$\kappa \;=\; x_1(t) \;+\; e_2(t) \tag{10}$$

where $x_1(t)$ is the mean of κ and $e_2(t)$ is the white noise process. From this and following argument used in (Ladde A. G. & Ladde G. S.,2013), (4) can be rewritten as:

$$\begin{aligned} dx_2(t) &= \gamma(x_1(t) + e_2(t) - x_2(t))dt \\ &= \gamma(x_1(t) - x_2(t))dt + \gamma e_2(t)dt \\ &= \gamma(x_1(t) - x_2(t))dt + \sigma\Big(t, x_2(t)\Big)dW_2(t). \end{aligned} \tag{11}$$

where $\sigma\Big(t, x_2(t)\Big)dW_2(t) \;=\; \gamma e_2(t)dt$ and $dW_2(t) \sim N(0, dt)$.

Again, following the argument used in the derivation of (11), the dynamic from (8) reduces to

$$dx_1(t) \;=\; \mu(\kappa - x_1(t))dt \;+\; \delta dW_1(t) \tag{12}$$

where $\delta > 0$ is a constant and $dW_1(t) \sim N(0, dt)$.

From (11) and (12), the mathematical model for the stochastic non-seasonal spot price process is described by the following system of differential equations:

$$dx_1 = \mu(\kappa - x_1)dt + \delta dW_1(t),$$
$$dx_2 = \gamma(x_1 - x_2)dt + \sigma(t, x_2)dW_2(t). \qquad (13)$$

2.3. Continuous Time Stochastic Volatility Model with Delay

When considering energy commodities, the measure of variation of the spot price process under random environmental perturbations is not predictable, because it depends on non-seasonal log of spot price. Bernard and Thomas (1989) in their work raised the issue of market's delayed response. They observed changes in drift returns that lead to two possible explanations. First explanation suggests that a part of the price influence response to new information is delayed. The second explanation suggests that researchers fail to adjust fully a raw return for risks, because the capital-asset-pricing model is used to calculate the abnormal return that is either incomplete or incorrect. In this paper, we incorporate the past history of non-seasonal log of spot price in the coefficient of diffusion parameter, that is, the volatility σ of the spot price that follows the GARCH model (Yuriy, Anatoliy & Jianhong, 2005). It is assumed that the measure of variation of random environmental perturbations on x_1 is constant. Under these assumptions, we propose an interconnected mean-reverting non-seasonal stochastic model for mean log-spot price, log-spot price, and volatility as follows:

$$dx_1 = \mu(\kappa - x_1)dt + \delta dW_1(t), \quad x_1(t_0) = x_{01}$$

$$dx_2 = \gamma(x_1 - x_2)dt + \sigma(t, x_2)dW2(t), \quad x_2(t_0) = x_{02} \qquad (14)$$

$$d\,\sigma^2(t,x_2) = \left[\alpha + \beta\left[\int_{t-\tau}^{t}\sigma(s,x)e^{-\gamma(t-s)}dW_2(s) + \delta\int_{t-\tau}^{t}\phi(s,t)dW_1(s)\right]^2 + c\,\sigma^2(t,x)\right]dt,$$
$$\sigma^2(t_0,x_{02}) = \sigma_0^2$$

where

$$\phi(a,b) = \frac{\gamma}{\mu-\gamma}\left(e^{-\gamma(b-a)} - e^{-\mu(b-a)}\right),\ \gamma,\ \mu \text{ are defined in}(4) \text{ and } (8), a, b \in \Re. \qquad (15)$$

For the sake of completeness, we assume the following:

$\boldsymbol{H_1}$: $x_{2t}(\theta) = x_2(t+\theta)$, $\theta \in [-\tau, 0]$, γ, μ, $\delta \in \Re^+$, $\alpha, c \in \Re$, (we will later show that $-2 < c < 0$), $\sigma : [0, T] \times C \to \Re$ is a continuous mapping, where C is the Banach space of continuous functions defined on $[-\tau, 0]$ into$\Re$ and equipped with the supremum norm; $W_1(t)$ and $W_2(t)$ are standard Wiener processes definedon a filtered prob ability space$(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$,where the filtration function $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, and each for $t \geq 0$, $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets. We know that system (14) can be re-written as

$$d\boldsymbol{x} = [A\,\boldsymbol{x} + \boldsymbol{p}]\ dt + \Sigma(t, x_2)\ d\boldsymbol{W}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \qquad (16)$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} -\mu & 0 \\ \gamma & -\gamma \end{bmatrix}, \boldsymbol{p}(t) = \begin{bmatrix} \mu\kappa \\ 0 \end{bmatrix}, \Sigma(t, x_2(t)) = \begin{bmatrix} \delta & 0 \\ 0 & \sigma(t, x_{2t}) \end{bmatrix},$$

$$\boldsymbol{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix},$$

Moreover, (16) can be considered as a system of nonlinear It$\hat{o}$-Doob type stochastic perturbed system of the following deterministic linear system of differential equations

$$dx = \boldsymbol{A}x dt. \tag{17}$$

In the following, we present an illustration to justify the structure of log spot price dynamic model.

2.4. Example

We present an example to illustrate the above described interconnected stochastic dynamic model for non-seasonal log spot price of energy commodity under the influence of random perturbations on mean-level and delayed volatility. We consider the Henry Hub Natural Gas Daily spot price from 1997 to 2011.

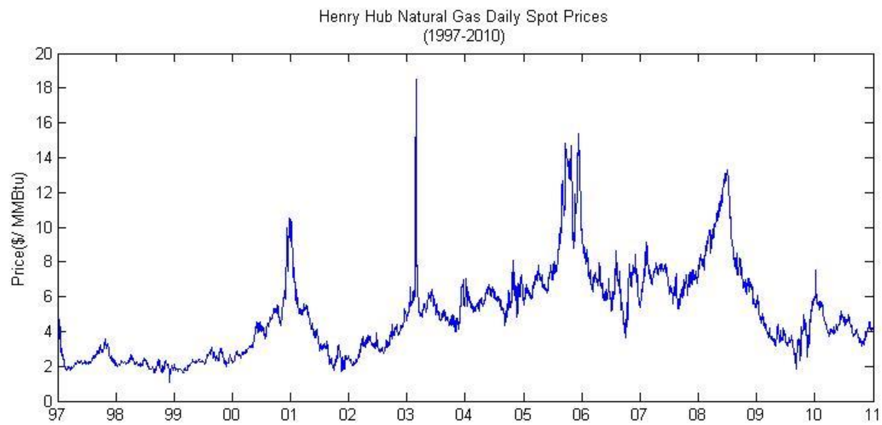

Figure 1: Plot of Henry Hub Daily Natural Gas Spot Prices, 1997-2011

We can clearly see that

- Price process appear as being randomly driven and clearly non-negative
- There is a tendency of spot prices to move back to their long term level (mean reversion).
- There are sudden large changes in spot prices (jumps/spikes).
- There is an unpredictability of spot price volatility

Table 1: Descriptive statistics of Henry Hub daily natural gas spot prices, 1997-2010

	Mean	Variance	Skewness	Kurtosis	Minimum	Maximum
S_t	4.9519	2.4966	1.0391	4.3491	1.05	18.48
$S_{t+1} - S_t$	-0.0001142	0.3189	-0.7735	191.8911	-8.01	6.50
$\ln(S_t)$	1.4754	0.5048	-0.0465	2.1540	0.0488	2.9167
$\ln[S_{t+1}/ S_t]$	2.8485e-5	0.0473	0.4814	22.0473	-0.56	0.5657

A summary of the statistic is presented in Table 1. We find that $\ln\left[\frac{S(t+1)}{S(t)}\right]$ has the smallest variance. Thus, it suggests a good candidate for our modeling. Hence, we use the logarithmic price, rather than the raw price data for our model.

3. Closed Form Solution

In this section, we find the solution representation of (16) in terms of the solution of unperturbed system of differential deterministic (17). This is achieved by employing method of variation of constants parameter (Ladde etal, 2013)

Theorem 1. ***(Closed Form Solution)***

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0; \boldsymbol{x}_0)$ and $\boldsymbol{y}(t, t_0; \boldsymbol{x}_0) = \boldsymbol{\Phi}(t, t_0)\, \boldsymbol{x}_0$ be the solutions of the perturbed and unperturbed system of differential equations (16) and (17), respectively. Then

$$\boldsymbol{x}(t) = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0,t) & e^{-\gamma(t-t_0)} \end{bmatrix} \boldsymbol{x}_0 + \begin{bmatrix} \kappa\left(1 - e^{-\mu(t-t_0)}\right) \\ \omega(t_0,t) \end{bmatrix} + \int_{t_0}^{t} \begin{bmatrix} \delta e^{-\mu(t-s)} dW_1(s) \\ \delta\phi(s,t)dW_1(s) + \sigma(s,x_{2s})e^{-\gamma(t-s)}dW_2(s) \end{bmatrix}. \tag{18}$$

where

$$\boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \tag{19}$$

$$\omega(a,b) = \kappa\left(\left[1 - e^{-\gamma(b-a)}\right] - \phi(a,b)\right), \quad a,b \in \Re, \tag{20}$$

and ϕ is defined in (15); the fundamental matrix solution, $\boldsymbol{\Phi}(t,t_0)$ of (17) is given by

$$\Phi(t,t_0) = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0,t) & e^{-\gamma(t-t_0)} \end{bmatrix} \tag{21}$$

Proof. The result follows by imitating the eigenvalue type method described in (Ladde A. G. & Ladde G. S, 2012; Ladde A. G. & Ladde G. S, 2013). Therefore

$$x_1(t) = e^{-\mu(t-t_0)}x_{01} + \kappa\left(1 - e^{-\mu(t-t_0)}\right) + \delta \int_{t_0}^{t} e^{-\mu(t-s)} dW_1(s), \tag{22}$$

$$x_2(t) = \phi(t_0,t)x_{01} + e^{-\gamma(t-t_0)}x_{02} + \omega(t_0,t) + \delta \int_{t_0}^{t} \phi(s,t)dW_1(s) + \int_{t_0}^{t} \sigma(s,x_2)e^{-\gamma(t-s)}dW_2(s). \tag{23}$$

In the following, we present the statistical properties of the solutions (22) and (23).

Theorem 2. Under the hypothesis of Theorem 1, we have

$$var[\boldsymbol{x}(t)] = \int_{t_0}^{t} \begin{bmatrix} \delta^2 e^{-2\mu(t-s)} ds \\ \delta^2\phi^2(s,t) + E\sigma^2(s,x(s))e^{-2\gamma(t-s)}ds \end{bmatrix}$$

Moreover,

$$\lim_{t\to\infty} \mathbb{E}\left[\, \boldsymbol{x}(t)\right] = \begin{bmatrix} \kappa \\ \kappa \end{bmatrix}$$

$$\lim_{t\to\infty} \boldsymbol{var}\,[\boldsymbol{x}(t)] = \begin{bmatrix} \dfrac{\delta^2}{2\mu} \\ \lim\limits_{t\to\infty} \dfrac{\mathbb{E}\,(\sigma^2(t,x_2(t))}{2\,\gamma} + \dfrac{\delta^2}{2\mu}\left[\dfrac{\gamma}{\mu+\gamma}\right] \end{bmatrix}.$$

Proof. From (18), we observe that

$$\mathbb{E}\,[\boldsymbol{x}(t)] = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0,t) & e^{-\gamma(t-t_0)} \end{bmatrix} \boldsymbol{x}_0 + \begin{bmatrix} \kappa\left(1 - e^{-\mu(t-t_0)}\right) \\ \omega(t_0,t) \end{bmatrix}.$$

Hence,
$$\mathbb{E}\,[x_1(t)] = x_{01}e^{-\mu(t-t_0)} + \kappa(1 - e^{-\mu(t-t_0)})$$

$$\mathbb{E}\,[x_2(t)] = x_{02}e^{-\gamma(t-t_0)} + \phi(t_0,t)x_{01} + \omega(t_0,t)$$

$$var\,[x_1(t)] = \frac{\delta^2}{2\mu}(1 - e^{-2\mu(t-t_0)})$$

$$var[x_2(t)] = \int_{t_0}^{t} \delta^2\phi^2(s,t) + \mathbb{E}\,\sigma^2(s,x(s))e^{-2\gamma(t-s)}\,ds.$$

The result follows by taking the limits as $t \rightarrow \infty$.

Remark 1. From Theorem 2, we observe that on the long-run, the mean-level of $x_1(t)$ and $x_2(t)$ are the same and it is given by κ.

4. Risk-Neutral Dynamics and Pricing

In order to minimize the risk of usage of mathematical model (16), we incorporate the risk neutral measure. From the dynamic nature of (16), it is known [Dai & Singelton, 2000; Duffie & Kan, 2000) that this model has affine multi-factor structure. In the following, we present a risk neutral measure induced by this type of model. This indeed leads to a risk neutral dynamic model with respect to (16). The general market price of risk $\Theta(t) = \left(\Theta_1(t),\ldots,\Theta_n(t)\right)$ with respect to the stochastic differential equation

$$dXt = \mu(t,X_t)dt + \sigma(t,X_t)dWt, \tag{24}$$

is given by

$$\Theta_i(t) = \frac{\frac{\eta(t,X_t)}{P(t,T)} - r(t)}{\frac{n\zeta_i(t,X_t)}{P(t,T)}}, i = 1,\ldots,n \tag{25}$$

where P(t, T) = G(X_t, t) is the zero-coupon bond price; $r(t)$ is the short-term rate factor for the risk-free borrowing or lending at time t over the interval $[t,\ t+dt]$; and $\eta(t,X_t)$, $\zeta(t,X_t)$ are defined by

$$\begin{cases} \eta(t,X_t) = \frac{\partial G(t,X_t)}{\partial t} + \frac{\partial G(t,X_t)}{\partial X}\mu(t,X_t) + \frac{1}{2}tr\left(\frac{\partial^2 G(t,X_t)}{\partial X^2}\sigma(t,X_t)\sigma'(t,X_t)\right), \\ \zeta(t,X_t) = \frac{\partial G(t,X_t)}{\partial X}\sigma(t,X_t) \end{cases} \tag{26}$$

with $X = (X_1, X_2, \ldots, X_n)$, $\frac{\partial G(t,X_t)}{\partial X} = \left(\frac{\partial G(t,X_t)}{\partial X_1}, \frac{\partial G(t,X_t)}{\partial X_2}, \ldots, \frac{\partial G(t,X_t)}{\partial X_n}\right)$.

In fact, since our price model (16), $\boldsymbol{X}(t) = (x_1(t),\ x_2(t))^T$ is an affine multi-factor model, the short-term rate factor r(t) and the zero-coupon bond price P(t, T) can be represented by

$$\begin{cases} r(t) = g + \boldsymbol{h}\,\boldsymbol{X}(t), \\ P(t,T) = \exp(a(t,T) + \boldsymbol{B}(t,T)\boldsymbol{X}(t)) \end{cases} \tag{27}$$

where $g \in \Re, \boldsymbol{h} = (h_1,h_2) \in \Re^2$, $a(t,T)$ and $\boldsymbol{B}(t,T) = \left(B_1(t,T), B_2(t,T)\right)$ are arbitrary smooth functions.

From (25) and (26), the market price of risk $\boldsymbol{\Theta}(t) = \left(\theta_1(t),\theta_2(t)\right)$ is given by

$$\boldsymbol{\Theta}(t) = \boldsymbol{a} + \boldsymbol{b}(t)\boldsymbol{X}(t) \tag{28}$$

where $\boldsymbol{a} = \begin{bmatrix} a_{10} \\ a_{20} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$

$$\begin{cases} a_{2,0}(t) = \dfrac{\frac{1}{2}B_2^2(t,T)\sigma^2(t,x_2) + \frac{da(t,T)}{dt} - g}{B_2(t,T)\sigma(t,x_2)}, \\ a_{2,1}(t) = \dfrac{\gamma B_2(t,T) - h_1}{B_2(t,T)\sigma(t,x_2)} \\ a_{2,2}(t) = \dfrac{-\gamma B_2(t,T) - h_2}{B_2(t,T)\sigma(t,x_2)} \\ a_{1,0}(t) = \dfrac{\frac{1}{2}B_1^2(t,T)\delta^2 + \mu\kappa B_1(t,T) + \frac{da(t,T)}{dt} - g}{B_1(t,T)\delta} \\ a_{1,1}(t) = \dfrac{-\mu B_1(t,T) - h_1}{B_1(t,T)\delta} \\ a_{1,2}(t) = 0 \end{cases}$$

In the following Lemma, we incorporate a market price of risk process that gives a risk-neutral dynamics of the same class as (16).

Lemma 3. Let us assume that A and B_i, $i = 1, 2$ are arbitrary constants. The market price of risk process reduces to;

$$\theta_1(t) = a_{1,0} + a_{1,1}x_1(t) + a_{1,2}x_2(t) \tag{29}$$

$$\theta_2(t) = a_{2,0} + a_{2,1}x_1(t) + a_{2,2}x_2(t) \tag{30}$$

In addition, let us assume that θ_i, $i = 1, 2$ satisfy the Novikov's condition with the $\overline{\mathbb{P}}$-Wiener process;

$$\begin{cases} \overline{W}_1(t) = W_1(t) + \int_{t_0}^{t} \theta_1(u)du \\ \overline{W}_2(t) = W_2(t) + \int_{t_0}^{t} \theta_2(u)du \end{cases} \tag{31}$$

and

$$\boldsymbol{C_1}: \begin{cases} h_1 + h_2 = 0, \\ a_{2,0}(t) = a_{1,2} = 0, \\ \bar{\gamma} = \dfrac{h_1}{B_2}, \bar{\mu} = \dfrac{h_2}{B_1} \\ \bar{\mu}\bar{\kappa} = \mu\kappa - \delta a_{1,0}, \end{cases} \tag{32}$$

where h_1 and h_2 are arbitrary real numbers; μ, κ and δ are defined in (14); $\theta_i, a_{i,j}, i = 1,2,$ $j = 0, 1, 2,$ are defined in (28).Then the risk-neutral dynamics of $x_1(t)$ and $x_2(t)$ remains within the same class,

$$d\,\boldsymbol{x} = [\overline{\boldsymbol{A}}\boldsymbol{x} + \overline{\boldsymbol{p}}]dt + \textstyle\sum(t, x_{2t})d\,\overline{\boldsymbol{W}}(t), \;\; \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{33}$$

where

$$\boldsymbol{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix},\bar{\boldsymbol{A}}=\begin{bmatrix}-\bar{\mu} & 0\\ \bar{\gamma} & -\bar{\gamma}\end{bmatrix},\bar{\boldsymbol{p}}(t)=\begin{bmatrix}\bar{\mu}\bar{\kappa}\\0\end{bmatrix},\Sigma\big(t,x_2(t)\big)=\begin{bmatrix}\delta & 0\\0 & \sigma\big(t,x_2(t)\big)\end{bmatrix},$$

$$\bar{W}=\begin{bmatrix}\bar{W}_1\\\bar{W}_2\end{bmatrix},x_0=\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix}.$$

Moreover, it satisfies Hypothesis $\boldsymbol{H_1}$. Hence,

$$\begin{cases}d\,x_1=\bar{\mu}(\bar{\kappa}-x_1)dt+\delta d\overline{W_1}(t)\\ d\,x_2=\bar{\gamma}(x_1-x_2)dt+\sigma(t,x_{2t})d\overline{W_2}(t)\end{cases}\tag{34}$$

Proof. The proof follows by substituting (31) and $\boldsymbol{C_1}$ into (16).

Remark 2. Under the assumption of Lemma 3, it is obvious that the solution to (33) is given by

$$\begin{cases}\boldsymbol{x}(t)=\begin{bmatrix}e^{-\bar{\mu}(t-t_0)} & 0\\ \bar{\phi}(t_0,t) & e^{-\bar{\gamma}(t-t_0)}\end{bmatrix}\boldsymbol{x}_0+\begin{bmatrix}\bar{\kappa}\big(1-e^{-\bar{\mu}(t-t_0)}\big)\\ \bar{\omega}(t_0,t)\end{bmatrix}\\ \qquad+\int_{t_0}^{t}\begin{bmatrix}\delta e^{-\bar{\mu}(t-s)}d\bar{W}_1(s)\\ \delta\bar{\phi}(s,t)d\bar{W}_1(s)+\sigma\big(s,x_2(s)\big)e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s)\end{bmatrix}\end{cases}\tag{35}$$

where $\bar{\phi}$and $\bar{\omega}$ are defined as

$$\begin{cases}\bar{\phi}(a,b)=\dfrac{\bar{\gamma}}{\bar{\mu}-\bar{\gamma}}\big(e^{-\bar{\gamma}(b-a)}-e^{-\bar{\mu}(b-a)}\big)\\ \bar{\omega}(a,b)\;=\bar{\kappa}\Big(\big[1-e^{-\bar{\gamma}(b-a)}\big]-\bar{\phi}(a,b)\Big)\end{cases}\tag{36}$$

In the following, we state a result with regard to (35).

Lemma 4. Under the assumption $\boldsymbol{H_1}$, (35) is equivalent to

$$\begin{cases}x(t)=\begin{bmatrix}e^{-\bar{\mu}\tau} & 0\\ \bar{\phi}(0,\tau) & e^{-\bar{\gamma}\tau}\end{bmatrix}\boldsymbol{x}(t-\tau)+\begin{bmatrix}\bar{\kappa}(1-e^{-\bar{\mu}\tau})\\ \bar{\omega}(0,\tau)\end{bmatrix}\\ \qquad+\int_{t-\tau}^{t}\begin{bmatrix}\delta e^{-\bar{\mu}(t-s)}d\bar{W}_1(s)\\ \delta\bar{\phi}(s,t)d\bar{W}_1(s)+\sigma\big(s,x_2(s)\big)e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s)\end{bmatrix}\end{cases}\tag{37}$$

$$\boldsymbol{x}(t-\tau)=\begin{bmatrix}x_1(t-\tau)\\x_2(t-\tau)\end{bmatrix}$$

Proof. The proof follows by changing the initial time t_0 in (35) to $t-\tau$.
Hence

$$x_1(t)=x_1(t-\tau)e^{-\bar{\mu}\tau}+\bar{\kappa}(1-e^{-\bar{\mu}\tau})+\delta\int_{t-\tau}^{t}e^{-\bar{\mu}(t-s)}d\bar{W}_1(s),\tag{38}$$

$$x_2(t)=x_1(t-\tau)\bar{\phi}(0,\tau)+x_2(t-\tau)e^{-\bar{\gamma}\tau}+\bar{\omega}(0,\tau)+\delta\int_{t-\tau}^{t}\bar{\phi}(s,t)dW_1(s)$$
$$+\int_{t-\tau}^{t}\sigma\big(s,x_2(s)\big)e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s).\tag{39}$$

Remark 3. From (39), we have

$$\delta\int_{t-\tau}^{t}\bar{\phi}(s,t)dW_1(s)+\int_{t-\tau}^{t}\sigma\big(s,x_2(s)\big)e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s)=x_2(t)-x_2(t-\tau)e^{-\bar{\gamma}\tau}$$
$$-x_1(t-\tau)\bar{\phi}(0,\tau)-\bar{\omega}(0,\tau).\tag{40}$$

The dynamics of volatility process under risk-neutral dynamic system is described by

$$d\,\sigma^2\big(t,x_2(t)\big)=\Bigg[\alpha+\beta\left[\int_{t-\tau}^{t}\sigma\big(s,x_2(s)\big)e^{-\bar{\gamma}(t-s)}dW_2(s)+\delta\int_{t-\tau}^{t}\bar{\phi}(s,t)dW_1(s)\right]^2 + c\,\sigma^2\big(t,x_2(t)\big)\Bigg]dt. \tag{41}$$

We set

$$u(t)=\mathbb{E}_{\overline{\mathbb{P}}}\left[\sigma^2\big(t,x_2(t)\big)\right]. \tag{42}$$

Taking the conditional expectation of both sides under the measure $\overline{\mathbb{P}}$, we obtain the following deterministic delay differential equation

$$\begin{cases}\dfrac{du}{dt}=\alpha+\beta\delta^2\displaystyle\int_{t-\tau}^{t}\bar{\phi}(s,t)^2ds\;+\beta\int_{t-\tau}^{t}u(s)e^{-2\bar{\gamma}(t-s)}ds\;+c\,u(t)\\ \quad\;\;=\alpha+\beta\delta^2 D+\beta\displaystyle\int_{t-\tau}^{t}u(s)e^{-2\bar{\gamma}(t-s)}ds\;+c\,u(t),\end{cases}$$

where

$$D=\int_{t-\tau}^{t}\bar{\phi}(s,t)^2ds\;=\left(\frac{\bar{\gamma}}{\bar{\mu}-\bar{\gamma}}\right)^2\left[\frac{1}{2\bar{\gamma}}\left(1-e^{-2\bar{\gamma}\tau}\right)-\frac{2}{\bar{\mu}+\bar{\gamma}}\left(1-e^{-(\bar{\mu}+\bar{\gamma})\tau}\right)+\frac{1}{2\bar{\mu}}\left(1-e^{-2\bar{\mu}\tau}\right)\right].$$

Hence,

$$\frac{du}{dt}=cu(t)+\beta\int_{t-\tau}^{t}u(s)e^{-2\bar{\gamma}(t-s)}ds\;+\nu, \tag{43}$$

where $\nu=\alpha+\beta\delta^2 D$.

Remark 4. The equilibrium solution process u^* of (43) satisfies the following integral equation

$$cu^*(t)+\beta\int_{t-\tau}^{t}u^*(s)e^{-2\bar{\gamma}(t-s)}ds+\nu=0, \tag{44}$$

since $\frac{du^*(t)}{dt}=0$. In particular, $u^*(t)$ is as follows;

$$u^*(t)=-\left[\frac{\nu}{c+\dfrac{\beta}{2\bar{\gamma}}\left(1-e^{-2\bar{\gamma}\tau}\right)}\right].$$

Using the transformation

$$v(t)=u(t)-u^*(t) \tag{45}$$

(44) is reduced to

$$\frac{dv(t)+u^*(t)}{dt}=c(v(t)+u^*(t))+\beta\int_{t-\tau}^{t}(v(s)+u^*(s))e^{-2\bar{\gamma}(t-s)}ds+\nu$$

$$\frac{dv(t)}{dt}=cv(t)+\beta\int_{t-\tau}^{t}v(s)e^{-2\bar{\gamma}(t-s)}ds+\left[cu^*(t)+\beta\int_{t-\tau}^{t}u^*(s)e^{-2\bar{\gamma}(t-s)}ds+\nu\right]$$

$$= cv(t) + \beta\int_{t-\tau}^{t} v(s)e^{-2\bar{\gamma}(t-s)}ds .$$

Hence,

$$\frac{dv(t)}{dt} = cv(t) + \beta\int_{-\tau}^{0} v(t+s)e^{2\bar{\gamma}s}ds \tag{46}$$

In order to find approximate solution representation, we need to investigate the behavior of (46). For this purpose, we present a result regarding its solution process. Our result is based on results of (Ladde, 1977; Ladde, 1979).

Definition 1

A non-constant solution v(t) of (46) is said to be

- oscillatory if $v(t)$ has arbitrary large number of zeros on $\Re_+ = [0,\infty)$, that is, there exists a sequence $\{t_n\}_1^\infty \ni t_n \in \Re_+,\ t_n \to \infty\ as\ n \to \infty$ and $v(t_n) = 0$.
- non-oscillatory if $v(t)$ is not oscillatory, that is, there exist a positive number T such that $v(t)$ is either positive or negative for all $t \geq T$.

Lemma 5. Under the following transformation

$$v(t) = e^{ct}z(t), \tag{47}$$

(46) is equivalent to

$$z'(t) = \beta\int_{-\tau}^{0} e^{(c+2\bar{\gamma})s}z(t+s)ds. \tag{48}$$

Moreover,

- (i) $for\,\beta < 0, and\ \frac{\beta\tau}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau} - 1\right] \le \frac{1}{e}$, every solution of (46) is non-oscillatory;
- (ii) $for\ \beta < 0, and,\ \frac{\beta\iota}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] > \frac{1}{e},\ \iota \in (0,\tau)$, every solution of (46) oscillates;
- (iii) *for* β > 0, (46) has non-oscillatory solutions.

Proof. To prove (i), suppose that

$$\frac{\beta\tau}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau} - 1\right] \le \frac{1}{e}, \beta < 0. \tag{49}$$

We observe that every solution of (46) is non-oscillatory if and only if every solution (48) is non-oscillatory. Therefore,we only need to show that (48) has non-oscillatory solution.

Suppose that a solution of (48) has the form

$$z(t) = e^{\lambda t} \tag{50}$$

where λ is an arbitrary constant which satisfies the following equation

$$\lambda = \beta\int_{-\tau}^{0} e^{(c+2\bar{\gamma}+\lambda)s}ds. \tag{51}$$

Define

$$G(\lambda) = \lambda - \beta\int_{-\tau}^{0} e^{(c+2\bar{\gamma}+\lambda)s}ds. \tag{52}$$

We show that G(λ) has at least one real root. From (49), (50) and the nature of β, we note that G(0) > 0 and for any s ϵ [−τ, 0],

$$G(\lambda)\le \lambda-\beta e^{-\lambda\tau}\int_{-\tau}^{0}e^{(c+2\bar{\gamma})s}ds$$
$$=-\beta\int_{-\tau}^{0}e^{(c+2\bar{\gamma})s}ds\left[-e+\exp\left[-e\tau\beta\int_{-\tau}^{0}e^{(c+2\bar{\gamma})s}ds\right]\right]$$
$$\le 0.$$

Therefore, (51) has at least one real root λ^* that lies between $\beta e\int_{-\tau}^{0}e^{(c+2\bar{\gamma})s}$ and 0. This shows that (46) has non-oscillatory solution.

Next, we out-line the proof for Lemma 5 (ii).

Proof: Suppose

$$\frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau}-e^{-(c+2\bar{\gamma})\iota}\right]>\frac{1}{e},\ \ \beta<0,\ \text{ for any } \iota\in(0,\tau). \tag{53}$$

We only need to show that (48) oscillates. To verify this, suppose on the contrary that $z(t)$ is a non-oscillatory solution of (48).Then for sufficiently large $t_0>0$ and without loss in generality, $z(t)>0$ for $t\ge t_1$,

Where $t_1=t_0-\tau$. Since $\beta<0$, $z'(t)<0$ for $t\ge t_1$. For any $\iota>0$ such that $-\tau<-\iota<0$, from (48), we have

$$z'(t)=\beta\int_{-\tau}^{0}e^{(c+2\bar{\gamma})s}z(t+s)ds. \tag{54}$$

$$z'(t)\le\beta\int_{-\tau}^{-\iota}e^{(c+2\bar{\gamma})s}z(t+s)ds,\ \ t\ge t_1 \tag{55}$$

Hence, for any $s\in(-\tau,-\iota), t-\tau<t+s<t-\iota<t$, (55) yields

$$z(t)<z(t-\iota)<z(t+s). \tag{56}$$

Define

$$w(t)=\frac{z(t-\iota)}{z(t)},\ \ t\ge t_1. \tag{57}$$

Note that $w(t)>1$. Dividing (55) by $z(t)$ and using (56), we have

$$\frac{z'(t)}{z(t)}-\frac{\beta}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\iota}-e^{-(c+2\bar{\gamma})\tau}\right]w(t)<\frac{z'(t)}{z(t)}-\beta\int_{-\tau}^{-\iota}e^{(c+2\bar{\gamma})s}\frac{z(t+s)}{z(t)}ds\le 0.$$

Integrating from $t-\iota$ to t, for $t\ge t_1$,

$$\log z(t)-\log z(t-\iota)-\frac{\beta}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\iota}-e^{-(c+2\bar{\gamma})\tau}\right]\int_{t-\iota}^{t}w(s)ds\le 0,$$

and hence

$$\log w(t)\ge\frac{\beta}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau}-e^{-(c+2\bar{\gamma})\iota}\right]\int_{t-\iota}^{t}w(s)ds,\ t\ge t_1 \tag{58}$$

Set

$$\liminf_{t\to\infty} w(t)=K. \tag{59}$$

Since $w(t)>1$, $K\ge 1$, K is either finite or infinite. Next we show that none of these cases are true.

Case 1. First, we assume that K is finite. There exist sequence $\{t_n\}$, $t_n\ge t_1\ni t_n\to\infty$ and $w(t_n)\to K$ as $n\to\infty$.

By integral meanvalue theorem, $\exists c_n \in (t_n - \iota, t_n)$ such that

$$\log w(t_n) \geq \frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] w(c_n). \tag{60}$$

We note that $\lim_{n\to\infty} w(c_n) = K_1$, where $K_1 \geq K$. Now, from (60) and taking limit, we obtain

$$\frac{\log K}{K} \geq \frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right]. \tag{61}$$

From (61), we have

$$\max_{K\geq 1} \frac{\log K}{K} = \frac{1}{e}, \tag{62}$$

and hence

$$\frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] \leq \frac{1}{e} \tag{63}$$

(63) contradicts with (53). Therefore K is not finite.

Case 2. Assume that K is infinite, from (57) and (59), we have

$$\lim_{t\to\infty}\left[\frac{z(t-\iota)}{z(t)}\right] = \infty \tag{64}$$

Choose $t_* = t - \alpha$, $\alpha > 0$, such that $t - \iota < t_* < t$ for $t \geq t_n$. Integrating both sides of (55) from t_* to t and $t - \iota$ to t_*, we have

$$z(t) - z(t_*) - \beta\int_{-\tau}^{-\iota} e^{(c+2\bar{\gamma})s}\left[\int_{t_*}^{t} z(u+s)du\right]ds \leq 0, \ t \geq t_1 \tag{65}$$

$$z(t_*) - z(t-\iota) - \beta\int_{-\tau}^{-\iota} e^{(c+2\bar{\gamma})s}\left[\int_{t-\iota}^{t_*} z(u+s)du\right]ds \leq 0, \ t \geq t_1 \tag{66}$$

respectively. We observe that for any $u \in [t_*, t]$, $s \in [-\tau, -\iota], u + s < t + s < t - \iota$, hence, $z(t-\iota) < z(t+s) < z(u+s)$, and for any $u \in [t-\iota, t_*]$, $u + s < t_* + s < t_* - \iota$, hence $z(t_* - \iota) < z(t_* + s) < z(u+s)$. Hence (65) and (66) become

$$z(t) + z(t-\iota)\frac{\beta\alpha}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] \leq z(t_*), \ t \geq t_1 \tag{67}$$

$$z(t_*) + z(t_*-\iota)\frac{\beta(\iota-\alpha)}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] \leq z(t-\iota), \ t \geq t_1. \tag{68}$$

Dividing (67) and (68) by $z(t)$ and $z(t_*)$ respectively, and using (53) and (64), we have

$$\lim_{t\to\infty}\frac{z(t_*)}{z(t)} = \lim_{t_*\to\infty}\frac{z(t-\iota)}{z(t_*)} = \infty. \tag{69}$$

Dividing (67) by $z(t_*)$ we have

$$\frac{z(t)}{z(t_*)} + \frac{z(t-\iota)}{z(t_*)}\frac{\beta\alpha}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] \leq 1, \ t \geq t_1 \tag{70}$$

which contradicts (53) and (69).

The proof of Lemma 5(iii) is similar to that of 5(i).

From Lemma 5, under conditions β < 0 an $\frac{\beta\tau}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau}-1\right]\leq\frac{1}{e}$,we can describe the asymptotic behavior of the steady/equilibrium state solution of (43). Moreover, we seek a solution in the form of $u(t)=\psi_1+\psi_2 e^{\rho t}$,where ψ_1,ψ_2 and ρ are arbitrary constants. In this case, the characteristic equation with respect to (43) is

$$h(\rho)=\rho-c-\beta\left[\frac{1-e^{-(\rho+2\bar{\gamma})\tau}}{\rho+2\bar{\gamma}}\right]=0. \tag{71}$$

From $u(t_0)=u_0$, we have

$$\psi_1=u^*=-\left[\frac{\nu}{c+\frac{\beta}{2\bar{\gamma}}(1-e^{-2\bar{\gamma}\tau})}\right], \tag{72}$$

$\psi_2=(u_0-\phi_1)e^{-\rho t_0}$.

However, using numerical simulation for equation (43), we observe that u(t) is asymptotically stable. From (46), the numerical scheme is defined as follows;

$$v_i=(1+c\Delta t+\beta(\Delta t)^2 e^{-2\bar{\gamma}})v_{i-1}+\beta(\Delta t)^2(v_{i-2}e^{-4\bar{\gamma}}+v_{i-3}e^{-6\bar{\gamma}}+\ldots+v_{i-l}e^{-2\bar{\gamma}}l) \tag{73}$$
$$u_i=v_i+u^*$$

where $v_i = v(t_i)$, $u_i = u(t_i)$ and $\{t_i\}_{i=1}^{m}$ is the time grid with a mesh of constant size Δt, l is the discrete-time delay analogueof τ.The solution is shown in Figure 2.

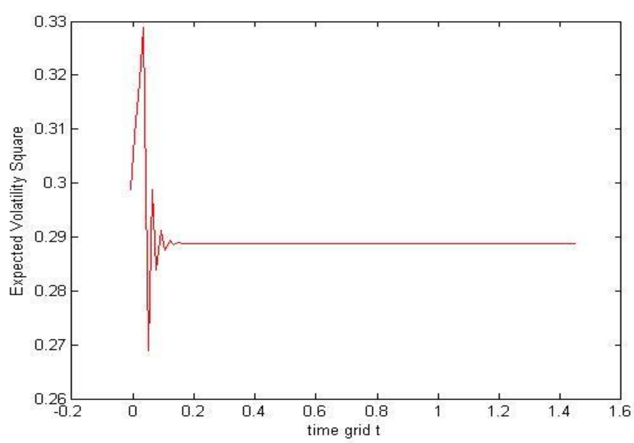

Figure 2: Solution of (43) with parameters in Table 2

5. Parameter Estimation

In this section, we find an expression for the forward price of energy commodity. Using the representation of forward price, we apply the Least-Square Optimization and Maximum Likelihood techniques to estimate the parameters defined in (2) and (35).

5.1. Derivation of Forward Price

Let $F(t,T)$ be the forward price at time t of an energy goods with maturity at time T. We define

$$F(t,T)=\mathbb{E}_{\overline{\mathbb{P}}}\left(S(T)\right) \tag{74}$$

where $S(T)$ is defined by (1); the expectation here is taken with respect to the risk neutral measure defined in (31).

Remark 5.

At maturity, it is expected that the forward price is equal to the spot price at that time i.e. $F(T,T)=S(T)$. This is the basic assumption of the risk neutral valuation method. From equation (35), the forward price $F(t,T)$ can be expressed as

$$\begin{aligned}F(t,T)&=\mathbb{E}_{\overline{\mathbb{P}}}\left(S(T)\right)=\mathbb{E}_{\overline{\mathbb{P}}}\left(\exp[f(T)+x_2(T)]\right)\\&=\exp\left[f(T)+e^{-\bar{\gamma}(T-t)}x_2(t)+\bar{\phi}(t,T)x_1(t)+\bar{\varpi}(t,T)+\Upsilon(t,T)\right],\end{aligned} \tag{75}$$

where $\Upsilon(t,T)$ is defined by

$$exp\left[\frac{\psi_1\, g(t,T,2\ \bar{\gamma}\ )+\ (u_0\ -\psi_1)e^{\rho\, T}\, g(t,T,\rho+2\bar{\gamma})+\left[\frac{\delta\,\bar{\gamma}}{\bar{\mu}-\bar{\gamma}}\right]^2\left(g(t,T,2\bar{\gamma})-2\, g(t,T,\bar{\mu}+\bar{\gamma})+g(t,T,2\bar{\mu})\right)}{2}\right]$$

$$g(t,T,a)=\frac{1-e^{-a(T-t)}}{a},\text{for any } a\in\Re \tag{76}$$

and ψ_1 is defined in (72). Hence

$$\begin{aligned}\log F(t,T)&=f(T)+e^{-\bar{\gamma}(T-t)}x_2(t)+\bar{\phi}(t,T)x_1(t)+\bar{\varpi}(t,T)+\Upsilon(t,T)\\&=f(T)+e^{-\bar{\gamma}(T-t)}\left(\log S(t)-f(t)\right)+\bar{\phi}(t,T)x_1(t)+\bar{\varpi}(t,T)+\Upsilon(t,T)\\&=A(t,T)+B(t.T)x_1(t)\end{aligned} \tag{77}$$

where $A(t,T)=f(T)+e^{-\bar{\gamma}(T-t)}\left(\log S(t)-f(t)\right)+\bar{\varpi}(t,T)+\Upsilon(t,T)$ and $B(t,T)=\bar{\phi}(t,T)$. Define

$$\begin{aligned}\epsilon_1&=(\bar{\mu},\bar{\kappa},\delta)\\\epsilon_2&=(\bar{\gamma},\alpha,\beta,c,\tau)\\\epsilon_3&=(A_0,A_1,A_2,B_1,B_2)\\\epsilon&=(\epsilon_1,\epsilon_2,\epsilon_3),\end{aligned} \tag{78}$$

where ϵ consists of risk-neutral parameters in (2) and (35). We can represent $\log F(t,T)$ as $\log F(t,T;\epsilon)$, $x_1(t)\equiv x_1(t;\epsilon_1), x_2(t)\equiv x_2(t;\epsilon_2), f(t)\equiv f(t;\epsilon_3)$. In the following subsection, we use the Least square optimization approach to estimate the parameters $\bar{\gamma},\bar{\mu},\bar{\kappa}$ and δ.

5.2. Least Squares Optimization Techniques

For time $t_i, i\in\{1,2,...,m\}=I(1,m)$, let $S(t_i)$ denote the historical spot price of commodity. For fixed $i\in I(1,m)$, $\tilde{F}(t_i,\mathrm{T}_j^i)$ represent a data for future price at a time t_i with delivery time T_j^i for $j\in I(1,n_i)$.

These data values are obtainable from the energy market. For each given quoted time t_i, we obtain $x_1(t;\epsilon_1)$

that minimizes the sum of squares

$$\text{sqdiff}(t_i,\epsilon)=\sum_{j=1}^{n_i}\left[\log F(t_i,T_j^i;\epsilon)-\log\tilde{F}(t_i,T_j^i)\right]^2, \tag{79}$$

where $\log F(t_i,T_j^i;\epsilon)$ is described in (77). Differentiating (79) with respect to $x_1(t;\epsilon_1)$ and setting the result to zero, we obtain the optimal value of $x_1(t;\epsilon_1)$ as a function of the parameter set. Moreover, we have

$$\tilde{x}_1(t_i;\bar{\epsilon})=\frac{\sum_{j=1}^{n_i}\left[B(t_i,T_j^i)\left(\log\tilde{F}(t_i,T_j^i)-A(t_i,T_j^i)\right)\right]}{\sum_{j=1}^{n_i}\left[B(t_i,T_j^i)\right]^2},\ i\in I(1,m), \tag{80}$$

Substituting this optimal value into the initial sum of squares (79), summing over the range of initial times $\{t_i\}_i^m$ and performing a nonlinear least-squares optimization as follows:

$$\text{sqdiff}(\epsilon)=arg\min_{\epsilon}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left[A_{t_i,T_j^i}+B_{t_i,T_j^i}x_1(t)(\bar{\epsilon})-\log\tilde{F}(t_i,T_j^i)\right]^2. \tag{81}$$

Using estimated parameters $\bar{\epsilon}$, simulated results $\{x_1(t_i)\}_{i=1}^m$, $\{x_2(t_i)\}_{i=1}^m$, $\{f(t_i)\}_{i=1}^m$and $\{S(t_i)\}_{i=1}^m$ are obtained.

In the case of real-world $\mathbb{P}$-parameter set $\{\gamma,\mu,\kappa,\delta\}$ estimation, the estimates of γ and κ are obtained using a Linear regression technique associated with the model . (80) contains an estimated hidden process $\overline{x_1}(t_i)$ which is obtained by the least square minimization approach. This estimated data is used in a regression of a one-factor mean reverting model $dx_1(t)=\mu(\kappa-x_1(t))+\delta dW_1(t)$ to obtain estimates for μ and δ . Weremark that this procedure is very stable.

5.3. Maximum Likelihood Approach

Now, following the approach in (Yuriy, Anatoliy & Jianhong, 2005) and using Maximum Likelihood technique, the time delay and the delay volatility parameters α, β and c are estimated. Our model contains two sources of randomness. One in the dynamic of log-spot price and the second in the expected log of spot price. Therefore, the presented model is an extension of GARCH model. An outline of the procedure is given below:

From equation (41), we have

$$\frac{d\sigma^2(t,x_2(t))}{dt}=\alpha+\beta\left[\int_{t-\tau}^{t}\sigma(s,x_2(s))e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s)+\delta\int_{t-\tau}^{t}\bar{\phi}(u,t)d\bar{W}_1(u)\right]^2+c\sigma^2(t,x_2(t)). \tag{82}$$

We define the discrete-time delay l to the continuous-time delay τ as $l=\left[\left|\frac{\tau}{\Delta}\right|\right]$ where Δ is the size of the mesh of the discrete-time grid, and $[|.|]$ is the floor function. Hence, we define

$$\varepsilon_i=\sigma_i\xi_i \\ \eta_i=\delta\zeta_i, \tag{83}$$

where ξ_i and $\zeta_i$are standard normal variates. The discrete-time delay model corresponding to (82) for volatilityis described by

$$\sigma_n^2=\alpha+\beta\Delta t\left[\sum_{i=1}^{l}\left(\varepsilon_{n-i}e^{-\bar{\gamma}i}+\eta_{n-i}\bar{\phi}(0,i)\right)\right]^2+r\sigma_{n-1}^2, \tag{84}$$

where $n\in I(1,\infty)$ and $r=1+c$. From (40), we further note that

$$\int_{t-\tau}^{t}\sigma(s,x_2(s))e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s)+\delta\bar{\phi}(s,t)d\bar{W}_1(s)=x_2(t)-x_2(t-\tau)e^{-\bar{\gamma}\tau}-x_1(t-\tau)\bar{\phi}(0,\tau)-\bar{\varpi}(0,\tau),$$

that is,

$$\sqrt{\Delta t}\sum_{i=1}^{l}\varepsilon_{n-i}e^{-\bar{\gamma}i}+\eta_{n-i}\bar{\phi}(0,i)=x_2(n)-x_2(n-1)e^{-\bar{\gamma}l}-x_1(n-1)\bar{\phi}(0,l)-\bar{\varpi}(0,l). \tag{85}$$

Define

$$P(n)=\left[x_2(n)-x_2(n-1)e^{-\bar{\gamma}l}-x_1(n-1)\bar{\phi}(0,l)-\bar{\varpi}(0,l)\right]^2, \tag{86}$$

This together with (84) yields

$$\sigma_n^2=\alpha+\beta P_n+r\sigma_{n-1}^2. \tag{87}$$

The solution of difference equation (87) is given by

$$\sigma_n^2=\begin{cases}\alpha F_n(r)+\beta G_n(r)+H_n(r), & n\geq l+1\\ \varepsilon_n^2 \quad\quad\quad\quad\quad\quad\quad\quad\quad , & n\leq l\end{cases} \tag{88}$$

where $r=1+c$,

$$F_n=\sum_{i=0}^{n-l-1}r^i, \tag{89}$$

$$G_n=\sum_{i=0}^{n-l-1}r^iP_{n-i}, \tag{90}$$

$$H_n=r^{n-l}\sigma_l^2. \tag{91}$$

We observe that the series F_n in (89) converges, if $|r| < 1$, that is, $|1 + c| < 1$. Hence,

$$-2<c<0. \tag{92}$$

From the definition of ε_n in (83), the probability density function f_{ε_n} of ε_n is

$$f_{\varepsilon_n}(y)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{y^2}{2\sigma_n^2}\right]. \tag{93}$$

Thus, the likelihood function $L(\alpha,\beta,c)$ of $f_\varepsilon, n\in I(1,N)$ for arbitrary large positive integer N is

$$L(\alpha,\beta,c)=\prod_{n=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{y^2}{2\sigma_n^2}\right]. \tag{94}$$

By applying the Maximum Likelihood method, we obtain the estimates $\alpha(l)$, $\beta(l)$, and $r(l)$ for $l\in I(1,p)$ for some arbitrary p.

6. *Some Results: Natural Gas*

In this section, we apply our model to the Henry Hub daily natural gas data set for the period 02/01/2001-09/30/2004. The data is collected from the United State Energy Information Administration website (www.eia.gov). Using the Henry Hub daily natural gas data set, we present the estimation results of our model. The parameter estimates of our model for the value of $l = 2$ are given. For this purpose, using a combination of direct search method and Nelder-Mead simplex algorithm, we search iteratively to find the parameters that maximizes the likelihood function. All codes are written in Matlab.

γ	μ	κ	δ	τ	α	β	c
1.8943	1.0154	1.5627	0.36	0.008	0.433	-0.07	-1.5

Table 2: Estimated Parameters of Henry Hub daily natural gas spot prices for the period 02/01/2001-09/30/2004.

Table 2 shows the risk-neutral parameter estimates of Henry Hub daily natural gas data set for the period 02/01/2001-09/30/2004.

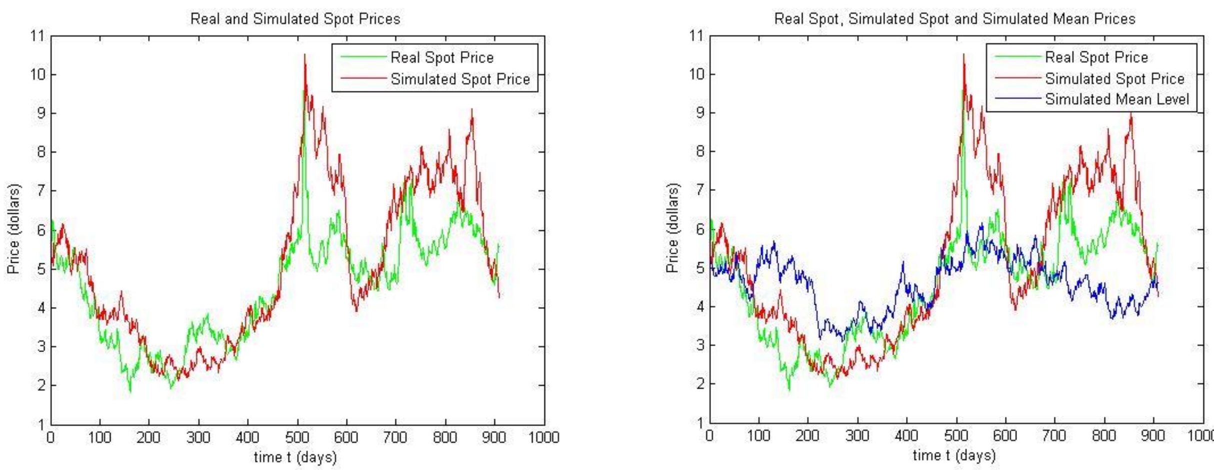

Figure 3: Real, Simulated and Forecasted Prices.

Figure 3 a shows the graphs of the real spot natural gas price data set together with the simulated spot price $exp(x_2(t))$. Figure 3 b shows and the graphs of the real spot natural gas price data set together with the simulated spot price $exp(x_2(t))$ and the simulated expected spot price $exp(x_1(t))$. We notice that in Figure 3 a, the simulated spot price captures the dynamics of the data. This shows that the simulation agrees with our mathematical model. Another observation is that the simulated mean level seems to be moving around the value 4.80 which is close to $exp(\bar{\kappa}) = exp(1.5627)$. This confirms the fact that κ is the equilibrium mean level.

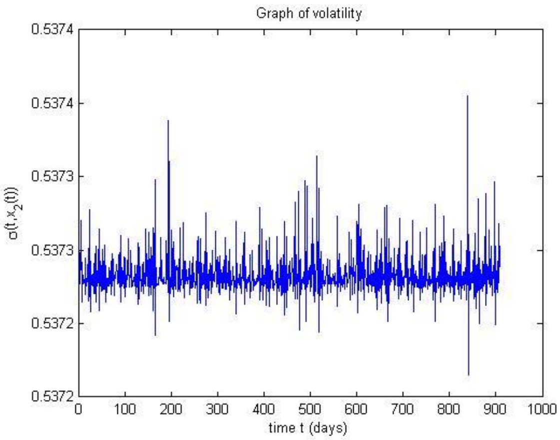

Figure 4: Simulated $\sigma(t,\ x_2(t))$.

Figure 4 shows the plot of volatility $\sigma(t,\ x_2(t))$ with time. It is clear from the graph that the solution is non-oscillatory. This is because Lemma 5 (i) is satisfied using the parameter estimates in Table 2.

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