12-2014

Positive Solutions of Boundary Value Dynamic Equations

Olusegun Michael Otunuga
Marshall University, otunuga@marshall.edu

Basant Karna
Marshall University, karna@marshall.edu

Bonita Lawrence
Marshall University, lawrence@marshall.edu

Recommended Citation

Otunuga O. M., Karna B., & Lawrence B. (2014). Positive Solutions of Boundary Value Dynamic Equations. American Review of Mathematics and Statistics, 2(2), 1-6.

American Review of Mathematics and Statistics
December 2014, Vol. 2, No. 2, pp. 55-89
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v2n2a4
URL: http://dx.doi.org/10.15640/arms.v2n2a4

Positive Solutions of Boundary Value Dynamic Equations

Olusegun M. Otunuga[1], Basant Karna[1] & Bonita Lawrence[1]

Abstract

In this paper, we deal with the existence of a positive solution for $2^{nd}$and 3^{rd}order boundary value problem by first defining their respective Green's function. TheGreen's function isused to derive the Green's function for the $2n^{th}$ and $3n^{th}$ order boundary value problem, respectively, where*n* is a positive integer. The Green's function is also used to derive conditions for positive solution of the $2n^{th}$ and $3n^{th}$ ordereigen value differentialequation, respectively.

Keywords: Positive solution, Green's function, Boundary Value Problem, Dynamical Equation

1 . Introduction

This paper focuses on determining eigen values λ, for which there exist positive solutions, with respect to a cone, of the nonlinear eigen value dynamice quation

$$y'' + \lambda f(t, y) = 0, \quad t \in [t_1, t_2],$$

Subject to the two-point boundary conditions

$$\alpha_{11} y(t_1) + \alpha_{12} y'(t_1) = 0,$$
$$\alpha_{21} y(t_2) + \alpha_{22} y'(t_2) = 0.$$

[1]Department of Mathematics, Marshall University, One John Marshall Drive, Huntington, WV, 25755, USA

Also,we consider the 3rd-order eigen value problem

$$y''' = \lambda f(t, y), \quad t \in [t_1, t_3]$$

subject to the three-point boundary conditions

$$\begin{cases} y(t_1) = \beta_1 \\ y(t_2) = \beta_2 \\ y(t_3) = \beta_3 \end{cases}$$

Boundary value problems for higher order differential equations play a role in both theory and applications. The existence of positive solutions for two-point eigenvalue problems has been studied by many researchers by using the Guo-Krasnosel'skii fixed point theorem. We refer readers to Davis, J.M. Henderson, J, Prasad, K.R. &Yin, W. (2000), Eloe, P.W. &Henderson, J (1998), Erbe L.H.&Wang H.(1994), Karna, Basant& Lawrence, Bonita (2007) for some recent results. However, few papers can befound in the literature for third order three-point boundary value problems (BVPs) (Prasad, K.R. and Rao, Kameswara (1991)). Some papers like Anderson, D.R. & Davis, J.M. (2002) deal with existence of positive solutions when the nonlinear term f is nonnegative. In this paper, we deal with the existence of a positive solution for the 2^{nd} and 3^{rd} order BVPs by first defining their respective Green's function. These Green's function are used to derive the Green's function for the $2n^{th}$ and $3n^{th}$ order BVP, respectively. The Green's function is also used to derive the condition for which a positive solution of the $2n^{th}$ order eigenvalue differential equation can be derived.

The rest of this paper is organized as follows:

In Section 2, we compute Green's function for a two-point boundary value problem on □ and also find conditions under which a positive solution will exist for the two-point problem. In Section 3, we derive Green's functions for even order BVPs and also compute the bounds for the Green's function. These bounds are used to proof the existence of positive solution(s) for $2n^{th}$ order BVPs. In Section 4, we find the conditions in which positive solution(s) will exist for the three-point boundary value problem.

2. Second Order Boundary Value Problem on $\mathbb{R}$

In this section, we consider the second order boundary value eigenvalue problem on $\mathbb{R}$.

2.1 Solution of the Second Order Differential Equation

Consider the second order eigenvalue BVP

$$y''(t) + \lambda f(t, y(t)) = 0, \qquad t \in [t_1, t_2] \tag{1}$$

$$\begin{cases} \alpha_{11} y(t_1) + \alpha_{12} y'(t_1) = 0 \\ \alpha_{21} y(t_2) + \alpha_{22} y'(t_2) = 0. \end{cases} \tag{2}$$

where $f : [t_1, t_2] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ are real constant. We will assume the following condition:

$\boldsymbol{A_1}$: $f : [t_1, t_2] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

We define the nonnegative numbers f_0, f^0, f_∞ and f^∞ by

$$\begin{cases} f_0 = \lim\limits_{y \to 0^+} \min\limits_{t \in [t_1, t_2]} \dfrac{f(t, y)}{y} \\ f^0 = \lim\limits_{y \to 0^+} \max\limits_{t \in [t_1, t_2]} \dfrac{f(t, y)}{y} \\ f_\infty = \lim\limits_{y \to \infty} \min\limits_{t \in [t_1, t_2]} \dfrac{f(t, y)}{y} \\ f^\infty = \lim\limits_{y \to \infty} \max\limits_{t \in [t_1, t_2]} \dfrac{f(t, y)}{y} \end{cases} \tag{3}$$

and assume that they all exist in the extended reals.

Now we are going to find the solution of the second order problem. We shall show that the solution $y(t)$ is of the form

$$y(t) = \int_{t_1}^{t_2} G(t,s)g(s)ds$$

where $G(t,s)$ will be defined later.

Writing $y''(t) = -g(t, y(t))$ where $g(t, y(t)) = \lambda f(t, y(t))$ and solving the differential equation (1) using Laplace transform, we have

$$L(y''(t)) = -L(g(t)).$$

This implies

$$s^2 L(y(t)) - sy(0) - y'(0) = -L(g(t)).$$

Hence,

$$L(y(t)) = \frac{1}{s} y(0) + \frac{1}{s^2} y'(0) - \frac{1}{s^2} L(g(t)).$$

Taking the inverse Laplace of both sides, we have

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \int_{t_1}^{t} (t-s)g(s)ds, \\ y'(t) &= y'(0) - \int_{t_1}^{t} g(s)ds \end{aligned}$$

Using the boundary conditions and solving for $y(0)$ and $y'(0)$, we have

$$\begin{cases} y(0) = \dfrac{-\beta_1 A}{D} \\ y'(0) = \dfrac{\alpha_{11} A}{D} \end{cases} \quad (4)$$

where

$$\begin{cases} \beta_i = \beta(t_i) = \alpha_{i1}t_i + \alpha_{i2}, \; i = 1,2 \\ A = \int_{t_1}^{t_2} (\beta_2 - \alpha_{21}s)g(s)ds \\ D = \alpha_{11}\beta_2 - \alpha_{21}\beta_1. \end{cases} \quad (5)$$

So,

$$y(t) = \frac{1}{D}\int_{t_1}^{t_2} (\beta_2 - \alpha_{21}s)(\alpha_{11}t - \beta_1)g(s)ds - \int_{t_1}^{t} (t-s)g(s)ds$$

Therefore,

$$y(t) = \int_{t_1}^{t_2} G(t,s)g(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{1}{D}(\beta_1 - \alpha_{11}s)(\alpha_{21}t - \beta_2) \; if \quad t_1 \le s \le t \le t_2; \\ \frac{1}{D}(\beta_2 - \alpha_{21}s)(\alpha_{11}t - \beta_1) \; if \quad t_1 \le t \le s \le t_2. \end{cases} \quad (6)$$

Throughout this section, we will require the following conditions:

$\boldsymbol{A}_2$: $\alpha_{11} > 0, \alpha_{21} > 0$;

$\boldsymbol{A}_3$: $m_1 \le t_1 \le t_2 \le m_2$, where $m_i = \dfrac{\beta(t_i)}{\alpha_{i1}} = \dfrac{\beta_i}{\alpha_{i1}}$, $i = 1, 2$.

Note: $\dfrac{\beta_1}{\alpha_{11}} \le t_1$ implies that $\beta_1 - \alpha_{11}t_1 \le 0$. Thus, $\alpha_{12} \le 0$. Also, $\dfrac{\beta_2}{\alpha_{21}} \ge t_2$ implies $\beta_2 - \alpha_{21}t_1 \ge 0$. Thus $\alpha_{22} \ge 0$.

Now, we establish some preliminary results that will be used later.

2.2 Properties of the function G(t, s)

We give some Lemma on theabove function $G(t,s)$.

Lemma 1. $G(t,s) > 0 \;\; for \;\; (t,s) \in [t_1, t_2] \times [t_1, t_2]$.

Proof.

For $t_1 \le s \le t \le t_2$, using conditions $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$, we have $\frac{\beta_1}{\alpha_{11}} \le s \le t \le \frac{\beta_2}{\alpha_{21}}$ so that $D = \alpha_{11}\beta_2 - \alpha_{21}\beta_1 > 0$ and $G(t,s) = \frac{1}{D}(\alpha_{21}t - \beta_2)(\beta_1 - \alpha_{11}s) > 0.$

Also, for $t_1 \le t \le s \le t_2$, we have $\frac{\beta_1}{\alpha_{11}} \le t \le s \le \frac{\beta_2}{\alpha_{21}}$ so that $G\ (t,s)\ >\ 0.$ Therefore, $G(t,s) > 0 \quad for \quad (t,s) \in [t_1,t_2] \times [t_1,t_2].$

Lemma 2. *The function $G(t,s)$ satisfies the homogeneous differential equation $-y'' = 0$ and the boundary conditions (2) for fixed s.*

Proof.

Since $G(t,s)$ is a polynomial of degree one, then it satisfies

$$\frac{\partial^2}{\partial t^2}G(t,s) = 0 \ \ \forall \ \ (t,s) \in [t_1,t_2] \times [t_1,t_2].$$

For $t_1 \le t \le s \le t_2, \frac{\partial}{\partial t}G(t,s) = \frac{1}{D}\alpha_{11}(\beta_2 - \alpha_{21}s)$ sothat $\alpha_{11}G(t_1,s) + \alpha_{12}\frac{\partial}{\partial t}G(t_1,s) = 0.$ Also for $t_1 \le s \le t \le t_2, \frac{\partial}{\partial t}G(t,s) = \frac{1}{D}\alpha_{21}(\beta_1 - \alpha_{11}s)$ so that $\alpha_{21}G(t_2,s) + \alpha_{22}\frac{\partial}{\partial t}G(t_2,s) = 0.$

Lemma 3. *For any fixed $s \in [t_1,t_2]$, the function $G(t,s)$ is continuous for every $t \in [t_1,t_2]$.*

Proof.

Clearly, $G(t,s)$ is continuous everywhere on $[t_1,t_2] \times [t_1,t_2]$ since it is continuous at the point $t\ = s.$ Hence, the proof is complete.

Lemma 4. $\frac{\partial}{\partial t}G(t,s) \equiv G'(t,s)$ has a jump discontinuity with a jump of factor -1 at the point $t = s$.

Proof.

Here, we show that the limit of $\frac{\partial}{\partial t}G(t,s)$ as t approaches s from above differ from its limit as t approaches s from below by -1.

$$G'(s^+,s) - G'(s^-,s) = \lim_{t\to s^+} G'(t,s) - \lim_{t\to s^-} G'(t,s)$$
$$= \frac{1}{D}(\alpha_{21}\beta_1 - \alpha_{21}\alpha_{11}s - \alpha_{11}\beta_2 + \alpha_{11}\alpha_{21}s)$$
$$= \frac{1}{D}(\alpha_{21}\beta_1 - \alpha_{11}\beta_2) = -1.$$

Lemma 5. *Define*

$$\gamma = \min\left\{ \min_{s\in[t_1,t_2]} \left\{ \frac{G(t_1,s)}{G(s,s)}, \frac{G(t_2,s)}{G(s,s)} \right\} \right\}, \quad (7)$$

then $0 < \gamma < 1$.

Proof.

The proof follows from simple algebra and simplification.

Theorem 1. *Assume that conditions* $A_1 - A_3$ *hold. Then,* $\gamma G(s,s) \le G(t,s) \le G(s,s)$, *where*

$$0 < \gamma = \min\left\{ \min_{s\in[t_1,t_2]} \left\{ \frac{G(t_1,s)}{G(s,s)}, \frac{G(t_2,s)}{G(s,s)} \right\} \right\} < 1. \quad (8)$$

Proof.

Case (i): For $t_1 \le s \le t \le t_2$, $G^{'}(t,s) = \frac{\alpha_{21}}{D}(\beta_1 - \alpha_{11}s) < 0,$ which implies that $G(t,s)$ is a decreasing function of *t* so that $G(t,s) \le G(s,s).$

Also for $t \le t_2$, $\frac{G(t,s)}{G(s,s)} \ge \frac{G(t_2,s)}{G(s,s)} \ge \gamma$ which implies $\gamma G(s,s) \le G(t,s).$

Case (ii): For $t_1 \le t \le s \le t_2$, $G'(t,s) = \frac{1}{D}\alpha_{11}(\beta_2 - \alpha_{21}s) > 0.$ This implies that $G(t,s)$ is an increasing function of *t*. Hence, $G(t,s) \le G(s,s).$

Also, for $t \ge t_1$, $\frac{G(t,s)}{G(s,s)} \ge \frac{G(t_1,s)}{G(s,s)} \ge \gamma$ and so we have $\gamma G(s,s) \le G(t,s).$

Therefore, $\gamma G(s,s) \le G(t,s) \le G(s,s) \quad for \quad t_1 \le t,s \le t_2.$

From Lemma2, 3 and 4, it follows that the function $G(t,s)$ is the Green's function for the equation

$$-y^{''}(t) = 0, \; t \in [t_1,t_2]$$

with boundary conditions

$$\begin{cases} \alpha_{11}y(t_1) + \alpha_{12}y'(t_1) = 0 \\ \alpha_{21}y(t_2) + \alpha_{22}y'(t_2) = 0. \end{cases} \quad (9)$$

2.3 Existence of Positive Solutions

In this Section, we find the range of λ for which there exist a positive solution for (1) satisfying (2).

Definition 1. Let X be a Banach space. A non empty closed convex set κ is called a **cone** of *X*, if it satisfies the following conditions:

(i) $\alpha_1 u + \alpha_2 v \in \kappa \quad \forall \; u,v \in \kappa$ and $\alpha_1, \alpha_2 \ge 0$.

(ii) $u \in \kappa$ and $-u \in \kappa$ implies $u = 0$.

Let $y(t)$ be the solution of the BVP (1) satisfying (2) given by

$$y(t) = \lambda \int_{t_1}^{t_2} G(t,s) f(s, y(s)) ds. \qquad (10)$$

Define
$X = \{u \mid u \in C[t_1, t_2]\},$

where $C[t_1, t_2]$ is the set of all continuous function on $[t_1, t_2]$ with norm

$$\| u \| = \max_{t \in [t_1, t_2]} | u(t) | .$$

Then, $(X, \|.\|)$ is a Banach space. Define a set κ by

$$\kappa = \{u \in X : u(t) \geq 0 \quad \text{on} \quad [t_1, t_2] \qquad (11)$$

and

$$\min_{t \in [t_1, t_2]} u(t) \geq \gamma \| u \| \}$$

where γ is defined in (7).

It follows that the set κ defined in (11) is a cone in X .

Define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \lambda \int_{t_1}^{t_2} G(t,s) f(s, y(s)) ds, \text{ for all } t \in [t_1, t_2]. \qquad (12)$$

If $y \in \kappa$ is a fixed point of T , then y satisfies (10) hence y is a positive solution of the BVP (1)-(2).

We seek a fixed point of the operator T in the cone κ .

The operator T defined in (12) preserves the cone κ, that is, $T:\kappa\to\kappa$. Furthermore, the operator T defined in (12) is completely continuous.
To establish the eigenvalue intervals where a fixed point exists in (1), we will employ the following Fixed Point Theorem due to Guo and Krasnosel'skii.

Theorem 2.(Guo-Krasnosel'skii Fixed Point Theorem) *Let X be a Banach space, $\kappa\subseteq X$ be a cone, and suppose that Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1\subset\Omega_2$ and $\overline{\Omega_1}\subset\Omega_2$. Suppose further that $T:\kappa\cap(\overline{\Omega_2}\setminus\Omega_1)\to\kappa$ is completely continuous operator such that either*

(i) $\|Tu\|\le\|u\|, u\in\kappa\cap\partial\Omega_1$ *and* $\|Tu\|\ge\|u\|, u\in\kappa\cap\partial\Omega_2$, *or*
(ii) $\|Tu\|\ge\|u\|, u\in\kappa\cap\partial\Omega_1$ *and* $\|Tu\|\le\|u\|, u\in\kappa\cap\partial\Omega_2$,

holds. Then T has a fixed point in $\kappa\cap(\overline{\Omega_2}\setminus\Omega_1)$.
We are going to present our first existence result.

Theorem 3. *Assume that conditions ($\mathbf{A}_1$) - ($\mathbf{A}_3$) are satisfied. Then, for each λ satisfying*

$$\frac{1}{\left[\gamma^2\int_{t_1}^{t_2}G(s,s)ds\right]f_\infty}<\lambda<\frac{1}{\left[\int_{t_1}^{t_2}G(s,s)ds\right]f^0}, \qquad (13)$$

there exist at least one positive solution of the BVP (1)- (2) in κ, where f_∞ and f^0 are as define in Section 2.1.
Proof.

Let λ be given as in (13). Now, let $\delta>0$ be chosen such that

$$\frac{1}{\left[\gamma^2\int_{t_1}^{t_2}G(s,s)ds\right](f_\infty-\delta)}\le\lambda\le\frac{1}{\left[\int_{t_1}^{t_2}G(s,s)ds\right](f^0+\delta)}.$$

Let T be the cone preserving, completely continuous operator defined in (12). By definition of f^0, there exists $H_1>0$ such that

$$\max_{t\in[t_1,t_2]}\frac{f(t,y)}{y}\le(f^0+\delta), \text{ for } 0<y\le H_1.$$

It follows that $f(t,y) \le (f^0 + \delta)y$, for $0 < y \le H_1$. Choose $y_1 \in \kappa$ with $\| y_1 \| = H_1$. Then, we have from the boundedness of $G(t,s)$ and the nature of λ, that

$$(Ty_1)(t) = \lambda \int_{t_1}^{t_2} G(t,s) f(s, y_1(s)) ds$$

$$\le \lambda \int_{t_1}^{t_2} G(s,s) f(s, y_1(s)) ds$$

$$\le \lambda \int_{t_1}^{t_2} G(s,s)(f^o + \delta) y_1(s) ds$$

$$\le \lambda \int_{t_1}^{t_2} G(s,s)(f^o + \delta) \| y_1 \| ds$$

$$\le \| y_1 \|.$$

Consequently, $\| Ty_1 \| \le \| y_1 \|$. So, if we define

$$\Omega_1 = \{u \in X : \| u \| < H_1\},$$

then,

$$\| Ty \| \le \| y \|, \text{ for } y \in \kappa \cap \partial\Omega_1. \quad (14)$$

By definition of f_∞, there exists $\overline{H_2} > 0$ such that

$$\min_{t \in [t_1, t_2]} \frac{f(t,y)}{y} \ge (f_\infty - \delta), \text{ for } y \ge \overline{H_2}.$$

It follows that

$$f(t,y) \ge (f_\infty - \delta) y, \text{ for } y \ge \overline{H_2}.$$

Let

$$H_2 = \max\{2H_1, \frac{1}{\gamma}\overline{H_2}\},$$

and let
$\Omega_2 = \{u \in X : \| u \| < H_2\}$.

Now, choose $y_2 \in \kappa \cap \partial\Omega_2$ with $\| y_2 \| = H_2,$ so that $\min_{t \in [t_1, t_2]} y_2(t) \geq \gamma \| y_2 \| \geq \overline{H_2}$. Then,

$$\begin{aligned}(Ty_2)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s, y_2(s)) ds \\ &\geq \lambda \int_{t_1}^{t_2} \gamma G(s,s) f(s, y_2(s)) ds \\ &\geq \lambda\gamma \int_{t_1}^{t_2} G(s,s)(f^{\infty} - \delta) y_2(s) ds \\ &\geq \gamma^2 \lambda \int_{t_1}^{t_2} G(s,s)(f^{\infty} - \delta) \| y_2 \| ds \\ &\geq \| y_2 \|.\end{aligned}$$

Thus,

$$\| Ty \| \geq \| y \|, \text{ for y} \in \kappa \cap \partial\Omega_2 \quad (15)$$

Applying Theorem 2(i), from (14) and (15), we have that T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega_2} \setminus \Omega_1)$. This fixed point is the positive solution of the BVP (1)-(2) for the given λ.

Another existence result applying Theorem 2(ii) is as follow:

Theorem 4:Assume that conditions ($\mathbf{A}_1$) - ($\mathbf{A}_3$) are satisfied. Then, for each λ satisfying

$$\frac{1}{\left[\gamma^2 \int_{t_1}^{t_2} G(s,s) ds\right] f_0} < \lambda < \frac{1}{\left[\int_{t_1}^{t_2} G(s,s) ds\right] f^{\infty}} \quad (16)$$

there exist at least one positive solution of the BVP (1) - (2) in κ.

The proof follows by imitating the statement of the proof in Theorem 3.

2.4 Example

Let's consider the example

$$y''+\lambda\frac{y(1+200y)}{1+y}=0,\ t\in[0,1]$$

with boundary conditions $\begin{cases} y(0)-y'(0)=0 \\ 2y(1)+3y'(1)=0 \end{cases}$

The Green's function is given by

$$G(t,s)=\begin{cases}\frac{1}{7}(-1-s)(-5+2t) & \text{if } 0\le s\le t; \\ \frac{1}{7}(5-2s)(1+t) & \text{if } 0\le t\le s.\end{cases}$$

We found $\gamma=\frac{1}{2}$, $f_\infty=200$ and $f^0=1$. Employing (13), there is a positive solution for all λ in the range $\left(\frac{3}{125},\frac{6}{5}\right)$.

3. Green's Function and Bounds for the $2n^{th}$ Order Boundary Value Differential Equation

Our interest in this section is finding positive solutions to all differential equation of the form

$$(-1)^{\frac{n}{2}}y^{(n)}=\lambda f(t,y(t)) \tag{17}$$

for even n, with boundary conditions

$$\begin{cases} \alpha_{11} y^{(2k)}(t_1) + \alpha_{12} y^{(2k+1)}(t_1) &= 0 \\ \alpha_{21} y^{(2k)}(t_2) + \alpha_{22} y^{(2k+1)}(t_2) &= 0, \ k = 0,1,2,\ldots \frac{n}{2} - 1. \end{cases} \quad (18)$$

Before we can do this, we need to be able to generate the Green's function of the homogeneous boundary value problem which we do in the following subsection.

3.1 Green's Function for the $2n^{(th)}$ Order DE

In this section, we will derive Green's function for $2n^{th}$ order homogeneous differential equation (17) satisfying (18).

Theorem 5. *Suppose that $G_2(t,s)$ is the Green's function satisfying*

$$-y''(t) = 0$$

with boundary conditions

$$\begin{cases} \alpha_{11} y(t_1) + \alpha_{12} y'(t_1) = 0 \\ \alpha_{21} y(t_2) + \alpha_{22} y'(t_2) = 0 \end{cases}$$

Then ,

$$G_n(t,s) = \int_{t_1}^{t_2} G_2(t,w) G_{n-2}(w,s)\,dw, \ \ n \in \{2k+2 : k \in \mathbb{N}\} \quad (19)$$

is the Green's function for

$$(-1)^{\frac{n}{2}} y^n(t) = 0, \ \ n \in \{2k+2 : k \in \mathbb{N}\}, \quad (20)$$

with boundary conditions (18).

Proof.

We shall show the proof by induction. First, we prove the case for $n = 4$.
Suppose $G_2(t,s)$ is the Green's function satisfying $-y''(t) = 0$, then

$$-y''(t) = g \ \ \Rightarrow \ \ y(t) = \int_{t_1}^{t_2} G_2(t,s) g(s)\,ds$$

so that

$$y''''(t) = g \Rightarrow (y'')'' = g\,.$$

Hence,

$$y''(t) = -\int_{t_1}^{t_2} G_2(t,s)g(s)ds = -H(t).$$

Thus

$$\begin{aligned} y(t) &= \int_{t_1}^{t_2} G_2(t,w)H(w)dw \\ &= \int_{t_1}^{t_2} G_2(t,w)\left\{\int_{t_1}^{t_2} G_2(w,s)g(s)ds\right\}dw \\ &= \int_{t_1}^{t_2}\left\{\int_{t_1}^{t_2} G_2(t,w)G_2(w,s)g(s)ds\right\}dw \\ &= \int_{t_1}^{t_2}\left\{\int_{t_1}^{t_2} G_2(t,w)G_2(w,s)dw\right\}g(s)ds \\ &= \int_{t_1}^{t_2} G_4(t,s)g(s)ds \end{aligned}$$

where

$$G_4(t,s) = \int_{t_1}^{t_2} G_2(t,w)G_2(w,s)dw.$$

From definition of $G_2(t,s), G_4(t,s) = \int_{t_1}^{t_2} G_2(t,w)G_2(w,s)dw$, y'' satisfies the boundary conditions (2).

Likewise, $G_4(t,s)$ satisfies boundary conditions (2) so that $y(t)$ satisfies the BC

$$\alpha_{11}y(t_1) + \alpha_{12}y'(t_1) = 0$$
$$\alpha_{21}y(t_2) + \alpha_{22}y'(t_2) = 0$$

$$\alpha_{11}y''(t_1) + \alpha_{12}y'''(t_1) = 0$$
$$\alpha_{21}y''(t_2) + \alpha_{22}y'''(t_2) = 0.$$

So, $G_4(t,s)$ is the Green's function for the equation

$$y''''(t) = 0,$$

satisfying the BCs

$\alpha_{11}y(t_1)+\alpha_{12}y'(t_1)=0$

$$\alpha_{21}y(t_2)+\alpha_{22}y'(t_2)=0$$

$\alpha_{11}y''(t_1)+\alpha_{12}y'''(t_1)=0$

$\alpha_{21}y''(t_2)+\alpha_{22}y'''(t_2)=0.$

Assume the case for $n=2k+2$ is true. Without loss of generality, assume k is odd. For $n=2k+4$, $(-1)^{k+2}y^{(2k+4)}=g$ implies $(-1)^{k+1}(y'')^{(2k+2)}=-g$. This implies

$$y''(t)=-\int_{t_1}^{t_2}G_{2k+2}(t,s)g(s)ds=-H_1(t).$$

Thus,

$$\begin{aligned} y(t) &= \int_{t_1}^{t_2}G_2(t,w)H_1(w)dw \\ &= \int_{t_1}^{t_2}G_2(t,w)\left[\int_{t_1}^{t_2}G_{2k+2}(w,s)g(s)ds\right]dw \\ &= \int_{t_1}^{t_2}\left[\int_{t_1}^{t_2}G_2(t,w)G_{2k+2}(w,s)dw\right]g(s)ds \\ &= \int_{t_1}^{t_2}G_{2k+4}(t,s)g(s)ds \end{aligned}$$

where $G_{2k+4}(t,s)=\int_{t_1}^{t_2}G_2(t,w)G_{2k+2}(w,s)dw$. This ends the proof.

3.2 Bounds for the Green's Function

Here, we find bound for the Green's function for the $2n^{th}$ order problem.

Theorem 6. *Assuming conditions ($\mathbf{A}_1$)-($\mathbf{A}_3$). Define*

$$C_n(s,s)=G_2(s,s)\left(\int_{t_1}^{t_2}G_2(x,x)dx\right)^{\frac{n}{2}-1}. \quad (21)$$

Then

$$\gamma^{\frac{n}{2}} C_n(s,s) \le G_n(t,s) \le C_n(s,s) \quad \text{for n } \in \{2k; k \in \square\}$$

Proof.

We shall show the proof by induction. For the case $n=4$, from previous theorem, $\gamma G_2(s,s) \le G_2(t,s) \le G_2(s,s) \ \forall \ (t,s) \in [t_1,t_2]\times[t_1,t_2]$.

So,

$$\begin{aligned} G_4(t,s) &= \int_{t_1}^{t_2} G_2(t,x) G_2(x,s)dx \\ &\le \int_{t_1}^{t_2} G_2(x,x) G_2(s,s)dx = C_4(s,s). \end{aligned}$$

Also

$$\begin{aligned} G_4(t,s) &= \int_{t_1}^{t_2} G_2(t,x) G_2(x,s)dx \\ &\ge \int_{t_1}^{t_2} \gamma^2 G_2(x,x) G_2(s,s)dx \\ &\ge \gamma^2 C_4(s,s). \end{aligned}$$

Hence,
$\gamma^2 C_4(s,s) \le G_4(t,s) \le C_4(s,s)$.

Suppose the case $n=k$ is true, that is

$$\gamma^{\frac{k}{2}} C_k(s,s) \le G_k(t,s) \le C_k(s,s). \quad (22)$$

For the case $n=k+2$,

$$\begin{aligned} G_{k+2}(t,s) &= \int_{t_1}^{t_2} G_2(t,x) G_k(x,s)dx \\ &\le \int_{t_1}^{t_2} G_2(x,x) C_k(s,s)dx = C_{k+2}(s,s). \end{aligned}$$

Likewise,

$$\begin{aligned} G_{k+2}(t,s) &= \int_{t_1}^{t_2} G_2(t,x)G_k(x,s)dx \\ &\geq \int_{t_1}^{t_2} \gamma G_2(x,x)\gamma^{k/2}C_k(s,s)dx \\ &\geq \gamma^{\frac{k+2}{2}}\int_{t_1}^{t_2} G_2(x,x)C_k(s,s)dx = \gamma^{\frac{k+2}{2}} C_{k+2}(s,s) \end{aligned}$$

The following theorem gives us the eigenvalue interval for which there exists positive solution(s) for even order problems.

Theorem 7. *For* $n \in \{2k; k \in \mathbb{N}\}$, *assuming that conditions (* A_1 *)-(* A_3 *) is satisfied, then for each* λ *satisfying*

$$\frac{1}{\left[\gamma^n \int_{t_1}^{t_2} C_n(s,s)ds\right] f_\infty} < \lambda < \frac{1}{\left[\int_{t_1}^{t_2} C_n(s,s)ds\right] f^0}, \quad (23)$$

there exist at least one positive solution of the BVP

$$(-1)^{\frac{n}{2}} y^n(t) = \lambda f(t, y(t)). \quad (24)$$

with boundary conditions

$$\begin{aligned} \alpha_{11}y^{(2k)}(t_1) + \alpha_{12}y^{(2k+1)}(t_1) &= 0 \\ \alpha_{21}y^{(2k)}(t_2) + \alpha_{22}y^{(2k+1)}(t_2) &= 0, \ k = 0,1,2,\ldots\frac{n}{2}-1. \end{aligned}$$

Proof.

The proof follows by using Theorem 2and changing γ to be $\gamma^{\frac{n}{2}}$ in (13) and (16).

3.3 Example

Using (19), we can easily generate the Green's function for the case where $n = 4, 6, 8, 10$, and so on. Below is one of such computed Green's function.

For the case where $n = 4$

$G_4(t,s)=$

$$\begin{cases} -\dfrac{(\beta_1-\alpha_{11}s)(s-t)(-\beta_2+\alpha_{21}t)\left(3\beta_1\left(-2\beta_2+\alpha_{21}(s+t)\right)+\alpha_{11}(3\beta_2(s+t)-2\alpha_{21}(s^2+st+t^2))\right)}{6D^2} \\ +\dfrac{(\beta_1-\alpha_{11}s)(\beta_1-\alpha_{11}t)((\beta_2-\alpha_{21}t)^3+(-\beta_2+\alpha_{21}t_2)^3)}{3\alpha_{21}D^2} \\ +\dfrac{(\beta_2-\alpha_{21}s)(\beta_2-\alpha_{21}t)((-\beta_1+\alpha_{11}s)^3+(\beta_1-\alpha_{11}t_1)^3)}{3\alpha_{11}D^2} \text{ if } t_1\le s\le t\le t_2; \\ \dfrac{(\beta_2-\alpha_{21}s)(s-t)(\beta_1-\alpha_{11}t)(-3\beta_1(-2\beta_2+\alpha_{21}(s+t))+\alpha_{11}(-3\beta_2(s+t)+2\alpha_{21}(s^2+st+t^2)))}{6D^2} \\ +\dfrac{(\beta_1-\alpha_{11}s)(\beta_1-\alpha_{11}t)((\beta_2-\alpha_{21}s)^3+(-\beta_2+\alpha_{21}t_2)^3)}{3\alpha_{21}D^2} \\ +\dfrac{(\beta_2-\alpha_{21}s)(\beta_2-\alpha_{21}t)((-\beta_1+\alpha_{11}t)^3+(\beta_1-\alpha_{11}t_1)^3)}{3\alpha_{11}D^2} \quad if \; t_1\le t\le s\le t_2. \end{cases}$$

is the Green's function satisfying

$y^{(4)}=0$

with boundary conditions

$$\begin{cases} \alpha_{11}y(t_1)+\alpha_{12}y'(t_1)=0 \\ \alpha_{21}y(t_2)+\alpha_{22}y'(t_2)=0 \\ \alpha_{11}y''(t_1)+\alpha_{12}y'''(t_1)=0 \\ \alpha_{21}y''(t_2)+\alpha_{22}y'''(t_2)=0. \end{cases} \quad (25)$$

For a specific case, consider the equation

$$y^{(4)}(t)=\lambda\frac{y(1+200y)}{1+y},\; t\in[0,1],$$

with boundary conditions $\begin{cases} y(0)-y'(0)=0 \\ 2y(1)+3y'(1)=0 \\ y''(0)-y'''(0)=0 \\ 2y''(1)+3y'''(1)=0 \end{cases}$

the Green's function is

$G_4(t,s)=$

$$\begin{cases} \frac{1}{147}\left(8-(1+s)^3\right)(5-2s)(5-2t)+\frac{1}{294}(1+s)(1+t)\left(125+(-5+2t)^3\right) \\ -\frac{1}{294}(1+s)(5-2t)(s-t)\left(-15(s+t)+4(s^2+st+t^2)+3(-10+2(s+t))\right) \ if \ \ t_1\le s\le t\le t_2; \\ \frac{1}{147}(5-2s)(8-(1+t)^3)(5-2t)+\frac{1}{294}(1+s)(125+(-5+2s)^3)(1+t) \\ +\frac{1}{294}(5-2s)(1+t)(s-t)\left(-15(s+t)+4(s^2+st+t^2)-3(10-2(s+t))\right) \text{ if } t_1\le t\le s\le t_2 \end{cases}$$

We found that $\gamma=\frac{1}{2}$, $f_\infty=200$, and $f^0=1$. Employing (13), we get the eigenvalue interval $\frac{72}{30625}<\lambda<\frac{36}{1225}$ for which there exists a positive solution.

4 . Third-Order Boundary Value Problem on □ with Green's Function and Bound

For this section, we are going to consider the third order eigenvalue problem on □. We are going to consider nonhomogeneous boundary conditions. In this section, we assume $f(t,y(t))$ to be as defined in Section 2.

4.1 Solving the Third Order Equation

Consider the boundary value problem

$$y'''(t)=\lambda f(t,y(t)),\ \ t\in[t_1,t_3] \tag{26}$$

with boundary conditions

$$\begin{cases} y(t_1) = \rho_1 \\ y'(t_2) = \rho_2 \text{ (27)} \\ y''(t_3) = \rho_3 \end{cases}$$

Defining $g(t) \equiv \lambda f(t, y(t))$, taking the Laplace transform of (26) and following the procedure used in finding the solution of (1)-(2), we have the solution of (26)-(27) as follows;

$$\begin{aligned} y(t) &= \rho_1 + (t-t_1)\rho_2 + \frac{1}{2}\left((t-t_2)^2 - (t_2-t_1)^2\right)\rho_3 \\ &\quad -\frac{1}{2}\int_{t_1}^{t_3}((t-t_2)^2 - (t_2-t_1)^2)g(s)ds \\ &\quad -\int_{t_1}^{t_2}(t_2-s)(t-t_1)g(s)ds + \frac{1}{2}\int_{t_1}^{t}(t-s)^2 g(s)ds. \end{aligned}$$

Define

$$z(t) \equiv \rho_1 + (t-t_1)\rho_2 + \frac{1}{2}((t-t_2)^2 - (t_2-t_1)^2)\rho_3, \text{ (28)}$$

we have

$$\begin{aligned} y(t) &= z(t) - \frac{1}{2}\int_{t_1}^{t_3}\left((t-t_2)^2 - (t_2-t_1)^2\right)g(s)ds \\ &\quad -\int_{t_1}^{t_2}(t_2-s)(t-t_1)g(s)ds + \frac{1}{2}\int_{t_1}^{t}(t-s)^2 g(s)ds, \end{aligned}$$

where $z(t)$ is the solution of the homogeneous boundary value differential equation

$$y'''(t) = 0,$$

with boundary conditions (27). Also,

$$G(t,s) = \begin{cases} \frac{1}{2}(s-t_1)^2 \text{ if } t_1 \le s \le t \le t_2 < t_3; \\ \frac{1}{2}\left[(s-t_1)^2 - (s-t)^2\right] \text{ if } t_1 \le t \le s \le t_2 < t_3; \\ \frac{1}{2}\left[(t_2-t_1)^2 - (t_2-t)^2\right] \text{ if } t_1 \le t \le t_2 \le s \le t_3; \\ \frac{1}{2}\left[(t_2-t_1)^2 - (t-t_2)^2 + (t-s)^2\right] \text{ if } t_1 < t_2 \le s \le t \le t_3; \\ \frac{1}{2}\left[(t_2-t_1)^2 - (t-t_2)^2\right] \text{ if } t_1 < t_2 \le t \le s \le t_3; \\ \frac{1}{2}(s-t_1)^2 \text{ if } t_1 \le s \le t_2 \le t \le t_3. \end{cases} \quad (29)$$

is the Green's function for the equation

$y'''(t) = 0,$ (30)

with boundary conditions

$$\begin{cases} y(t_1) = 0 \\ y'(t_2) = 0 \quad (31) \\ y''(t_3) = 0. \end{cases}$$

For the rest of this Section, we define $G(t,s) \equiv G_3(t,s)$. From (28), $z(t)$ has zeroes t' and t'', where

$$\begin{cases} t' = \dfrac{(\rho_3 t_2 - b_2) + \sqrt{A}}{\rho_3}, \\ t'' = \dfrac{(\rho_3 t_2 - b_2) - \sqrt{A}}{\rho_3}, \text{ and} \\ A = [\rho_3(t_1 - t_2) + \rho_2]^2 - 2\rho_1\rho_3. \end{cases} \quad (32)$$

We assume the following conditions on t_1, t_2, t_3 and ρ_1, ρ_2, ρ_3 throughout this Section:

$$\mathbf{B}_1 : t_2 > \frac{t_1 + t_3}{2},\ t_3 < t''$$

$$\mathbf{B}_2 : \rho_1 > 0,\ \rho_3 < 0,\ (t_2 - t_1)\rho_3 < \rho_2 < (t_2 - t_3)\rho_3 .$$

Note: $\mathbf{B}_1$ is derived from the fact that $G(t_3, s)$ must be nonnegative on the interval $t_1 < t_2 \le t \le s \le t_3$ .We choose $t_3 < t''$ so that $(t_1, t_3) \subset (t', t'')$. $\mathbf{B}_2$ is derived such that $t_1 < t_2 - \frac{\rho_2}{\rho_3} < t_3$, where $t_2 - \frac{\rho_2}{\rho_3}$ is the maximum point of $z(t)$. Also, we make $\rho_3 < 0$ because we want $z(t)$ to be concave down and $\rho_1 > 0$ since we want a positive solution for $y(t)$.

4.2 Bounds for the Green's Function

In this section, we find the bounds for the Green's function (29).

Theorem 8. *Given that condition ($\mathbf{B}_1$) holds,* $G(t,s) > 0$ *for (t,s)* $\in (t_1, t_3] \times (t_1, t_3]$.

Proof.

For $t_1 \le s \le t \le t_2 < t_3$, $G(t,s) > 0$ since $s \ne t_1$.
For $t_1 < t < s \le t_2 < t_3$, since $t_1 < t < s$, we have $s - t_1 > s - t > 0$ and so $G(t,s) = \frac{1}{2}\left[(s - t_1)^2 - (s - t)^2\right] > 0$. Also, if $t = s$, then $G(t,s) = \frac{1}{2}(s - t_1)^2 > 0$. Hence, $G(t,s) > 0$.
For $t_1 < t < t_2 \le s \le t_3$, since $t_1 < t < t_2$, we have $t_2 - t_1 > t - t_1$ and so $G(t,s) = \frac{1}{2}\left[(t_2 - t_1)^2 - (t_2 - t)^2\right] > 0$. Also, if $t = t_2, G(t,s) = \frac{1}{2}(t_2 - t_1)^2 > 0$. Therefore $G(t,s) > 0$.

For $t_1 < t_2 \le s \le t \le t_3$, since $t_2 > \frac{t_1+t_3}{2}$, we have $t_2 - t_1 > t_3 - t_2 > t - t_2$.

So, $G(t,s) = \frac{1}{2}\left[(t_2-t_1)^2 - (t-t_2)^2 + (t-s)^2\right] > 0$.

For $t_1 < t_2 \le t \le s \le t_3$, $G(t,s) > 0$ since $t_2 > \frac{t_1+t_3}{2}$.

Lastly, for $t_1 < s \le t_2 \le t < t_3$, $G(t,s) > 0$ since $s \ne t_1$.

In the next theorem, we find the bounds for the Green's function (29). This bound is later used to find the range of λ values for which (26) -(27) has a positive solution.

Theorem 9. *For a fixed s,*

$$G(t,s) \le \frac{1}{2}(s-t_1)^2 \text{ for all (t,s) } \in (t_1,t_3]\times(t_1,t_3].$$

$$G(t,s) \ge \frac{1}{2}((t_2-t_1)^2 - (t_3-t_2)^2) \text{ for all (t,s) } \in [t_2,t_3]\times[t_2,t_3].$$

Proof.

For $t_1 \le t < s < t_2 < t_3$, $G'(t,s) = s - t > 0$ which implies that $G(t,s)$ is an increasing function of t. So, $G(t,s) < G(s,s)$ for $t < s$.

For $t_1 \le t \le t_2 \le s \le t_3$, $G'(t,s) = t_2 - t \ge 0$. Hence, $G(t,s)$ is a non-decreasing function of t and

$$G(t,s) \le G(t_2,s) = \frac{1}{2}(t_2-t_1)^2 \le \frac{1}{2}(s-t_1)^2 \text{ for } t \le t_2 \le s.$$

Likewise, for $t_1 < t_2 \le s \le t \le t_3$, $G'(t,s) = t_2 - s \le 0$, so $G(t,s)$ is a non-increasing function of t and $G(t,s) \le G(s,s) = \frac{1}{2}\left[(t_2-t_1)^2 - (s-t_2)^2\right] \le \frac{1}{2}(t_2-t_1)^2 \le \frac{1}{2}(s-t_1)^2$.

For $t_1 < t_2 \le t \le s \le t_3$, $t < t_3$ and $-(t-t_2)^2 > -(t_3-t_2)^2$. Hence

$$G(t,s)=\frac{1}{2}\left((t_2-t_1)^2-(t-t_2)^2\right)\geq\frac{1}{2}\left((t_2-t_1)^2-(t_3-t_2)^2\right).$$

Lastly,for $t_1 < t_2 \leq s \leq t \leq t_3$,

$$\begin{aligned} G(t,s) &= \frac{1}{2}\left((t_2-t_1)^2-(t-t_2)^2+(t-s)^2\right)\geq\frac{1}{2}\left((t_2-t_1)^2-(t-t_2)^2\right) \\ &\geq \frac{1}{2}\left((t_2-t_1)^2-(t_3-t_2)^2\right) \end{aligned}$$

4.3 Existence of Positive Solution.

In this subsection, we find the range of λ for which (26)-(27) has positive solution.Let $y(t)$ be the solution of the BVP (26)-(27), given by

$$y(t)=z(t)+\lambda\int_{t_1}^{t_3}G(t,s)\,f(s,y(s))ds \qquad (33)$$

Defining

$$v(t)\equiv y(t)-z(t),$$

(33) can be re-written as

$$v(t)=\lambda\int_{t_1}^{t_3}G(t,s)\,f(s,v(s))ds, \qquad (34)$$

which is the solution of the homogeneous boundary value differential equation

$$v'''(t)=\lambda f(t,v(t)),\ \ t\in[t_1,t_3], \qquad (35)$$

with boundary conditions

$$\begin{cases} v(t_1)=0 \\ v'(t_2)=0 \\ v''(t_3)=0. \end{cases} \quad (36)$$

Also $G(t,s)$ is the Green's function for the differential equation

$$v'''(t)=0,\ \ t\in[t_1,t_3]$$

with boundary conditions (36).

Define a set X by
$X=\{u\,|\,u\in C[t_1,t_3]\}$

with norm

$$\|u\|=\max_{t\in[t_1,t_3]}|u(t)|,$$

Then $(X,\|.\|)$ is a Banach space.

Let

$$m=\min\left\{\min_{t_2\le s\le t}\left\{\frac{(t_2-t_1)^2-(t_3-t_2)^2+(t_3-s)^2}{(t_2-t_1)^2+(t_2-s)^2}\right\},\frac{(t_2-t_1)^2-(t_3-t_2)^2}{(t_2-t_1)^2}\right\}. \quad (37)$$

We first show that $0<m<1$.

Since for $t_1<t_2\le s\le t\le t_3$, we have $G'(t,s)=t_2-s\le 0$. Hence $G(t,s)$ is a decreasing function of t and $G(t_3,s)<G(t_2,s)$.

Also, for $t_1<t_2\le t\le s\le t_3$, we have $G'(t,s)=t_2-t\le 0$, so $G(t,s)$ is a decreasing function of t and $G(t_3,s)<G(t_2,s)$.

Define a set κ by

$$\kappa=\{u\in X:u(t)\ge 0 \text{ on } [t_1,t_2] \text{ and } \min_{t\in[t_2,t_3]}u(t)\ge m\|u\|\}.$$

It follows that κ is a cone. Using condition ($\mathbf{B}_2$),

$$z(t)>0 \text{ for } t\in(t',t''),$$

where t' and t'' are as define in (32).

From the fact that $z(t')=0$ and $z(t_1)=\rho_1>0$, we conclude that $t'<t_1$ since z(t) is concave down. Also, since $t_3<t''$ then $(t_1,t_3)\subseteq(t',t'')$.So, we conclude that

$z(t)\geq 0$ for $t\in[t_1,t_3]$.

Define the operator T : $\kappa\rightarrow X$ by

$$(Tv)(t)=\lambda\int_{t_1}^{t_3}G(t,s)\,f(s,v(s))ds,\ \ \forall\ \ t\in[t_1,t_3] \tag{38}$$

It follows that T preserves κ .If $v\in\kappa$ is a fixed point of *T*, then v satisfies ((35) and hence v is a positive solution of the BVP (35)-(36). We seek a fixed point of the operator ,T , in the cone κ .

Now, we find the range of λ that gives a positive solution for (34)

Theorem 10. *Assume that conditions ($\mathbf{B}_1$),($\mathbf{B}_2$) is satisfied. Then, for each λ satisfying*

$$\frac{1}{\left[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)ds\right]f_\infty}<\lambda<\frac{1}{\left[\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2ds\right]f^0}, \tag{39}$$

there exist at least one positive solution of the BVP (35)-(36)) in κ where m is defined in (37).

Proof.

Let λ be given as in (39). Now, let $\delta>0$ be chosen such that

$$\frac{1}{\left[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\ ds\right](f_\infty-\delta)}\leq\lambda\leq\frac{1}{\left[\int_{t_1}^{t_3}G(s,s)ds\right](f^0+\delta)}.$$

Let T be the cone preserving, completely continuous operator defined in (38). By definition of f^0, there exist $H_1 > 0$ such that

$$\max_{t\in[t_1,t_3]} \frac{f(t,v)}{v} \le (f^0+\delta), \text{ for } 0<v\le H_1.$$

It follows that, $f(t,v) \le (f^0+\delta)v$, for $0<v\le H_1$. So choosing $v_1 \in \kappa$ with $\| v_1 \| = H_1$. Then, we have from the boundedness of $G(t,s)$ that

$$\begin{aligned}(Tv_1)(t) &= \lambda\int_{t_1}^{t_3} G(t,s) f(s,v_1(s))ds \\ &\le \lambda\int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 f(s,v_1(s))ds \\ &\le \lambda\int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 (f^0+\delta)v_1(s)ds \\ &\le \lambda\int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 (f^0+\delta)\| v_1 \|ds \\ &\le \| v_1 \|.\end{aligned}$$

Consequently, $\| Tv \| \le \| v \|$. So, if we define

$$\Omega_1 = \{u \in X : \| u \| < H_1\},$$

Then

$$\| Tv \| \le \| v \|, \text{ for } v \in \kappa \cap \partial\Omega_1. \quad (40)$$

By definition of f_∞, there exists an $\overline{H_2} > 0$ such that

$$\min_{t\in[t_1,t_3]} \frac{f(t,v)}{v} \ge (f_\infty-\delta), \text{for } v \ge \overline{H_2}.$$

It follows that $f(t,v) \ge (f_\infty-\delta)v$, for $v \ge \overline{H_2}$.

Let $$H_2 = \max\{2H_1, \frac{1}{m}\overline{H_2}\},$$

and $$\Omega_2 = \{u \in X : \| u \| < H_2\}.$$

Now choose $v_2 \in \kappa \cap \partial\Omega_2$ with $\| v_2 \| = H_2,$ so that $\min_{t\in[t_1,t_2]} v_2(t) \geq m \| v_2 \| \geq \overline{H_2}.$

Consider,

$$\begin{aligned}
T(v_2)(t) &= \lambda\int_{t_1}^{t_3} G(t,s) f(s,v_2(s))\, ds \\
&\geq \lambda\int_{t_2}^{t_3} \frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2) f(s,v_2(s))\, ds \\
&\geq \lambda\int_{t_2}^{t_3} \frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)(f^{\infty}-\delta)v_2(s)\, ds \\
&\geq m\lambda\int_{t_2}^{t_3} \frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)(f^{\infty}-\delta)\| v_2 \|\, ds \\
&\geq \| v_2 \|.
\end{aligned}$$

Thus,

$$\| Tv \| \geq \| v \|, \text{ for } v \in \kappa \cap \partial\Omega_2 \tag{41}$$

Applying Theorem 2 to (40) and (41) yields a fixed point for $Tv(t) \in \kappa \cap (\overline{\Omega_2} \setminus \Omega_1)$. This fixed point is the positive solution of the BVP (35)-(36) for the given λ .

Next, we prove other range for λ for which a positive solution exists.

Theorem 11. *Assume that conditions (*$\mathbf{B}_1$*)-(*$\mathbf{B}_2$*) is satisfied. Then, for each* λ *satisfying*

$$\frac{1}{\left[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)ds\right]f_0} < \lambda < \frac{1}{\left[\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2\, ds\right]f^{\infty}} \tag{42}$$

there exist at least one positive solution of the BVP (35)-(36) in κ .

Proof.

The proof is similar to the proof given in Theorem 10.

4.4 Green's Function and Bound for the $3n^{(th)}$ Order BVP

Our interest in this Section is to find positive solutions to all differential equations of the form

$$y^{(n)} + \lambda f(t, y(t)) = 0 \quad (43)$$

subject to some boundary conditions

$$\begin{cases} y^{(3k)}(t_1) & = \rho_1 \\ y^{(3k+1)}(t_2) & = \rho_2 \\ y^{(3k+2)}(t_3) & = \rho_3,\ k = 0,1,2,\ldots,\dfrac{n}{3}-1. \end{cases} \quad (44)$$

We generate the Green's function of the homogeneous boundary value problem (43)-(44)

Theorem 12. *Suppose that $G_3(t,s)$ is the Green's function of (30)-(31). Then ,*

$$G_n(t,s) = \int_{t_1}^{t_3} G_3(t,w) G_{n-3}(w,s)\,dw, \quad n \in \{3k+3 : k \in \mathbb{N}\} \quad (45)$$

is the Green's function for

$$y^n(t) = 0,\ n \in \{3k+3 : k \in \mathbb{N}\}, \quad (46)$$

with boundary conditions

$$\begin{cases} y^{(3k)}(t_1) & = 0 \\ y^{(3k+1)}(t_2) & = 0 \\ y^{(3k+2)}(t_3) & = 0,\ k = 0,1,2,\ldots,\dfrac{n}{3}-1. \end{cases} \quad (47)$$

Proof.

The proof is similar to the proof given in Theorem 5.

4.5 Bounds for the Green's Function

In this section, we find the bounds for Green's function, $G_n(t,s)$, $n \in \{3k; k \in \mathbb{N}\}$.

Theorem 13. *Assuming conditions* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *, then for* $n \in \{3k; k \in \mathbb{N}\}$,

$$\left(\frac{1}{2}\right)^{\frac{n}{3}}\left(t_3-t_1\right)^{\frac{n}{3}-1}\left((t_2-t_1)^2-(t_3-t_2)^2\right)^{\frac{n}{3}} \le G_n(t,s) \text{ for all } (t,s)\in[t_2,t_3]\times[t_2,t_3].$$

$$G_n(t,s) \le 3\left(\frac{1}{6}\right)^{\frac{n}{3}}(t_3-t_1)^{n-3}(s-t_1)^2 \text{ for all } (t,s)\in[t_1,t_3]\times[t_1,t_3].$$

Proof.

We shall show the proof by induction. From Theorem 9,

$$G_3(t,s) \le \frac{1}{2}(s-t_1)^2 \text{ for all } (t,s)\in[t_1,t_3]\times[t_1,t_3], \text{and}$$

$$G_3(t,s) \ge \frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2) \text{ for all } (t,s)\in[t_2,t_3]\times[t_2,t_3].$$

Assuming the case for $n=k$ is true, that is,

$$\left(\frac{t_3-t_1}{2}\right)^{\frac{k}{3}-1}\left((t_2-t_1)^2-(t_3-t_2)^2\right)^{\frac{k}{3}} \le G_k(t,s) \text{ for all } (t,s)\in[t_2,t_3]\times[t_2,t_3].$$

$$G_k(t,s) \le 3\left(\frac{1}{6}\right)^{\frac{k}{3}}(t_3-t_1)^{k-3}(s-t_1)^2 \text{ for all } (t,s)\in[t_1,t_3]\times[t_1,t_3],$$

For $n=k+3$,

$$\begin{aligned} G_{k+3}(t,s) &= \int_{t_1}^{t_3} G_3(t,w)G_k(w,s)dw \\ &\le \int_{t_1}^{t_3} \frac{3}{2}(w-t_1)^2\left(\frac{1}{6}\right)^{\frac{k}{3}}(t_3-t_1)^{k-3}(s-t_1)^2 dw \\ &= 3\left(\frac{1}{6}\right)^{\frac{k+3}{3}}(t_3-t_1)^k(s-t_1)^2. \end{aligned}$$

Also,

$$\begin{aligned} G_{k+3}(t,s) &= \int_{t_1}^{t_3} G_3(t,w)G_k(w,s)dw \\ &\ge \int_{t_1}^{t_3} \frac{1}{2}\left((t_2-t_1)^2-(t_3-t_2)^2\right)\left(\frac{t_3-t_1}{2}\right)^{\frac{k}{3}-1}\left((t_2-t_1)^2-(t_3-t_2)^2\right)^{\frac{k}{3}} dw \\ &\ge \left(\frac{t_3-t_1}{2}\right)^{\frac{k}{3}}\left((t_2-t_1)^2-(t_3-t_2)^2\right)^{\frac{k+3}{3}}. \end{aligned}$$

By defining the two functions

$$\begin{aligned} F_n(s,s) &= \left(\frac{1}{3}\right)^{\frac{n-3}{3}}\left(\frac{1}{2}\right)^{\frac{n}{3}}(t_3-t_1)^{n-3}(s-t_1)^2, \\ E_n(s,s) &= \left(\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\right)^{\frac{n}{3}}(t_3-t_1)^{\frac{n-3}{3}} \end{aligned}$$

and using (39) and (42), we can state the following theorems.

Theorem 14. *Assume that conditions ($\mathbf{B}_1$),($\mathbf{B}_2$) are satisfied. Then, for each λ satisfying*

$$\frac{1}{\left[m\int_{t_2}^{t_3} E_n(s,s)ds\right]f_\infty} < \lambda < \frac{1}{\left[\int_{t_1}^{t_3} F_n(s,s)ds\right]f^0},$$

there exist at least one positive solution of the BVP (46)-(47) in κ .

Proof.

The proof is similar to that of Theorem 10.

Theorem 15. *Assume that conditions ($\mathbf{B}_1$),($\mathbf{B}_2$) are satisfied. Then, for each λ satisfying*

$$\frac{1}{\left[m\int_{t_2}^{t_3} E_n(s,s)ds\right] f_0} < \lambda < \frac{1}{\left[\int_{t_1}^{t_3} F_n(s,s)ds\right] f^{\infty}}$$

there exist at least one positive solution of the BVP (46)-(47) in κ .

Proof.

The proof is similar to that of Theorem 11.

4.6 Example

Consider the third order boundary value problem

$$y'''(t) + \lambda y(200 - 199.5e^{-7y}) = 0,\ \ t \in [0,1],$$

with boundary conditions

$$\begin{cases} y(1) & = 1 \\ y'(2.6) & = 0 \\ y''(4) & = -1. \end{cases}$$

The Green's function is given by

$$G(t,s) = \begin{cases} \frac{1}{2}(s-1)^2 \;\; if \;\; 1 \le s \le t \le 2.6 < 4; \\ \frac{1}{2}(-1+2s-t)(-1+t) \;\; if \;\; 1 \le t \le s \le 2.6 < 4; \\ \frac{1}{2}(4.2-t)(t-1) \;\; \text{if} \;\; 1 \le t \le 2.6 \le s \le 4; \\ \frac{1}{2}(-4.2+s^2+5.2t-2st) \;\; if \;\; 1 < 2.6 \le s \le t \le 4; \\ \frac{1}{2}(4.2-t)(-1.+t) \;\; if \;\; 1 < 2.6 \le t \le s \le 4; \\ \frac{1}{2}(s-1)^2 \;\; if \;\; 1 \le s \le 2.6 \le t \le 4 \end{cases}$$

For this particular example,

$$z(t) = 1 + \frac{1}{2}(2.56 - (t-2.6)^2), \; m = 0.132743, \; f_\infty = 200, \; f^0 = \frac{1}{2}.$$

Using (39), positive solution exists for all λ in the interval (0.0897,0.2222).

References

Anderson, D.R. and Davis, J.M. (2002).Multiple Solutions and Eigenvalues for Third Order Right Focal Boundary Value Problems. J Math. Appl, pp. 135-157.

Chyan C. J., Davis J. M., Henderson J., and Yin W. K. C. (1998). Eigenvalue Comparisons for Differential Equations on a measure chain. Electron. J. Differential Equations, 35),1-7.

Chyan C. J. and Henderson J. (2000).Eigenvalue Problems for Nonlinear Differential Equations ona Measure Chain. J. Math. Anal. Appl., 547-559.

Chyan C. J., Henderson J., and Lo C. J. (1999).Positive Solutions in an Annulus for NonlinearDifferential Equations on a Measure Chain.Tamkang J. Math., 30(3),231-240.

Clark S. and Hinton D. B. (1998).Hinton.A Lyapunov Inequality for Linear Hamiltonian Systems.Math.Inequal. Appl., 1(2),201-209.

Coddington E. A. and Levinson N. (1955). Theory of Ordinary Differential Equations. McGraw-HillBook Company, Inc., New York,.

Davis, J.M. Henderson, J, Prasad, K.R. and Yin, W. (2000).Eigen Intervals for Non-Linear Right Focal Problems. Appl. Anal. 74, 215-231.

Eloe, P.W. Henderson, J (1998).Positive Solutions and Nonlinear (k, n-k) Conjugate Eigenvalue Problem. Diff. Eqns.dyn.Sys. 6, 309-317.

ErbeL.H.and Wang H.On the Existence of Positive Solutions of Ordinary Differential Equations.Proc.Amer.Math. Soc. 120(1994),743-748.

deFigueiredo D.G., Lions P. L., and Nussbaum R. D. (1992). A Prior Estimates and Existence of Positive Solutions of Semilinear elliptic equations. J. Math.PuresAppl, 61, 41-63.

Henderson J. and Kaufmann E. R., (1997). Multiple Positive Solutions for Focal Boundary Value Problems. Comm. Appl. Anal. 1, 53-60.

Henderson J and Wang H., (1997). Positive Solutions for Nonlinear Eigenvalue. J. Math. Anal. Appl. 208, 252-259.

Karna, Basant and Lawrence, Bonita, (2007).Existence of Positive Solutions for Multi-point Boundary Value Problems.Electron. J. Qual. Theory Differ. Equ., No. 26, 11 pp.

Krasnosel'skii (1964). Positive Solutions of Operator Equation. P. Noordhoff Ltd, Groningen, The Netherlands..

Kuiper H.J(1971). Positive Solutions of Nonlinear Elliptic Eigenvalue Problems. Rend. Circ. Mat. Palermo . 20, 113-138.

Lian W. C, Wong F. H. and Yeh C. C. (1996). On the Existence of Positive Solutions of Nonlinear Second Order Differential Equations. Proc. Amer. Math. Soc. 124, 1117-1126.

Prasad, K.R. and Rao, Kameswara. (1991). Solvability of a Nonlinear Third Order General Three-point Eigenvalue Problem. Mathematics subject classification., pp.7-15.