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5-3-2019

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# On monochromatic pairs with nondecreasing diameters

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Submitted: Jul 14, 2018; Accepted: Apr 18, 2019; Published: May 3, 2019  
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## Abstract

Let  $n, m, r, t$  be positive integers and  $\Delta : [n] \rightarrow [r]$ . We say  $\Delta$  is  $(m, r, t)$ -permissible if there exist  $t$  disjoint  $m$ -sets  $B_1, \dots, B_t$  contained in  $[n]$  for which

- (a)  $|\Delta(B_i)| = 1$  for each  $i = 1, 2, \dots, t$ ,
- (b)  $\max(B_i) < \min(B_{i+1})$  for each  $i = 1, 2, \dots, t - 1$ , and
- (c)  $\max(B_i) - \min(B_i) \leq \max(B_{i+1}) - \min(B_{i+1})$  for each  $i = 1, 2, \dots, t - 1$ .

Let  $f(m, r, t)$  be the smallest such  $n$  so that all colorings  $\Delta$  are  $(m, r, t)$ -permissible. In this paper, we show that  $f(2, 2, t) = 5t - 4$ .

**Mathematics Subject Classifications:** 05D10, 11B75, 11B50

## 1 Introduction

In this paper we study a Ramsey-type integer coloring problem posed by Bialostocki et al. [2] in 1995, which we restate using the definition of *permissibility* given below.

**Definition 1.** Let  $n, m, r$ , and  $t$  be positive integers and  $\Delta : [n] \rightarrow [r]$  be a coloring. We say  $\Delta$  is  $(m, r, t)$ -permissible if there exists a collection of disjoint  $m$ -sets  $\{B_1, B_2, \dots, B_t\}$  each contained in  $[n]$  such that

- $|\Delta(B_i)| = 1$  for each  $i = 1, 2, \dots, t$ ,
- $\max(B_i) < \min(B_{i+1})$  for each  $i = 1, 2, \dots, t - 1$ , and
- $\max(B_i) - \min(B_i) \leq \max(B_{i+1}) - \min(B_{i+1})$  for each  $i = 1, 2, \dots, t - 1$ .

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We say the collection  $\{B_1, B_2, \dots, B_t\}$  is *permissible* in  $\Delta$  if they satisfy the above conditions, thereby realizing  $\Delta$  as  $(m, r, t)$ -permissible.

The first condition requires the  $m$ -sets be monochromatic; note two  $m$ -sets can be associated with different colors. The second condition is a non-overlapping property establishing precedence between the  $m$ -sets, and the third condition requires the ranges of each  $m$ -set, called their *diameters*, form a nondecreasing sequence.

In this language the question posed by Bialostocki, Erdős, and Lefmann [2] is as follows:

**Question 2.** Given positive integers  $m, r,$  and  $t,$  does there exist an integer  $n$  such that for all  $\Delta : [n] \rightarrow [r], \Delta$  is  $(m, r, t)$ -permissible? If so, what is the minimum possible value for  $n$ ?

The answer to the first question is yes; it follows from a result of van der Waerden involving arithmetic progressions [11]. Bialostocki et al. define  $f(m, r, t)$  as the smallest integer  $n$  for which all  $r$ -colorings of  $[n]$  are  $(m, r, t)$ -permissible. Several infinite families of parameters have  $f(m, r, t)$  determined or bounded.

**Observation 3.** Note that for all positive integers  $m, r,$  and  $t, f(m, r, t) \geq mt$  since the union of  $t$  disjoint  $m$ -sets realizing a coloring as  $(m, r, t)$ -permissible must have cardinality no less than  $f(m, r, t)$ . This inequality is an equality if and only if  $m = 1$  or  $r = 1$ . Additionally  $f(m, r, 1) = (m - 1)r + 1$  by the pigeonhole principle.

Based on the above observation, interest in this question focuses on parameter families in which  $m, r,$  and  $t$  are at least 2.

**Theorem 4** (Bialostocki et al. [2]). *Let  $m, r,$  and  $t$  be integers at least 2.*

(a)  $f(m, 2, 2) = 5m - 3.$

(b)  $f(m, 3, 2) = 9m - 7.$

(c) *If  $r \geq 4,$  then*

$$3r(m - 1) + 3 \leq f(m, r, 2) < ((2m - 2) - r + 1) \cdot (2 + \log_2 r) - 1.$$

(d)  $f(m, 2, t) \leq cmt^2$  for some constant  $c.$

(e)  $8m - 4 \leq f(m, 2, 3) \leq 10m - 6.$

(f)  $f(2, r, t) \leq (r(t - 1) + 1)(r + 1).$

Also in 1995 Bollobás, Erdős, and Jin [5] discovered results for the analogous question involving strictly increasing diameters, as well as established bounds for  $f(2, r, 2)$  and gave asymptotic results for  $f(2, r, t)$ :

**Theorem 5** (Bollobás et al. [5]).

(a) Let  $r \geq 2$ . Then  $4r - 5 \log_2 r < f(2, r, 2) \leq 4r + 1$ .

(b) For a fixed  $t$ ,  $f(2, r, t)$  is linear in  $r$ .

In 2005 Gryniewicz showed the lower bound of the inequality in Theorem 4(c) is an equality when  $r = 4$ .

**Theorem 6** (Gryniewicz [9]). Let  $m$  be a positive integer at least 2. Then

$$f(m, 4, 2) = 12m - 9.$$

In 2015 Bernstein et al. found the value of  $f(m, 2, 3)$ , first bounded in Theorem 4(e).

**Theorem 7** (Bernstein et al. [1]). Let  $m$  be a positive integer at least 2. Then

$$f(m, 2, 3) = 8m - 5 + \left\lfloor \frac{2m - 2}{3} \right\rfloor + \delta,$$

where  $\delta = 1$  if  $m \in \{2, 5\}$  and  $\delta = 0$  otherwise.

Recent work has been done on generalizations of Question 2, as well as investigations into its relationship to a theorem by Erdős, Ginzburg, and Ziv (see [3, 4, 6, 7, 8, 10] for examples). With possibly the only exceptions being the bounds given in Theorem 4(d) and (f), as well as Theorem 5(b), all recent work in evaluating  $f(m, r, t)$  focuses on parameter families with  $t = 2$  and more recently  $t = 3$  [1]. In our work we compute values for  $f(2, 2, t)$  for arbitrarily large  $t$  which, as far as we can tell, is the first known set of actual values of  $f(m, r, t)$  with  $m, r \geq 2$  and arbitrarily large  $t$ , rather than asymptotic results.

Note that  $f(2, 2, t)$  has been computed when  $t \leq 3$ ;  $f(2, 2, 1) = 3$  by Observation 3,  $f(2, 2, 2) = 7$  by Theorem 4(a), and  $f(2, 2, 3) = 12$  by Theorem 7. Additionally,  $f(2, 2, t) \leq 6t - 3$  by Theorem 4(f). Note that the upper bound given in Theorem 4(f) is not an attempt at a sharp bound. The inequality is found by using the pigeonhole principle and, in fact, guarantees the existence of a permissible collection of  $t$  2-sets which all have the same diameter.

In this paper we prove the following:

**Theorem 8.** Let  $t \geq 4$  be a positive integer. Then  $f(2, 2, t) = 5t - 4$ .

In Section 2 we introduce notation for the problem and present some constructions which are useful in later proofs for finding collections of permissible sets. In Section 3 we first prove that  $f(2, 2, t) \geq 5t - 4$  by presenting a coloring of  $[5t - 5]$  which is not  $(2, 2, t)$ -permissible, then prove a weak upper bound on  $f(2, 2, t)$  which gives the existence of a permissible collection of sets with additional structure. We finish the section with a proof that all colorings of  $[5t - 4]$  with  $t \geq 4$  are  $(2, 2, t)$ -permissible by showing there does not exist a coloring of  $[5t - 4]$  which is not  $(2, 2, t)$ -permissible.

## 2 Definitions and Constructions

We begin this section with a series of definitions, terminology, and notation related to colorings with 2 colors (using the color set  $\{a, b\}$  rather than  $[2]$ ) and the identification of collections of 2-sets (abbreviated *pairs*) which are permissible.

Let  $n$  be a positive integer and  $\Delta : [n] \rightarrow \{a, b\}$ . At times we present  $\Delta$  as the string  $x_1x_2 \cdots x_n$ , where  $x_i = \Delta(i)$  for each  $i \in [n]$ . We say  $x_ix_{i+1} \cdots x_{i+k-1}$ , where  $x_j = \Delta(j)$  for each  $j = i, i+1, \dots, i+k-1$ , is a *string of length  $k$  in  $\Delta$* . For  $x \in \{a, b\}$ , we use the abbreviation  $x^n$  to denote the string  $xx \cdots x$  ( $n$  times) and similarly define  $w^n = ww \cdots w$  ( $n$  times), where  $w$  is any word from the alphabet  $\{a, b\}$ . We say  $\{i, i+1, \dots, i+k-1\}$  is a  *$k$ -tuple* in  $\Delta$  if  $\Delta(i) = \Delta(i+1) = \cdots = \Delta(i+k-1)$ . If  $\Delta(i-1) \neq \Delta(i)$  (or  $i = 1$ ) and  $\Delta(i+k-1) \neq \Delta(i+k)$  (or  $i+k-1 = n$ ), we say the  $k$ -tuple is *isolated*. Specifically, a *double* is a 2-tuple and a *triple* is a 3-tuple. A string  $x_ix_{i+1} \cdots x_{i+k-1}$  in  $\Delta$  of length  $k$  is *alternating* if  $\Delta(j) \neq \Delta(j+1)$  for each  $j = i, i+1, \dots, i+k-2$ . A pair  $\{i, i+k\}$  in  $[n]$  is a *monochromatic pair of diameter  $k$* , abbreviated  $D_k$ , in  $\Delta$  if  $\Delta(i) = \Delta(i+k)$ . Note that a  $D_1$  is a double and a triple contains two  $D_1$ s and a  $D_2$ .

We now define the *alternating substring/triples* partition of  $[n]$  with respect to  $\Delta$  (abbreviated the *AST* partition of  $\Delta$ ) and give the process for constructing it. As we see later, this partition yields a permissible collection of pairs which have diameters 1 or 2, which is instrumental going forward.

**Construction 9.** Let  $n > 0$  and  $\Delta : [n] \rightarrow \{a, b\}$ . By the following steps, we partition  $[n]$  into alternating substrings and triples.

1. Find the maximum number of pairwise disjoint triples in  $\Delta$ . Suppose that there are  $w$  of them; call them  $T_1, T_2, \dots, T_w$ , and suppose the minimal element in  $T_i$  is  $\tau_i$  for each  $i \in [w]$ . We additionally require that for all  $i \in [w]$ , if  $\tau_i > 1$ , then either  $\Delta(\tau_i - 1) \neq \Delta(\tau_i)$  or  $\tau_i - 1 \in T_{i-1}$ . This ensures that we “frontload” the triples. In other words, we read through the string from left to right and define our triples in a greedy method. We define the collection of triples as  $\mathcal{T} := \{T_1, T_2, \dots, T_w\}$ .
2. Let  $\mathcal{U} = \{U_0, U_1, \dots, U_w\}$  be a collection of subsets of  $[n]$  defined as  $U_0 = [\min(T_1) - 1]$ ,  $U_i = [\max(T_i) + 1, \min(T_{i+1}) - 1]$  for each  $i = 1, 2, \dots, w-1$ , and  $U_w = [\max(T_w) + 1, n]$ . Note that some of these sets may be empty if  $\Delta$  contains consecutive triples, or begins or ends with a triple. It follows that  $\mathcal{U} \cup \mathcal{T}$  is a partition of  $[n]$ .
3. Partition each  $U_i$  into consecutive alternating substrings of maximal length. Note that if there are  $s$  such maximal substrings in  $U_i$ , then there will be  $s - 1$  doubles contained in  $U_i$ . Thus in total we have  $v + w + 1$  such maximal alternating substrings  $S_i$ ,  $i \in [v + w + 1]$ , where we use the convention that an empty set  $U_i$  corresponds to an empty substring. Thus the substrings  $S_i$  partition the terms of  $\bigcup_{i=0}^w U_i$  as  $\bigcup_{i=1}^{v+w+1} S_i$ . Let  $\mathcal{S} = \{S_1, S_2, \dots, S_{v+w+1}\}$  be the collection of these alternating substrings, and for each  $i \in [v + w + 1]$ , let  $k_i = |S_i|$  be the *length* of  $S_i$ .

Observe that  $\mathcal{S} \cup \mathcal{T}$  is a partition of  $[n]$ , and so we define  $(\mathcal{S}, \mathcal{T}; v, w)$  as the *AST partition of  $\Delta$* . Since  $\mathcal{S} \cup \mathcal{T}$  is a partition of  $[n]$ , note  $n = 3w + k_1 + \cdots + k_{v+w+1}$ .



In a similar manner there are 8 canonical  $D_2$ s in  $\Delta_2$ , which are illustrated below.

$$\Delta_2 : \frac{abababababa}{S_1} \frac{abab}{S_2} \frac{aaa}{T_1} \frac{ba}{S_3} \frac{ab}{S_4} \frac{baba}{S_5} \frac{bbb}{T_2} \frac{aaa}{S_6} \frac{a}{T_3} \frac{ab}{S_7} \frac{ab}{S_8}$$

### 3 Main Results

We prove Theorem 8 in two stages. First, we give what will be the sharp lower bound on  $f(2, 2, t)$ , then produce a weaker upper bound by showing when  $n$  is large enough, there is a permissible collection of pairs with bounded diameters contained in any 2-coloring of  $[n]$ . We conclude the section by establishing properties that a non-permissible coloring of  $[5t - 4]$  must have, then showing that no coloring satisfies all of the properties.

#### 3.1 Bounds on $f(2, 2, t)$

We first show that  $f(2, 2, t) \geq 5t - 4$  by demonstrating the existence of a coloring of  $[5t - 5]$  which is not  $(2, 2, t)$ -permissible. This is derived by setting  $m = r = 2$  in the following theorem.

**Theorem 17.** *Let  $m, r, t \geq 2$  and  $\Delta$  be the  $r$ -coloring of  $[(mr + 1)(t - 1)]$  given by  $((12 \cdots r)^{m1})^{t-1}$ . Then  $\Delta$  is not  $(m, r, t)$ -permissible.*

*Proof.* The result is trivial for  $t = 2$ . Assume that  $((12 \cdots r)^{m1})^{t-2}$  is not  $(m, r, t - 1)$ -permissible for some  $t \geq 3$  and  $((12 \cdots r)^{m1})^{t-1}$  is  $(m, r, t)$ -permissible. Let  $\{B_1, \dots, B_t\}$  be a permissible collection of pairs in  $((12 \cdots r)^{m1})^{t-1}$ .

Since  $((12 \cdots r)^{m1})^{t-2}$  is not  $(m, r, t - 1)$ -permissible, we have  $\min(B_2) \leq mr + 1$ ; otherwise  $\{B_2, \dots, B_t\}$  is a permissible collection of  $t - 1$  pairs in  $\Delta$  restricted to  $[mr + 2, (mr + 1)(t - 1)]$ , implying that  $((12 \cdots r)^{m1})^{t-2}$  is  $(m, r, t - 1)$ -permissible. Hence  $B_1 = \{i, r + i, \dots, (m - 1)r + i\}$  for some  $i \in [r]$ , and thus  $\text{diam}(B_1) = m(r - 1)$ . Similarly, we have that  $\max(B_{t-1}) > (rm + 1)(t - 2)$  and hence  $\text{diam}(B_t) = r(m - 1)$  for reasons identical to those above. Therefore  $\text{diam}(B_2) = r(m - 1)$ , and it follows that if  $\min(B_2) = j$  for some  $j \in [r]$  (and necessarily  $r(m - 1) + 2 \leq \min(B_2) \leq mr + 1$ ):

$$\Delta(\max(B_2)) = \Delta(\min(B_2) + r(m - 1)) = \begin{cases} j - 1 & \text{if } j \neq 1 \text{ and} \\ r & \text{if } j = 1. \end{cases}$$

Thus  $B_2$  is not monochromatic, giving a contradiction. So  $((12 \cdots r)^{m1})^{t-1}$  is not  $(m, r, t)$ -permissible. It follows that  $f(2, 2, t) > 5t - 5$ .  $\square$

We now show that  $f(2, 2, t) \leq 5t - 2$ , a weaker result than what Theorem 8 would guarantee, but we show this bound can be realized by a permissible collection of pairs with bounded diameter.

**Theorem 18.** *Let  $t \geq 1$  be an integer and  $\Delta : [5t - 2] \rightarrow \{a, b\}$ . Then  $\Delta$  is  $(2, 2, t)$ -permissible, and there is a permissible collection of  $t$  pairs having diameter at most 2.*

*Proof.* Let  $(\mathcal{S}, \mathcal{T}; v, w)$  be the AST partition of  $\Delta$ . By Observation 13 if  $v + w \geq t$ , then  $\Delta$  is  $t$ -permissible as it contains at least  $t$  disjoint  $D_1$ s. Now suppose  $v + w \leq t - 1$ . Recall that for each  $i \in [v + w + 1]$ ,  $k_i = |S_i|$ , and define  $k'_i \in \{0, 1, 2\}$  so that  $k'_i = k_i \pmod 3$ . Note  $(k_i - k'_i)/3 = \lfloor k_i/3 \rfloor$  for each  $i \in [v + w + 1]$ . Let  $\sigma$  denote the number of canonical  $D_2$ s of  $\Delta$ . Recall  $5t - 2 = 3w + k_1 + k_2 + \cdots + k_{v+w+1}$  and  $\sigma = w + \lfloor k_1/3 \rfloor + \lfloor k_2/3 \rfloor + \cdots + \lfloor k_{v+w+1}/3 \rfloor$ . So

$$\begin{aligned} 3\sigma &= 3w + (k_1 - k'_1) + (k_2 - k'_2) + \cdots + (k_{v+w+1} - k'_{v+w+1}) \\ &= (5t - 2) - (k'_1 + k'_2 + \cdots + k'_{v+w+1}) \\ &\geq 5t - 2 - 2(v + w + 1) \\ &\geq 5t - 2 - 2t = 3t - 2. \end{aligned} \tag{1}$$

So  $\sigma \geq t - \frac{2}{3}$ , and since  $\sigma$  and  $t$  are integers,  $\sigma \geq t$ . By choosing any  $t$  of the  $\sigma$  canonical  $D_2$ s of  $\Delta$ , we have a permissible collection of  $t$  pairs in  $\Delta$ . So  $\Delta$  is  $(2, 2, t)$ -permissible and is realized by a permissible collection of pairs with diameter at most 2.  $\square$

### 3.2 Colorings of $[5t - 4]$ which are not $(2, 2, t)$ -permissible

For the remainder of this section, let  $\Delta$  be a coloring of  $[5t - 4]$ ,  $(\mathcal{S}, \mathcal{T}; v, w)$  be the AST partition of  $\Delta$ ,  $\mathcal{D}$  be the set of canonical  $D_2$ s of  $\Delta$ , and  $\sigma = |\mathcal{D}|$ . We now outline a series of properties that  $\Delta$  must satisfy to *not* be  $(2, 2, t)$ -permissible. We conclude this section by showing no coloring can satisfy all such conditions, which establishes  $f(2, 2, t) \leq 5t - 4$ , and thus proves Theorem 8.

As it will be relevant in the proof of Theorem 8, take note that all of the following lemmas and observations are valid for  $t \geq 2$ , and we indicate where appropriate what the largest diameter for a realization of permissibility is for a coloring. This will be leveraged in the final proof. We begin an observation about the number of canonical  $D_2$ s in  $\Delta$  provided it is not  $(2, 2, t)$ -permissible.

**Observation 19.** Let  $t \geq 2$  and suppose  $\Delta$  is not  $(2, 2, t)$ -permissible. By a similar argument found in (1), we find  $3\sigma \geq 5t - 4 - 2(v + w + 1)$ . So  $\sigma \geq t - \frac{4}{3}$  and hence  $\sigma \geq t - 1$ . By Observation 13, we have  $\sigma = t - 1$ , and hence  $\Delta$  contains exactly  $t - 1$  canonical  $D_2$ s.

The previous observation allows us to classify the value of  $v + w$  in the AST partition of  $\Delta$  given it is not  $(2, 2, t)$ -permissible.

**Lemma 20.** *Let  $t \geq 2$ . If  $\Delta$  is not  $(2, 2, t)$ -permissible, then  $v + w = t - 1$ . Therefore  $\Delta$  has exactly  $t$  alternating substrings in its AST partition.*

*Proof.* By Observation 13,  $v + w \leq t - 1$ . Suppose  $v + w \leq t - 2$ . Then in a manner similar to what is shown in (1) and Observation 19, we find

$$3\sigma \geq 5t - 4 - 2(v + w + 1) \geq 5t - 4 - 2(t - 2) = 3t.$$

So  $\sigma \geq t$  and therefore  $\Delta$  contains at least  $t$  canonical  $D_2$ s. Hence  $\Delta$  is  $(2, 2, t)$ -permissible with a realization whose largest diameter is 2.  $\square$



Adapting the proof for Theorem 18 allows us to determine the congruence for  $|S_i|$  for each  $i = 1, 2, \dots, v + w + 1$ .

**Lemma 21.** *Let  $t \geq 2$ . If  $\Delta$  is not  $(2, 2, t)$ -permissible, there exists  $j \in \{1, 2, \dots, v + w + 1\}$  such that  $|S_j| \equiv 1 \pmod{3}$  and  $|S_k| \equiv 2 \pmod{3}$  whenever  $k \neq j$ ,  $1 \leq k \leq v + w + 1$ .*

*Proof.* By Lemma 20, we have  $t = v + w + 1$  and therefore  $\Delta$  has  $t$  alternating substrings in its AST partition. Again for each  $i \in [t]$ , define  $k'_i \in \{0, 1, 2\}$  so that  $k'_i \equiv k_i \pmod{3}$ . By Observation 19 we have  $\sigma = t - 1$ . Combining this with a similar computation to (1) we get

$$3(t - 1) = 3\sigma = 3w + \sum_{i=1}^t (k_i - k'_i) = 3w + \sum_{i=1}^t k_i - \sum_{i=1}^t k'_i = 5t - 4 - \sum_{i=1}^t k'_i.$$

So  $k'_1 + \dots + k'_t = 2t - 1$ . Therefore exactly one term in this sum is 1 while the remaining terms are each 2. Hence  $\Delta$  is  $(2, 2, t)$ -permissible with a realization whose largest diameter is 2.  $\square$

We now classify the  $k$ -tuples which may exist in  $\Delta$ .

**Lemma 22.** *If  $\Delta$  is not  $(2, 2, t)$ -permissible, then  $\Delta$  does not contain an isolated 4-tuple or isolated  $k$ -tuple for each  $k \geq 6$ .*

*Proof.* First, observe that if  $\Delta$  contains a  $k$ -tuple for some  $k \geq 6$ , then  $\Delta$  has at least two consecutive triples in its AST partition. Hence there is an empty substring  $S_i \in \mathcal{S}$  for some  $i \in [v + w + 1]$ , giving that  $k_i \equiv 0 \pmod{3}$ . So  $\Delta$  is  $(2, 2, t)$ -permissible by Lemma 21 with a realization whose largest diameter is 2 – a contradiction.

Now suppose  $\Delta$  contains the isolated 4-tuple  $\{\ell, \ell + 1, \ell + 2, \ell + 3\}$  for some  $\ell \in [n - 3]$ . For each  $i \in [w]$ , let  $T'_i$  be the set containing the smallest two elements in  $T_i$ . It follows from Construction 9 that  $T_j = \{\ell, \ell + 1, \ell + 2\}$  for some  $j \in \{1, 2, \dots, w\}$  and  $\ell + 3$  does not belong to any double or triple identified in the AST partition. Then  $\{P_1, P_2, \dots, P_v, T'_1, T'_2, \dots, T'_w, \{\ell + 2, \ell + 3\}\}$  is a permissible collection of  $v + w + 1$  doubles, and since  $v + w + 1 = t$  by Lemma 20,  $\Delta$  is  $(2, 2, t)$ -permissible with a realization whose largest diameter is 1 – a contradiction.  $\square$

With the previous two lemmas, we now establish that  $\Delta$  can have at most one triple and, if  $\Delta$  contains a triple, give conditions on the lengths of the alternating substrings surrounding the triple.

**Lemma 23.** *If  $\Delta$  is not  $(2, 2, t)$ -permissible and  $\Delta$  contains any triples, then  $\Delta$  contains  $(\bar{1}, \tau, \bar{2})$  or  $(\bar{2}, \tau, \bar{1})$ . Furthermore  $\Delta$  contains at most one triple.*

*Proof.* Observe that  $\Delta$  cannot begin with a triple, end with a triple, or have two consecutive triples in its AST partition; otherwise one of its alternating substrings has length 0, which contradicts Lemma 21. Therefore every triple in the AST partition of  $\Delta$  is preceded and followed by a nonempty alternating substring. Furthermore by Lemma 21, it follows

that if  $\Delta$  contains a triple in its AST partition, then  $\Delta$  must contain  $(\bar{1}, \tau, \bar{2})$ ,  $(\bar{2}, \tau, \bar{1})$ , or  $(\bar{2}, \tau, \bar{2})$ .

Assume  $\Delta$  contains  $(x, \tau, y)$ , where  $x \equiv y \equiv 2 \pmod{3}$ , and let  $S_L$ ,  $T$ , and  $S_R$  denote the corresponding alternating substrings and triple, and let  $i$  be the integer such that  $T = \{i, i+1, i+2\}$ . Since  $\Delta(i-2) \neq \Delta(i-1)$ , it follows by the AST construction that  $\Delta(i-1) \neq \Delta(i)$ . Since  $\Delta(i+3) \neq \Delta(i+4)$ , it follows by Lemma 22 that  $\Delta(i+2) \neq \Delta(i+3)$ . Thus  $S_L \cup \{i\}$  and  $S_R \cup \{i+2\}$  are alternating substrings of lengths  $x+1$  and  $y+1$ , respectively. Hence  $S_L \cup \{i\}$  contains a permissible collection of  $\lfloor x/3 \rfloor + 1$   $D_2$ s, and similarly  $S_R \cup \{i+2\}$  contains a permissible collection of  $\lfloor y/3 \rfloor + 1$   $D_2$ s. Removing those  $D_2$ s from  $\mathcal{D}$  contained in  $S_L$ ,  $S_R$ , and  $T$ , then combining them with the permissible collections of  $D_2$ s contained in  $S_L \cup \{i\}$  and  $S_R \cup \{i+2\}$  produces a permissible collection of  $t$   $D_2$ s, which is a contradiction. So  $\Delta$  does not contain  $(\bar{2}, \tau, \bar{2})$ . So if  $\Delta$  contains any triples, then  $\Delta$  contains  $(\bar{2}, \tau, \bar{1})$  or  $(\bar{1}, \tau, \bar{2})$ .

By Lemma 21, there is exactly one substring in the AST partition of  $\Delta$  which has length congruent to  $1 \pmod{3}$ . Therefore  $w \leq 2$ , and if  $w = 2$  then  $\Delta$  must contain  $(\bar{2}, \tau, \bar{1}, \tau, \bar{2})$ . Assume that  $w = 2$  and  $\Delta$  contains  $(x, \tau, y, \tau, z)$ , where  $y \equiv 1 \pmod{3}$  and  $x \equiv z \equiv 2 \pmod{3}$ . Let  $S_L, \{i, i+1, i+2\}, S_M, \{j, j+1, j+2\}, S_R$  be the associated alternating substrings and triples. In a manner similar to the previous argument,  $S_L \cup \{i\}, \{i+2\} \cup S_M \cup \{j\}$ , and  $\{j+2\} \cup S_R$  are alternating substrings. By removing the  $D_2$ s in  $\mathcal{D}$  contained in  $S_L \cup \{i, i+1, i+2\} \cup S_M \cup \{j, j+1, j+2\} \cup S_R$  and adding those contained in  $S_L \cup \{i\}, \{i+2\} \cup S_M \cup \{j\}$ , and  $\{j+2\} \cup S_R$ , we produce a permissible collection of  $t$   $D_2$ s in  $\Delta$  – a contradiction. Thus  $w \leq 1$ .

Note that in each of the above alterations, each coloring was found to be  $(2, 2, t)$ -permissible with a realization whose largest diameter is 2.  $\square$

We now give an observation which classifies the possible final substring in  $\Delta$  and is useful in the arguments which follow.

**Observation 24.** Suppose  $\Delta$  is not  $(2, 2, t)$ -permissible.

(a)  $\Delta(5t-4) \neq \Delta(5t-6)$ . Otherwise  $\Delta|_{[5t-7]}$  is a coloring which is  $(2, 2, t-1)$ -permissible with a permissible collection of  $(t-1)$  pairs of diameter at most 2 by Theorem 18. The collection of these  $(t-1)$  sets with  $\{5t-6, 5t-4\}$  yields a permissible collection of  $t$  pairs in  $\Delta$  with a largest diameter of 2.

With this in mind, since  $\Delta$  cannot end with a triple, the alternating substring which ends  $\Delta$  cannot have length exceeding 2. So  $k_{v+w+1} = 1$  or  $k_{v+w+1} = 2$  and thus  $\Delta$  must end with type (1) or (2).

(b)  $\Delta(5t-11) \neq \Delta(5t-9)$ . Otherwise  $\Delta|_{[5t-9]}$  is  $(2, 2, t-1)$ -permissible with a permissible collection of  $t-1$  pairs whose largest diameter is at most 2. Since either  $\{5t-8, 5t-6\}$ ,  $\{5t-8, 5t-4\}$ , or  $\{5t-6, 5t-4\}$  is a monochromatic pair, we can produce a permissible collection of  $t$  pairs, which means that  $\Delta$  is  $(2, 2, t)$ -permissible, and this realization has a largest diameter of either 2 or 4.

Now we give a proof for the main result.

*Proof of Theorem 8.* Suppose  $\Delta$  is not  $(2, 2, t)$ -permissible. By Observation 24(a),  $\Delta$  must end with type (1) or (2). By Lemma 23,  $\Delta$  may contain at most 1 triple and if  $\Delta$  contains a triple, then  $(\bar{2}, \tau, \bar{1})$  or  $(\bar{1}, \tau, \bar{2})$  is contained in  $\Delta$ . That said there are 12 types with which  $\Delta$  may end; see Figure 1.

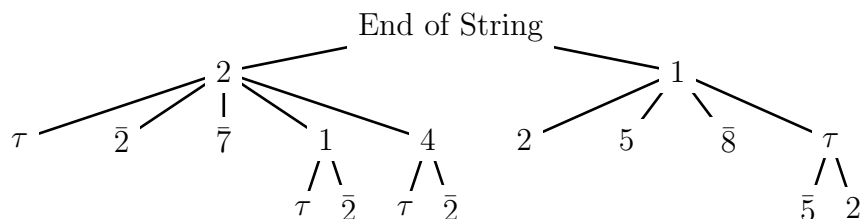


Figure 1: The possible endings for a coloring which is not  $(2,2,t)$ -permissible. For example, the left-most leaf corresponds to type  $(\tau, 2)$ , while the right-most leaf corresponds to type  $(2, \tau, 1)$ .

We now show in each case above that  $\Delta$  is, in fact,  $(2, 2, t)$ -permissible and therefore prove Theorem 8. Without loss of generality, we assume  $\Delta(5t - 4) = b$ . Suppose  $\Delta$  ends with type:

- $(\tau, 2)$ . Since  $\Delta$  cannot end with  $bbbab$  by Observation 24(a), we have  $\Delta$  ends with  $aaaab$ . Since  $t \geq 2$ , there is a nonempty alternating substring  $S_{t-1}$  which precedes the triple. If the last letter in  $S_{t-1}$  is an  $a$ , then by the AST construction  $\Delta$  has two consecutive triples, which violates Lemma 23. So  $\Delta$  ends with  $baaaab$ , in which case we have an isolated quadruple in  $\Delta$ , which contradicts Lemma 22. Thus a coloring with this end type is  $(2, 2, t)$ -permissible with a realization having a largest diameter at most 2.
- $(\bar{2}, 2)$ . Then  $\Delta$  ends with  $baab$ , and all of the corresponding indices in  $[5t - 7, 5t - 4]$  are preceded by the  $t - 1$  canonical  $D_2$ s of  $\Delta$ . Then  $\mathcal{D} \cup \{\{5t - 7, 5t - 4\}\}$  is a permissible collection of  $t$  pairs and hence  $\Delta$  is  $(2, 2, t)$ -permissible with a realization whose largest diameter is 3.
- $(\bar{7}, 2)$ ,  $(\bar{8}, 1)$ , or  $(\bar{5}, \tau, 1)$ . By Observation 24(b),  $\Delta$  is  $(2, 2, t)$ -permissible with a realization having a largest diameter at most 4. Note this argument implies that, if  $\Delta$  is a coloring of  $[5t - 4]$  with  $t \geq 3$  (rather than 4) and ending with any of these three types, then  $\Delta$  is also  $(2, 2, t)$ -permissible. This will be relevant in the last subcase of this proof.
- $(\tau, 1, 2)$ . Since  $t \geq 3$ , by Lemmas 21 and 23,  $\Delta$  ends with type  $(\bar{2}, \tau, 1, 2)$ . So  $\Delta$  ends with either  $abaaaaab$ ,  $baaaaaab$ ,  $babbbaab$ , or  $abbbbaab$ . Then  $\Delta$  is  $(2, 2, t)$ -permissible by Observation 24(b) for the first or third case and by Lemma 22 for the second or fourth case. Thus  $\Delta$  is  $(2, 2, t)$ -permissible with a realization having a largest diameter at most 4.

- $(\bar{2}, 1, 2)$ . In this case  $\Delta$  ends with  $baaab$ , meaning the triples were not correctly identified in its AST partition. So  $\Delta$  cannot end with this type.
- $(\tau, 4, 2)$ . Since  $t \geq 3$ , by Lemmas 21 and 23,  $\Delta$  must end with type  $(\bar{2}, \tau, 4, 2)$ . Then  $\Delta$  ends with  $abaaababaab$ ,  $abbbbabaab$ ,  $baaaababaab$ , or  $babbbbabaab$ . Note the second and third cases are not possible as the triples were not selected properly for its partition. In the first case,  $\{5t - 12, 5t - 10\}$  and  $\{5t - 9, 5t - 7\}$  belong to  $\mathcal{D}$ . The collection the pairs in  $\mathcal{D}$ , excluding  $\{5t - 12, 5t - 10\}$  and  $\{5t - 9, 5t - 7\}$ , combined with  $\{5t - 14, 5t - 12\}$ ,  $\{5t - 11, 5t - 8\}$ , and  $\{5t - 7, 5t - 4\}$  is permissible, demonstrating that  $\Delta$  is  $(2, 2, t)$ -permissible. Finally  $\Delta$  is  $(2, 2, t)$ -permissible by Observation 24(b) for the fourth case. Thus  $\Delta$  is  $(2, 2, t)$ -permissible with a realization having a largest diameter at most 4.
- $(\bar{2}, 4, 2)$  or  $(5, 1)$ . In the latter case, since  $t \geq 3$ , by Lemmas 21 and 23,  $\Delta$  must end with type  $(\bar{2}, 5, 1)$ . So  $\Delta$  ends with  $abbabaab$  or  $abbababb$ . In a similar manner to the above case, the collection of pairs in  $\mathcal{D}$ , excluding  $\{5t - 9, 5t - 7\}$ , combined with  $\{5t - 11, 5t - 8\}$  and  $\{5t - 7, 5t - 4\}$  is permissible, again demonstrating that  $\Delta$  is  $(2, 2, t)$ -permissible with a realization having a largest diameter at most 3. Again note this argument holds for  $t \geq 3$ , similar to the colorings which end with  $(\bar{8}, 1)$ .
- $(2, 1)$ . By Lemmas 21 and 23 and since  $t \geq 3$ ,  $\Delta$  must end with type  $(\bar{2}, 2, 1)$ . Then  $\Delta$  ends with  $baabb$ . Note that each of the  $(t - 1)$  sets in  $\mathcal{D}$  precedes  $\{5t - 8, 5t - 5\}$ . Hence  $\mathcal{D} \cup \{\{5t - 8, 5t - 5\}\}$  is a set of  $t$  permissible sets, which demonstrates that  $\Delta$  is  $(2, 2, t)$ -permissible with a realization having a largest diameter at most 3. Again note this argument holds for  $t \geq 3$ , similar to the colorings which end with  $(\bar{8}, 1)$  or  $(5, 1)$ .
- $(2, \tau, 1)$ . By Lemma 21 and since  $t \geq 4$ ,  $\Delta$  must end with type  $(\bar{2}, \bar{2}, 2, \tau, 1)$ . So  $\Delta$  ends with  $baabaaab$ ,  $baabbbbb$ ,  $abbaaaab$ , or  $abbabbbb$ . Again note the second and third cases are not possible as the triples were not selected properly for its partition. In the first case,  $\Delta|_{[5t-9]}$  ends with type  $(\bar{2}, 1)$ , or rather  $(2, 1)$ ,  $(5, 1)$ , or  $(\bar{8}, 1)$ .  
 Since  $t - 1 \geq 3$ , we have  $\Delta|_{[5t-9]}$  is therefore  $(2, 2, t - 1)$ -permissible with a permissible collection of  $t$  pairs with largest diameter 4; let  $\mathcal{D}'$  denote this collection. So  $\mathcal{D}' \cup \{\{5t - 8, 5t - 4\}\}$  is a permissible collection of  $t$  pairs, demonstrating the  $(2, 2, t)$ -permissibility of  $\Delta$  with a realization having a largest diameter at most 4. In the fourth case,  $\Delta$  is  $(2, 2, t)$ -permissible with a realization having a largest diameter 1 by Lemma 22.  $\square$

We conclude with the observation that, in this work, we heavily use the binary nature of 2-colorings for constructing the AST partition used in our arguments. Hence we do not believe this method will translate nicely to  $[r]$ -colorings with  $r \geq 3$  or the discovery of permissible  $m$ -sets with  $m \geq 3$ .

## Acknowledgements

We also thank our anonymous referee for their careful reading, detailed comments, and thoughtful suggestions for improving our paper.

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