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Abstract

Let a, b, c, d, and e be positive integers. In 1982 Heinrich showed the existence of a partitioned incomplete Latin square (PILS) of type (a, b, c) and (a, b, c, d) if and only if $a = b = c$ and $2a \geq d$. For PILS of type (a, b, c, d, e) with $a \leq b \leq c \leq d \leq e$, it is necessary that $a+b+c \geq e$, but not sufficient. In this paper we prove an additional necessary condition and classify the existence of PILS of type $(a, b, c, d, a + b + c)$ and PILS with three equal parts. Lastly, we show the existence of a family of PILS in which the parts are nearly the same size.

1 Introduction

Let $a, n \in \mathbb{Z}^+$ and S be a symbol set of order n. Let $[n] = \{1, \ldots, n\}$, $a+S = \{a+s \mid s \in S\}$, and aS be the multiset in which each element of S occurs a times. A *partial Latin square* of order n over S is a partially filled $n \times n$ matrix in which symbols from S occur at most once in each row and column. Unless otherwise stated, we assume that $S = [n]$. We use PLS(n) to denote the set of partial Latin squares of order n.

Let $P \in$ PLS(n). The content of cell (i, j) of P for each $i, j \in [n]$ is denoted $P(i, j)$ and either $P(i, j) \in S$ or $P(i, j) = \varnothing$. If $P(i, j)$ is nonempty for each $i, j \in [n]$, then

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P is called a *Latin square* and $P \in \text{LS}(n)$, where $\text{LS}(n)$ is the set of Latin squares of order n. We will often treat P as a subset of $[n] \times [n] \times [n]$, where $(i, j, k) \in P$ if and only if $k = P(i, j)$.

We say that P can be completed over $[n]$ if there exists $L \in \text{LS}(n)$ over $[n]$ containing P. There are numerous results in the literature on completing partial Latin squares.

Let S_n be the symmetric group acting on $[n]$. For $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$, we use $\theta(P) \in \text{PLS}(n)$ to denote the matrix in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively. The mapping θ is called an *isotopism*, and P and $\theta(P)$ are said to be *isotopic*. It is well known, and follows from a straightforward argument, that for any $\theta \in S_n \times S_n \times S_n$, $\theta(P) \in \text{PLS}(n)$ and P can be completed if and only if $\theta(P)$ can be completed.

In what follows we review two collections of objects: partitioned incomplete Latin squares (PILS) and Latin squares realizing partitions. The former is well-documented (see III.1.5 in [3]) and we establish that the latter objects are equivalent.

Let $H = \{S_1, S_2, ..., S_m\}$ be a partition of $[n]$, where $|S_i| = s_i$ for each $i \in [m]$. A *partitioned incomplete Latin square* (PILS) of order n, having partition H, is a partial Latin square P of order n indexed by $[n]$ satisfying the following properties:

(1) The cells $S_i \times S_i$ are empty for each $i \in [m]$.
(2) For each $j \in S_i$, row j and column j of P contains every element of $[n] \backslash S_i$ exactly once.

If P is a PILS of order n with partition H, then $\theta(P)$ is also a PILS of order n for each $\theta \in S_n \times S_n \times S_n$. It may be that $\theta(P)$ is under a new partition $H' = \{S'_1, S'_2, ..., S'_m\}$. However, it holds that $|S'_i| = s_i$ for each $i \in [m]$. Thus, we may assume that $S_i = (s_1 + s_2 + \ldots + s_{i-1}) + [s_i]$ for each $i \in [m]$. The set of all PILS of order n with partition H is denoted as $\text{PILS}(s_1, \ldots, s_m)$. We also say that an element of $\text{PILS}(s_1, \ldots, s_m)$ is a PILS of type $(s_1, \ldots, s_m)$. The purpose of this paper is to find necessary and sufficient conditions under which $\text{PILS}(s_1, \ldots, s_m)$ is nonempty when $m = 5$. (The reason we focus on $m = 5$ is because for $m \le 4$, necessary and sufficient conditions are known; see Theorem 1.1 below.)

	5	4	2	3
4		5	3	1
5	4		1	2
2	3	1		
3	1	2		

1	5	4	2	3
4	2	5	3	1
5	4	3	1	2
2	3	1	4	5
3	1	2	5	4

Figure 1: A PILS of type $(1, 1, 1, 2)$ and the corresponding LS realizing $(1^3, 2)$.

Let $L \in \text{LS}(n)$ and $q < n$ be a positive integer. If M is a $q \times q$ sub-matrix of L and is itself a Latin square, then M is called a *subsquare* of L. Subsquares of L are *disjoint* if they do not share a row, column, or symbol. In Figure 1, four pairwise disjoint subsquares are highlighted in the Latin square.

Let $(n_1, \ldots, n_k)$ be an integer partition of n. We may abbreviate this if parts of the partition are identical; if $n = \alpha_1 n_1 + \ldots + \alpha_\ell n_\ell$ then we may also let $(n_1^{\alpha_1}, \ldots, n_\ell^{\alpha_\ell})$

denote the partition of n with α_i copies of n_i for each $i \in [\ell]$. We say that $(n_1, \ldots, n_k)$ is *realized* if there is a Latin square of order n with pairwise disjoint subsquares of order n_i for each $i \in [k]$. For example, the Latin square in Figure 1 realizes $(1^3, 2)$. Observe that $P \in \mathrm{PILS}(s_1, ..., s_m)$ if and only if the completion of P is a realization of $(s_1, ..., s_m)$. Hence there is a one-to-one correspondence between PILS and Latin squares that realize integer partitions. Many of the results stated below were originally stated as realizations of integer partitions. We will state them here as existence results of PILS.

It is straightforward that $\mathrm{PILS}(a, n-a)$ is empty for each $a \in [n]$. Heinrich [6] classified the existence of PILS having partitions with exactly 3 or 4 parts.

Theorem 1.1. *Let a, b, c, and d be positive integers.*

1. $\mathrm{PILS}(a, b, c)$ *is nonempty if and only if* $a = b = c$.
2. $\mathrm{PILS}(a, b, c, d)$ *is nonempty if and only if* $a = b = c$ *and* $1 \leq d \leq 2a$ *(perhaps after relabeling).*
3. $\mathrm{PILS}(a^k)$ *is nonempty if and only if* $k \neq 2$.

In [7], Heinrich constructed Latin squares realizing partitions with exactly two distinct parts and proved statements 1 and 2 in Theorem 1.2 below. In [10], Kuhl and Schroeder proved the third statement in Theorem 1.2, which together with Theorem 1.1 completely characterizes PILS of type (a^s, b^t).

Theorem 1.2. *Let a and b be any positive integers.*

1. *If* $s \geq 3$ *and* $t \geq 3$, *then* $\mathrm{PILS}(a^s, b^t)$ *is nonempty.*
2. $\mathrm{PILS}(a^s, b)$ *is nonempty if and only if* $(s-1)a \geq b$ *and* $s \geq 3$.
3. $\mathrm{PILS}(a^s, b^2)$ *is nonempty if and only if* $sa \geq b$ *and* $s \geq 3$.

A necessary condition for a PILS of type having k parts is that each part is bounded by the sum of $k-2$ other parts, but it is not sufficient. In particular, it is known that PILS of type $(n_1, n_2, n_3, n_3, n_1 + n_2 + n_3)$ do not exist [9], and in [9] it is stated that there are numerous examples of PILS of five parts that do not exist. This leads us to believe that characterizing the existence of these PILS is difficult. In Section 2 we define outline squares and illustrate their relevance to finding PILS. In Section 3 we give a new necessary condition for PILS of five parts to exist. In Section 4 we use outline squares to prove necessary and sufficient conditions under which PILS of type $(n_1, n_2, n_3, n_4, n_1 + n_2 + n_3)$ and PILS with three equal parts exist. In Section 5, we use an inductive technique to show the existence of PILS where the parts are close in size. Finally, we end with a conjecture and a discussion on how the existence problem can be translated into a integer programming problem. We also give some comparisons between the results of this paper to some by Colbourn in a recent publication [2].

2 Outline Squares

We begin with arrays in which cells contain more than one symbol and symbols may occur more than once in a cell, row, and column. The following constructions are attributed to Hilton [8], though his constructions are more general. Definitions and notations come from [8]. Let $Q = (q_1, \ldots, q_u)$ be a partition of n. Let $L \in \mathrm{LS}(n)$. The *reduction modulo* (Q,Q,Q) of L is made by amalgamating rows $(q_1+\cdots+q_{i-1})+[q_i]$, columns $(q_1+\cdots+q_{i-1})+[q_i]$, and symbols $(q_1+\cdots+q_{i-1})+[q_i]$ for each $i \in [u]$. Thus, the reduction modulo (Q,Q,Q) is a $u \times u$ array whose cells are filled from $[u]$ such that symbol k occurs t times in cell (i,j) if and only if symbols from $(q_1+\cdots+q_{k-1})+[q_k]$ occur exactly t times in the corresponding sub-matrix of L. In Figure 2, an element of LS(5) is given with a reduction modulo (Q,Q,Q), where $Q = (1^3, 2)$.

1	5	4	2	3
4	2	5	3	1
5	4	3	1	2
2	3	1	4	5
3	1	2	5	4

1	4	4	2 3
4	2	4	1 3
4	4	3	1 2
2 3	1 3	1 2	4 4 4 4

Figure 2: An element of LS(5) and its reduction modulo (Q,Q,Q), where $Q = (1^3, 2)$.

Let R be a $u \times u$ array on symbol set $[u]$ in which symbols may occur more than once in a cell. For each $i, j \in [u]$, let $R(i,j)$ denote the multiset of symbols in cell (i,j) of R, and $|R(i,j)|$ denote the number of symbols appearing in cell (i,j) of R (with repetition). We say that R is a *outline square* (abbreviated OS) associated to (Q,Q,Q), or R is an OS with partition Q, if the following hold:

1. symbol $i \in [u]$ occurs $q_i n$ times;
2. $|R(i,j)| = q_i q_j$ for each $i, j \in [u]$; and
3. the number of times each symbol i occurs in row (column) j is $q_i q_j$ for each $i, j \in [u]$.

Denote the set of outline squares with partition Q as OS(Q). Note that an outline square is not a new object; it is a special case of an *outline rectangle* (see [8] for its definition and applications).

The following theorem is a corollary of a more general result in [8].

Theorem 2.1. *To each outline square R there is a Latin square L and partition Q such that R is the reduction of L modulo (Q,Q,Q).*

If $R \in \mathrm{OS}(Q)$ is the reduction of $L \in \mathrm{LS}(n)$ modulo (Q,Q,Q), then we say that R *lifts* to L. If $R \in \mathrm{OS}(Q)$ lifts to a realization of Q, then we also say that R lifts to an element of PILS(Q) – this is through producing an element of LS(n), then removing the diagonal subsquares.

The utility of an outline square in OS(Q) and a PILS(Q) is given by the following definition, which we will exploit heavily in our construction of PILS.

Definition 2.2. Let $t \geq 3$, $Q = (q_1, q_2, \ldots, q_t)$ be a partition and $R \in \text{OS}(Q)$ with symbol set $[t]$. Suppose that each diagonal cell contains one distinct symbol – that is, without loss of generality, R_{ii} contains q_i^2 copies of i for each $i \in [t]$. Then the lift of R will be a Latin square in which the associated subarrays to the diagonal cells of R consist of disjoint subsquares of orders corresponding to the parts of Q. Hence the lift of R is a PILS(Q). We denote the set of such outline squares in OS(Q) with this additional diagonal condition as OS$^*(Q)$, and note that OS$^*(Q)$ is nonempty precisely when PILS(Q) is nonempty. Hence as we go forward, we focus on the existence of elements in OS$^*(Q)$.

3 Necessary Conditions

As stated in the introduction, a condition necessary for the existence of a PILS with type having five parts is that the largest part must be bounded by the sum of the smallest three parts. In this section we find additional necessary conditions.

The first is a necessary condition for all nonempty PILS, and the following condition relates to those parameters for which the largest part is the sum of the smallest three parts.

Lemma 3.1. *Let a, b, c, d, and e be positive integers and suppose that* PILS(a, b, c, d, e) *is nonempty. Then*

$$(a + b + c + d + e)^2 \geq 3(a + b + c)(d + e) + a^2 + b^2 + c^2 + d^2 + e^2.$$

Proof. Observe that the above condition simplifies to $2ab+2ac+2bc \geq (a+b+c)(d+e) - 2de$.

Let $L \in \text{PILS}(a, b, c, d, e)$ and $R \in \text{OS}^*(a, b, c, d, e)$ be the reduction of L modulo (a, b, c, d, e). We deduce the inequality by counting the occurrences of symbols 4 and 5 in the six off-diagonal cells in the upper-left 3×3 subarray of R. Let $\sigma_k(R_{ij})$ denote the number of occurrences of symbol k in cell R_{ij}. Observe that since R is an outline square, we have the following equations from the occurrences of 4 and 5 in rows 1, 2, and 3, and columns 4 and 5:

$$\begin{aligned}
ad &= \sigma_4(R_{12}) + \sigma_4(R_{13}) + \sigma_4(R_{15}) & de &= \sigma_4(R_{15}) + \sigma_4(R_{25}) + \sigma_4(R_{35}) \\
bd &= \sigma_4(R_{21}) + \sigma_4(R_{23}) + \sigma_4(R_{25}) \\
cd &= \sigma_4(R_{31}) + \sigma_4(R_{32}) + \sigma_4(R_{35}) \\
\\
ae &= \sigma_5(R_{12}) + \sigma_5(R_{13}) + \sigma_5(R_{14}) & de &= \sigma_5(R_{14}) + \sigma_5(R_{24}) + \sigma_5(R_{34}) \\
be &= \sigma_5(R_{21}) + \sigma_5(R_{23}) + \sigma_5(R_{24}) \\
ce &= \sigma_5(R_{31}) + \sigma_5(R_{32}) + \sigma_5(R_{34})
\end{aligned}$$

It follows that

$$\sum_{k \in \{4,5\}} \sum_{\substack{i,j \in [3] \\ i \neq j}} \sigma_k(R_{ij}) = ad + bd + cd + ae + be + ce - 2de = (a + b + c)(d + e) - 2de.$$

In other words, symbols 4 and 5 appear a total of $(a+b+c)(d+e)-2de$ times in the six off-diagonal cells of R, in the first three rows and columns. There are $2ab+2ac+2bc$ symbols which occur total in these six cells, hence the inequality follows. □

Since the sum of any three parts is at least as large as each remaining part, we conclude with the following corollary:

Corollary 3.2. *Let a, b, and c be positive integers and suppose that* $\mathrm{PILS}(a,a,a,b,c)$ *is nonempty. Then $b \le 3a$, $c \le 3a$, and $6a^2 \ge 3a(b+c)-2bc$.*

The lemma below gives necessary conditions for when a PILS exists and the largest part is maximal with respect to the first four parts. For its proof we use the classical completion result given by Ryser's Theorem [11], which is a generalization of the famous Hall's Theorem [5].

Theorem 3.3. *Let r and s be positive integers such that $r, s \le n$. Let $P \in \mathrm{PLS}(n)$ such that all nonempty cells occur in a filled $r \times s$ sub-matrix R. Then P can be completed if and only if each of the n symbols occur at least $r+s-n$ times in R.*

Lemma 3.4. *Let (n_1,n_2,n_3,n_4,n_5) be a partition of n for which $n_1 \le n_2 \le n_3 \le n_4 \le n_5$ and $n_1+n_2+n_3 = n_5$. If* $\mathrm{PILS}(n_1,n_2,n_3,n_4,n_5)$ *is nonempty, then either*

- *$n_1 = n_2 = n_3$ and $n_4 \le 3n_1$,*
- *$n_4 = n_1+n_3$, $n_1 = n_2$, and $n_3 \le 2n_1$, or*
- *$n_4 = n_1+n_3$ and $n_2 = n_3$.*

That is, if $\mathrm{PILS}(Q)$ *is nonempty for a partition with 5 parts in which the three smallest parts sum to the largest part, either*

- *$Q = (a,a,a,b,3a)$ with $1 \le a \le b \le 3a$ or*
- *$Q = (a,a,b,a+b,2a+b)$ with $1 \le b \le 2a$ (with no explicit ordering for a and b).*

Proof. By assumption, $\mathrm{PILS}(n_1,n_2,n_3,n_4,n_5)$ is nonempty; let $A \in \mathrm{PILS}(n_1,n_2,n_3,n_4,n_5)$ and $R \in \mathrm{OS}^*(n_1,n_2,n_3,n_4,n_5)$ be its reduction over [5]. Let R' be the sub-array of R obtained by removing the last row and column. Let $\sigma_k(R_{ij})$ denote the number of occurrences of symbol k in cell R_{ij}, and similarly define σ_k for R'.

Observe that each occurrence of 5 in row 4 belongs to either R_{41}, R_{42}, or R_{43}. Hence $\sigma_5(R_{41})+\sigma_5(R_{42})+\sigma_5(R_{43}) = n_4n_5$, and therefore

$$n_4n_5 = n_4n_1+n_4n_2+n_4n_3 = |R_{41}|+|R_{42}|+|R_{43}| \ge \sigma_5(R_{41})+\sigma_5(R_{42})+\sigma_5(R_{43}) = n_4n_5.$$

So every symbol in R_{41}, R_{42}, and R_{43} (and similarly R_{14}, R_{24}, and R_{34}) is 5. So in R', symbol i (where $\{i,j,k\} = \{1,2,3\}$) can only be found in R_{ii}, R_{jk}, and R_{kj}. From Ryser's Theorem, it follows that i, for each $i \in [5]$, occurs at least $2(n-n_5)-n = n_4$ times in R', so $\sigma_i(R') \ge n_in_4$. So for each $\{i,j,k\} = \{1,2,3\}$,

$$\sigma_i(R') = \sigma_i(R_{ii})+\sigma_i(R_{jk})+\sigma_i(R_{kj}) \ge n_in_4, \text{ and hence } \sigma_i(R_{jk})+\sigma_i(R_{kj}) \ge n_i(n_4-n_i). \tag{1}$$

Observe that $\sigma_5(R') = n_5(n - n_5)$ and hence

$$|R'| = \sum_{i=1}^{5} \sigma_i(R') \geq n_1n_4 + n_2n_4 + n_3n_4 + n_4^2 + n_5(n - n_5) = (n_1 + n_2 + n_3 + n_4)^2 = |R'|.$$

So $\sigma_i(R') = n_in_4$ for each $i \in [4]$. This implies that the inequalities in (1) are equalities. Furthermore, it implies $n_in_j = \sigma_k(R_{ij}) + \sigma_5(R_{ij})$ where $\{i, j, k\} = \{1, 2, 3\}$ – in other words only symbols k and 5 appear in cell R_{ij}. Observe that

$$\begin{aligned} n_jn_5 &= \sigma_5(R_{ij}) + \sigma_5(R_{kj}) + \sigma_5(R_{4j}) \text{ and} \\ &= \sigma_5(R_{ji}) + \sigma_5(R_{jk}) + \sigma_5(R_{j4}), \end{aligned}$$

and since $\sigma_5(R_{j4}) = \sigma_5(R_{4j}) = n_jn_4$, it follows that

$$\sigma_5(R_{ij}) + \sigma_5(R_{kj}) = \sigma_5(R_{ji}) + \sigma_5(R_{jk}) = n_j(n_5 - n_4) = n_j(n_i + n_j + n_k - n_4). \quad (2)$$

We now focus on the cells A_{ij} and A_{kj}, where $\{i, j, k\} = \{1, 2, 3\}$. Observe that only k and 5 occur in both R_{ij} and R_{ji} and only i and 5 occur in R_{kj} and R_{jk}. This, along with (2), implies that

$$\sigma_k(R_{ij}) + \sigma_i(R_{kj}) = \sigma_i(R_{jk}) + \sigma_k(R_{ji}). \quad (3)$$

From (3) and the equality in (1), it follows that

$$\begin{aligned} 2[\sigma_k(R_{ij}) + \sigma_i(R_{kj})] &= \sigma_k(R_{ij}) + \sigma_i(R_{kj}) + \sigma_i(R_{jk}) + \sigma_k(R_{ji}) \\ &= [\sigma_k(R_{ij}) + \sigma_k(R_{ji})] + [\sigma_i(R_{kj}) + \sigma_i(R_{jk})] \\ &= n_k(n_4 - n_k) + n_i(n_4 - n_i). \end{aligned}$$

Hence the number of symbols R_{ij} and R_{kj} can be computed in two ways:

$$\begin{aligned} n_j(n_i + n_k) &= |R_{ij}| + |R_{kj}| \\ &= \sigma_5(R_{ij}) + \sigma_k(R_{ij}) + \sigma_5(R_{kj}) + \sigma_i(R_{kj}) \\ &= [\sigma_5(R_{ij}) + \sigma_5(R_{kj})] + [\sigma_k(R_{ij}) + \sigma_i(R_{kj})] \\ &= n_j(n_i + n_j + n_k - n_4) + [n_k(n_4 - n_k) + n_i(n_4 - n_i)]/2. \end{aligned}$$

The equation above simplifies to $2n_j(n_4 - n_j) = n_i(n_4 - n_i) + n_k(n_4 - n_k)$. In a similar manner by considering the entries in R_{ij} and R_{ik}, we have that $2n_i(n_4 - n_i) = n_j(n_4 - n_j) + n_k(n_4 - n_k)$. From these two equations, we can conclude that $n_i(n_4 - n_i) = n_j(n_4 - n_j)$, or rather $n_4(n_j - n_i) = (n_j - n_i)(n_j + n_i)$. Hence either $n_i = n_j$ or $n_i + n_j = n_4$.

Therefore either $n_4 = n_1 + n_3$ or $n_1 = n_3$, either $n_4 = n_2 + n_3$ or $n_2 = n_3$, and either $n_4 = n_1 + n_2$ or $n_1 = n_2$. So the only possible PILS with five parts and a maximal fifth part must relate to partitions of the form $(n_1, n_1, n_1, n_4, 3n_1)$, $(n_1, n_1, n_3, n_1 + n_3, 2n_1 + n_2)$, or $(n_1, n_3, n_3, n_1 + n_3, n_1 + 2n_3)$.

In the first case, the trivial necessary condition for the existence of a PILS$(n_1, n_1, n_1, n_4, 3n_1)$ requires that $n_4 \leq 3n_1$. For the second case, we focus on the entries in R_{12} and R_{21}. We have that $n_4 - n_3 = n_1$ and thus using (1) we have

$$2n_1n_2 = |R_{12}| + |R_{21}| \geq \sigma_3(R_{12}) + \sigma_3(R_{21}) = n_3(n_4 - n_3) = n_1n_3.$$

Hence $2n_2 \geq n_3$ and therefore $n_3 \leq 2n_1$. □

4 Some Characterizations of PILS with 5 parts

We begin with a classification of PILS with five parts in which three parts are of equal size. We show that the necessary condition in Corollary 3.2 is also sufficient.

Theorem 4.1. *Let a, b, and c be positive integers. Then* $\mathrm{PILS}(a, a, a, b, c)$ *is nonempty if and only if* $b \le 3a$, $c \le 3a$, *and* $6a^2 \ge 3ab + 3ac - 2bc$.

Proof. The forward direction follows from Lemma 3.1. Suppose $b \le 3a$, $c \le 3a$, and $6a^2 \ge 3ab + 3ac - 2bc$. Define positive integers $\delta, \epsilon \in \{0, 1\}$ and x such that $bc = 3x + \delta + \epsilon$, where $\delta \ge \epsilon$. Define $y^-, y^+, z^-, z^+, w, w'$ as the integers below:

$$\begin{aligned} y^- &= \lfloor (ab - x)/2 \rfloor, & y^+ &= \lceil (ab - x)/2 \rceil, \\ z^- &= \lfloor (ac - x)/2 \rfloor, & z^+ &= \lceil (ac - x)/2 \rceil, \\ w &= a^2 - y^+ - z, & w' &= a^2 - y^- - z^+. \end{aligned}$$

Observe that $y^- + y^+ + x = ab$ and $z^- + z^+ + x = ac$. Through tedious case analysis, one can show that $w, w' \ge 0$ follows from $6a^2 \ge 3ab + 3ac - 2bc$. Furthermore one can show that $y^-, z^- \ge 0$ and $y^+, z^+ \ge \delta$ follow from $b, c \le 3a$. Define R over $[5]$ as given in Figure 3. From the previous observations, each symbol in each cell of R

$1 : a^2$	$3 : w$ $4 : y^+ - \delta$ $5 : z^- + \delta$	$2 : w' + \delta$ $4 : y^-$ $5 : z^+ - \delta$	$2 : y^-$ $3 : y^+$ $5 : x$	$2 : z^+ - \delta$ $3 : z^-$ $4 : x + \delta$
$3 : w'$ $4 : y^- + \delta$ $5 : z^+ - \delta$	$2 : a^2$	$1 : w + \epsilon$ $4 : y^+ - \delta$ $5 : z^- + \delta - \epsilon$	$1 : y^+ - \epsilon$ $3 : y^-$ $5 : x + \epsilon$	$1 : z$ $3 : z^+$ $4 : x$
$2 : w\delta$ $4 : y^+ - \delta$ $5 : z^-$	$1 : w' + \epsilon$ $4 : y^- + \delta - \epsilon$ $5 : z^+ - \delta$	$3 : a^2$	$1 : y^-$ $2 : y^+ - \delta$ $5 : x + \delta$	$1 : z^+ - \epsilon$ $2 : z^-$ $4 : x + \epsilon$
$2 : y^+ - \delta$ $3 : y^-$ $5 : x + \delta$	$1 : y^-$ $3 : y^+$ $5 : x$	$1 : y^+ - \epsilon$ $2 : y$ $5 : x + \epsilon$	$4 : b^2$	$1 : x + \epsilon$ $2 : x + \delta$ $3 : x$
$2 : z$ $3 : z^+$ $4 : x$	$1 : z^+ - \epsilon$ $3 : z^-$ $4 : x + \epsilon$	$1 : z^-$ $2 : z^+ - \delta$ $4 : x + \delta$	$1 : x + \epsilon$ $2 : x + \delta$ $3 : x$	$5 : c^2$

Figure 3: The outline square R used in Theorem 4.1. Note that $a : t$ indicates t copies of a.

has a nonnegative occurrence, and one can check directly that $R \in \mathrm{OS}^*(a, a, a, b, c)$. Hence $\mathrm{PILS}(a, a, a, b, c)$ is nonempty. □

We give an observation which will be useful in a later classification.

Observation 4.2. Let a, b, and c be positive integers for which $c = 3a$ and $b \ge a$. Then $3ab + 3ac - 2bc = 9a^2 - 3ab \le 6a^2$. Hence it follows from Theorem 4.1 that $\mathrm{PILS}(a, a, a, b, 3a)$ is nonempty.

We now turn our attention to the classification of PILS of five parts in which the largest part is maximal; that is the sum of the smallest three parts. We show that the necessary conditions in Lemma 3.4 are also sufficient.

Theorem 4.3. *If Q is a partition with 5 parts in which the three smallest parts sum to the largest part, then* PILS(Q) *is nonempty if and only if*

- $Q = (a, a, a, b, 3a)$ *with* $1 \leq a \leq b \leq 3a$ *or*
- $Q = (a, a, b, a + b, 2a + b)$ *with* $1 \leq b \leq 2a$ *(with no explicit ordering for* a *and* b*).*

Proof. It is sufficient to show that if a and b are positive integers with $1 \leq b \leq 2a$, then PILS$(a, a, b, a+b, 2a+b)$ is nonempty; the result then follows from Observation 4.2 and Lemma 3.4.

Define y^-, y^+, z^-, z^+ as the integers below:

$$\begin{aligned} y^- &= \lfloor ab/2 \rfloor, & y^+ &= \lceil ab/2 \rceil, \\ z^- &= \lfloor a^2 - ab/2 \rfloor, & z^+ &= \lceil a^2 - ab/2 \rceil. \end{aligned}$$

Observe that $y^- + y^+ = ab$ and $y^- + z^+ = y^+ + z^- = a^2$. Since $b \leq 2a$, each of these integers is nonnegative. Define R over $[5]$ as given in Figure 4. One can check that $R \in \mathrm{OS}^*(a, a, b, a + b, 2a + b)$, and therefore PILS$(a, a, b, a + b, 2a + b)$ is nonempty. □

$1 : a^2$	$3 : y^-$ $5 : z^+$	$2 : y^+$ $5 : y^-$	$5 : a(a+b)$	$2 : z^-$ $3 : y^+$ $4 : a(a+b)$
$3 : y^+$ $5 : z^-$	$2 : a^2$	$1 : y^-$ $5 : y^+$	$5 : a(a+b)$	$1 : z^+$ $3 : y^-$ $4 : a(a+b)$
$2 : y^-$ $5 : y^+$	$1 : y^+$ $5 : y^-$	$3 : b^2$	$5 : b(a+b)$	$1 : y^-$ $2 : z^+$ $4 : b(a+b)$
$5 : a(a+b)$	$5 : a(a+b)$	$5 : b(a+b)$	$4 : (a+b)^2$	$1 : a(a+b)$ $2 : a(a+b)$ $3 : b(a+b)$
$2 : z^+$ $3 : y^-$ $4 : a(a+b)$	$1 : z^-$ $3 : y^+$ $4 : a(a+b)$	$1 : y^+$ $2 : y^-$ $4 : b(a+b)$	$1 : a(a+b)$ $2 : a(a+b)$ $3 : b(a+b)$	$5 : (2a+b)^2$

Figure 4: The outline square R used in Theorem 4.3. Note that $a : t$ indicates t copies of a.

5 An Inductive Technique

In the next results, we show the existence of PILS in which the five parts are close in size, and the fifth part does not reach its maximum. First, we start with the following lemma.

Lemma 5.1. *Let $n_i \in [3]$ for each $i \in [5]$. Then* $\mathrm{PILS}(n_1, n_2, n_3, n_4, n_5)$ *is nonempty.*

Proof. Let $Q = (n_1, n_2, n_3, n_4, n_5)$. It follows from Theorems 1.1, 1.2, and 4.3 that $\mathrm{PILS}(Q)$ is nonempty except possibly when Q is $(1, 1, 2, 2, 3)$, $(1, 1, 2, 3, 3)$, or $(1, 2, 2, 3, 3)$. By lifting the outline squares in Figure 5, we have that $\mathrm{PILS}(Q)$ is nonempty for each Q in this list. □

1 0 0 0 0	0 0 1 0 0	0 0 0 0 2	0 1 0 0 1	0 0 1 2 0
0 0 1 0 0	0 1 0 0 0	0 0 0 1 1	0 0 0 0 2	1 0 1 1 0
0 0 0 1 1	0 0 0 0 2	0 0 4 0 0	1 0 0 0 3	1 2 0 3 0
0 0 0 0 2	1 0 0 0 1	0 1 0 0 3	0 0 0 4 0	1 1 4 0 0
0 1 1 1 0	0 0 1 2 0	2 1 0 3 0	1 1 4 0 0	0 0 0 0 9

1 0 0 0 0	0 0 1 0 0	0 1 0 0 1	0 0 1 0 2	0 0 0 3 0
0 0 1 0 0	0 1 0 0 0	0 0 0 0 2	1 0 1 0 1	0 0 0 3 0
0 1 0 1 0	0 0 0 2 0	0 0 4 0 0	0 0 0 0 6	2 1 0 3 0
0 0 0 0 3	0 0 0 0 3	2 1 0 0 3	0 0 0 9 0	1 2 6 0 0
0 0 1 2 0	1 0 1 1 0	0 0 0 6 0	2 3 4 0 0	0 0 0 0 9

1 0 0 0 0	0 0 0 1 1	0 1 0 1 0	0 1 0 0 2	0 0 2 1 0
0 0 0 0 2	0 4 0 0 0	2 0 0 2 0	0 0 2 0 4	0 0 2 4 0
0 0 0 1 1	1 0 0 1 2	0 0 4 0 0	0 3 0 0 3	1 1 0 4 0
0 1 2 0 0	1 0 2 0 3	0 0 0 0 6	0 0 0 9 0	2 5 2 0 0
0 1 0 2 0	0 0 2 4 0	0 3 0 3 0	3 2 4 0 0	0 0 0 0 9

Figure 5: Elements of $\mathrm{OS}^*(1, 1, 2, 2, 3)$, $\mathrm{OS}^*(1, 1, 2, 3, 3)$, and $\mathrm{OS}^*(1, 2, 2, 3, 3)$. Note, the ith entry in a cell corresponds to the number of copies of i in a cell, where $i \in [5]$.

Theorem 5.2. *Let q and r_i be positive integers and $n_i = 5q + r_i$ for each $i \in [5]$. If* $\mathrm{PILS}(r_1, r_2, r_3, r_4, r_5)$ *is nonempty, then* $\mathrm{PILS}(n_1, n_2, n_3, n_4, n_5)$ *is nonempty.*

Proof. Set $n = n_1 + n_2 + n_3 + n_4 + n_5$ and $r = r_1 + r_2 + r_3 + r_4 + r_5$, and let $R = (r_1, r_2, r_3, r_4, r_5)$, $Q = (n_1, n_2, n_3, n_4, n_5)$, and $Q_i = (q^{i-1}, (q + r_i), q^{5-i})$, where $i \in [5]$. From Theorem 1.2 (2), PILS of type Q_i exist for each $i \in [5]$. For $i \in [5]$, let $M_i \in \mathrm{OS}^*(Q_i)$ be the reduction of a PILS of type Q_i and let $M_6 \in \mathrm{OS}^*(R)$ be the reduction of a PILS of type R. We assume that symbol i occurs in cell (i, i) of M_j for each $j \in [6]$.

Consider the 5×5 array obtained by taking the union of 5 copies of each M_i for each $i \in [5]$, as well as a copy of M_6 with its diagonal removed, that is

$$T = \left(\bigcup_{i=1}^{5} 5M_i \right) \cup \left(M_6 \setminus \{ M_6(i, i) \mid i \in [5] \} \right).$$

In what follows, we show that $T \in \mathrm{OS}^*(Q)$. First, we show each symbol appears the correct number of times in T. Let $i \in [5]$. Observe the following:

- symbol i in M_i occurs $(5q + r_i)(q + r_i)$ times,
- symbol $j \neq i$ in M_i occurs $(5q + r_i)q$ times, and

- symbol i in $M_6 \setminus \{M_6(i,i) \mid i \in [5]\}$ occurs $r_i(r - r_i)$ times.

Therefore,

$$\begin{aligned}\sigma_i(T) &= 5(5q + r_i)(q + r_i) + 5\sum_{j \neq i}(5q + r_j)q + r_i(r - r_i) \\ &= 125q^2 + 25qr_i + 5q\sum_{j=1}^{5} r_j + r_i r = nn_i.\end{aligned}$$

Second, we show each cell of T has the correct number of symbols. Observe that in T, for each distinct $i, j \in [5]$, cell (i, j) contains

$$\underbrace{5(q+r_i)q}_{\text{from } M_i} + \underbrace{5(q+r_j)q}_{\text{from } M_j} + \underbrace{r_i r_j}_{\text{from } M_6} + \underbrace{15q^2}_{\text{from others}} = 25q^2 + 5qr_i + 5qr_j + r_i r_j = n_i n_j$$

symbols, and for each $i \in [5]$, cell (i, i) contains

$$\underbrace{5(q+r_i)^2}_{\text{from } M_i} + \underbrace{0}_{\text{from } M_6} + \underbrace{20q^2}_{\text{from others}} = 25q^2 + 10qr_i + r_i^2 = (5q_i + r_i)^2 = n_i^2$$

symbols. Last, we count the number of times symbol i occurs in column or row j of T. Observe the following: symbol i occurs

- $(q + r_i)^2$ times in column (or row) i of M_i,
- $(q + r_i)q$ times in column (or row) j of M_i, where $i \neq j$,
- $(q + r_j)q$ times in column (or row) j of M_j, where $i \neq j$,
- q^2 times in column (or row) j of M_k for $k \notin \{i, j\}$, and
- $r_i r_j$ times in column (or row) j of M_6.

Therefore i occurs

$$\underbrace{5(q+r_i)^2}_{\text{from } M_i} + \underbrace{0}_{\text{from } M_6} + \underbrace{20q^2}_{\text{from others}} = n_i^2$$

times in column (or row) i of T and

$$\underbrace{5q(q+r_i)}_{\text{from } M_i} + \underbrace{5q(q+r_j)}_{\text{from } M_j} + \underbrace{r_i r_j}_{\text{from } M_6} + \underbrace{15q^2}_{\text{from others}} = n_i n_j$$

times in column (or row) j of T, where $i \neq j$. So $T \in \mathrm{OS}(Q)$. Additionally, $T(i,i)$ consists of $(5q + r_i)^2$ copies of symbol i; therefore $T \in \mathrm{OS}^*(Q)$. So T can be lifted to an element of $\mathrm{PILS}(n_1, n_2, n_3, n_4, n_5)$. □

We conclude with some corollaries. First, we may deduce the existence of PILS with close part sizes from Lemma 5.1 and Theorem 5.2.

Corollary 5.3. *Let q be a positive integer and $n_i = 5q + r_i$, where $r_i \in [3]$ and $i \in [5]$. Then* $\mathrm{PILS}(n_1, n_2, n_3, n_4, n_5)$ *is nonempty.*

The next corollary follows from Theorem 4.3 and 5.2.

Corollary 5.4. *Let q, a, and b be positive integers. Then* $\mathrm{PILS}(5q+a, 5q+a, 5q+b, 5q+a+b, 5q+2a+b)$ *is nonempty.*

Finally, we conclude with a corollary showing that a PILS with five consecutive part sizes exist, which follows from Theorem 5.2 and the outline squares given in Figure 6.

Corollary 5.5. *Let n be a positive integer. Then* $\mathrm{PILS}(n, n+1, n+2, n+3, n+4)$ *is nonempty.*

1 0 0 0 0	0 0 2 0 0	0 2 0 0 1	0 0 0 0 4	0 0 1 4 0
0 0 2 0 0	0 4 0 0 0	0 0 0 2 4	2 0 0 0 6	0 0 4 6 0
0 2 0 0 1	0 0 0 2 4	0 0 9 0 0	2 0 0 0 10	1 4 0 10 0
0 0 0 0 4	0 0 2 0 6	0 2 0 0 10	0 0 0 16 0	4 6 10 0 0
0 0 1 4 0	2 0 2 6 0	3 2 0 10 0	0 8 12 0 0	0 0 0 0 25

4 0 0 0 0	0 0 5 1 0	0 1 0 3 4	0 2 0 0 8	0 3 3 6 0
0 0 3 3 0	0 9 0 0 0	2 0 0 2 8	2 0 3 0 10	2 0 6 10 0
0 2 0 2 4	0 0 0 4 8	0 0 16 0 0	3 5 0 0 12	5 5 0 14 0
0 0 2 0 8	2 0 3 0 10	3 5 0 0 12	0 0 0 25 0	5 10 15 0 0
0 4 3 5 0	4 0 4 10 0	3 6 0 15 0	5 8 17 0 0	0 0 0 0 36

9 0 0 0 0	0 0 8 4 0	0 2 0 4 9	0 6 0 0 12	0 4 7 10 0
0 0 8 4 0	0 16 0 0 0	0 0 0 6 14	4 0 6 0 14	8 0 6 14 0
0 2 0 6 7	2 0 0 6 12	0 0 25 0 0	6 8 0 0 16	7 10 0 18 0
0 0 4 0 14	4 0 4 0 16	8 10 0 0 12	0 0 0 36 0	6 14 22 0 0
0 10 3 8 0	6 0 8 14 0	7 8 0 20 0	8 10 24 0 0	0 0 0 0 49

16 0 0 0 0	0 0 4 6 10	0 7 0 9 8	0 6 8 0 14	0 7 12 13 0
0 0 6 8 6	0 25 0 0 0	5 0 0 9 16	9 0 8 0 18	6 0 16 18 0
0 9 0 7 8	4 0 0 10 16	0 0 36 0 0	6 12 0 0 24	14 9 0 25 0
0 0 10 0 18	9 0 12 0 14	7 11 0 0 24	0 0 0 49 0	12 24 20 0 0
0 11 8 13 0	7 0 14 19 0	12 12 0 24 0	13 17 26 0 0	0 0 0 0 64

25 0 0 0 0	0 0 7 10 13	0 11 0 12 12	0 6 14 0 20	0 13 14 18 0
0 0 13 10 7	0 36 0 0 0	5 0 0 14 23	14 0 10 0 24	11 0 19 24 0
0 9 0 11 15	7 0 0 15 20	0 0 49 0 0	12 16 0 0 28	16 17 0 30 0
0 4 13 0 23	14 0 13 0 21	8 20 0 0 28	0 0 0 64 0	18 24 30 0 0
0 17 9 19 0	9 0 22 23 0	22 11 0 30 0	14 26 32 0 0	0 0 0 0 81

Figure 6: Elements of $\mathrm{OS}(n, n+1, n+2, n+3, n+4)$ for each $n \in [5]$. Note, the ith entry in a cell corresponds to the number of copies of i in a cell, where $i \in [5]$.

6 Concluding Remarks

We end with two brief remarks, both of which concern the problem of existence of elements of $\mathrm{PILS}(n_1, n_2, n_3, n_4, n_5)$. The first remark translates the problem of existence to an integer programming problem involving outline squares. We have yet to determine whether this is a useful translation. The second remark is on whether the necessary condition in Lemma 3.1 is in fact sufficient.

Let $Q = (n_1, n_2, n_3, n_4, n_5)$ be a partition in which $n = n_1 + n_2 + n_3 + n_4 + n_5$, and let $P \in \mathrm{OS}^*(Q)$ over symbol set [5]. Recall that $\sigma_k(P_{ij})$ denotes the number of times symbol k occurs in cell P_{ij}. For each cell P_{ij}, where $i \neq j$ and $i, j \in [5]$, and for each $k \in [5]$, we have the following equations which arise from the number of expected symbols of type k in cell (i, j), row k, and column k:

$$\sum_{\ell=1}^{5} \sigma_\ell(P_{ij}) = n_i n_j, \ \sum_{j=1}^{5} \sigma_k(P_{ij}) = n_i, \text{ and } \sum_{i=1}^{5} \sigma_k(P_{ij}) = n_j.$$

Since the content of P_{ii} is known for each $i \in [5]$, there are altogether 60 linear equations in 60 variables of the form $\sigma_k(P_{ij})$, where $k \in [5]$ and $i, j \in [5]\backslash\{k\}$. There is a nonnegative integer solution to the system if and only if $\mathrm{PILS}(n_1, n_2, n_3, n_4, n_5)$ is nonempty. This admits a rank 46 system of linear equations, which may be difficult to solve in general. We do not believe this is a worthwhile translation for constructing PILS with an arbitrary number of parts, but for PILS with 5 parts, this may be tractable.

Lastly, we have yet to find an example of a PILS in which the necessary condition in Lemma 3.1 is not sufficient. From our analysis, it seems that when the difference in part sizes is 'big enough', the PILS will not exist. Furthermore, it seems that the necessary condition in Lemma 3.1 keeps part sizes 'close enough' so as to guarantee existence. Thus, we end with the following conjecture:

Conjecture 6.1. *Let a, b, c, d, e be positive integers. A* PILS *of type (a, b, c, d, e) exists if and only if $(a+b+c+d+e)^2 \geq 3(a+b+c)(d+e) + a^2 + b^2 + c^2 + d^2 + e^2$, for all 10 possible re-orderings of a, b, c, d, and e.*

We note that Theorems 4.1, 1.1, and 1.2 imply that Conjecture 6.1 holds when at least three parts are of equal size, and Theorems 4.1 and 4.3 also imply that Conjecture 6.1 holds whenever the largest part is the sum of the three smallest parts.

We close with a comparison between our results and those of a recent publication by Colbourn [2]. There is a natural correspondence between self-conjugate partial incomplete Latin squares and group divisible designs. For background on this, see [4], [1], and [9]. In [2], more advances are made on the existence of PILS through constructions arising from group divisible designs and other objects of the same flavor. Colbourn presents a lemma with a necessary condition, two conjectures, and two results which are relevant to our work in this paper.

Lemma 6.2 ([2], Lemma 1.6)**.** *If a* $\mathrm{PILS}(n_1, n_2, \ldots, n_k)$ *exists, then for any partition* $\{D, E\}$ *of* $[k]$*, we have that*

$$(n_1 + \cdots + n_k)^2 - (n_1^2 + \cdots + n_k^2) \geq 3\left(\sum_{i\in D} n_i\right)\left(\sum_{j\in E} n_j\right).$$

Conjecture 6.3 ([2], Conjecture 1.8)**.** *Let $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_{k-2} = n_{k-1} = n_k$ be integers. Then* $\mathrm{PILS}(n_1, \ldots, n_k)$ *is nonempty.*

Conjecture 6.4 ([2], Conjecture 1.10). *Let $k \geq 5$ and $n_1 \leq n_2 \leq \cdots \leq n_k \leq (n-2)n_1$ be integers. Then* $\mathrm{PILS}(n_1, \ldots, n_k)$ *is nonempty.*

Lemma 6.5 ([2], Lemma 3.7). *If $k \in \{5, 6\}$ and $n_1 \leq \cdots \leq n_{k-2} = n_{k-1} = n_k$, then* $\mathrm{PILS}(n_1, \ldots, n_k)$ *is nonempty.*

Theorem 6.6 ([2], Theorem 5.6). *Let $k \geq 5$, $\mu \geq 1$, and $\mu \leq n_1 \leq \cdots \leq n_k \leq 3\mu$. Then* $\mathrm{PILS}(n_1, \ldots, n_k)$ *is nonempty except possibly when $k = 5$ and $\mu = 6$.*

Note that Lemma 6.2 is a generalization of Lemma 3.1. In the case when $n = 5$, Theorem 4.1 is a stronger result than Lemma 6.5. Conjecture 6.4 involves PILS with part sizes which are bounded, and Theorem 6.6 is a stronger result than Theorem 5.2.

The generalization of Conjecture 6.1 to larger numbers of parts, as indicated by the necessary condition provided in Lemma 6.2, is false. For example, $\mathrm{PILS}(29^2, 10^2, 2^1, 1^7)$ is empty while each part is bounded by the sum of any other 10 parts, as well as the necessary conditions from Lemma 6.2 are satisfied (see [2] for the argument). This suggests that as the number of parts increase, the number and complexity of necessary conditions for the existence of a PILS will increase. However, at this point we have not produced a partition in 5 parts meeting our necessary conditions which failed to have a PILS of the same type. So with this in mind, in the absence of a counterexample, we put our conjecture forward.

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