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Abstract

Let $r, c, s \in \{1, 2, \dots, n\}$ and let P be a partial latin square of order n in which each nonempty cell lies in row r, column c, or contains symbol s. We show that if $n \notin \{3, 4, 5\}$ and row r, column c, and symbol s can be completed in P, then a completion of P exists. As a consequence, this proves a conjecture made by Casselgren and Häggkvist. Furthermore, we show exactly when row r, column c, and symbol s can be completed.

1 Introduction

Let n be a positive integer and S a symbol set of cardinality n. A *partial latin square* of order n is an $n \times n$ matrix partially filled with symbols from S such that each symbol occurs at most once in each row and column. Rows and columns are indexed with the set $[n] = \{1, 2, \dots, n\}$ and $S = [n]$, unless otherwise stated. The set of all partial latin squares of order n is denoted $\mathrm{PLS}(n)$.

Let $P \in \mathrm{PLS}(n)$. The symbol located in cell (i, j) of P, if such a symbol exists, is denoted $P(i, j)$. We will often write P as a subset of $[n] \times [n] \times S$ in which $(i, j, k) \in P$ if and only if $P(i, j) = k$. If cell (i, j) of P is nonempty for all $i, j \in [n]$, then P is called a *latin square* and we write $P \in \mathrm{LS}(n)$. We say that P is *completable* if there is a latin square $L \in \mathrm{LS}(n)$ in which $P(i, j) = L(i, j)$ for each nonempty cell (i, j) of P, or alternatively, $P \subseteq L$.

The problem of completing partial latin squares is notoriously difficult. For an arbitrary $P \in \mathrm{PLS}(n)$, determining whether or not it is completable is NP-complete [3]. However, there are known families of completable partial latin squares. Marshall Hall [6] proved the following.

Theorem 1.1. *Let r and n be integers such that $r \leqslant n$. Let $P \in \mathrm{PLS}(n)$ in which the cells of r rows (columns) are filled and the remaining cells empty. Then P can be completed to an element of $\mathrm{LS}(n)$.*

Theorem 1.1 essentially says that if P contains exactly r filled rows or r filled columns, then P can be completed. The same can be said of symbols. If P contains exactly r symbols, each occurring n times, then P can be completed. Furthermore, by rearranging the completed $n - r$ rows (columns) from Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let $P \in \mathrm{PLS}(n)$ with exactly r filled rows (columns) and one filled column (row). Then P is completable.*

Other families of completable partial latin squares come from solutions to the famous Evans Conjecture [4].

Theorem 1.3. *All partial latin squares of order n with at most $n - 1$ nonempty cells are completable.*

Proofs of Theorem 1.3 were given independently by Häggkvist for $n \geqslant 1111$ [5], and by Andersen and Hilton [1] and Smetaniuk [7] for all n. The upper bound on the number of nonempty cells is sharp. In [1], the authors determine all incompletable partial latin squares of order n with exactly n nonempty cells. One such square has symbol 1 in the first k diagonal cells and symbols $2, 3, \ldots, n-k+1$ in the last $n-k$ cells of column $k+1$. We name this partial latin square $B_{k,n}$.

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$. For $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$, we use $\theta(P) \in \mathrm{PLS}(n)$ to denote the array in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively. The mapping θ is called an *isotopism*, and P and $\theta(P)$ are said to be *isotopic*. A *conjugate* of P is an array in which the coordinates of each triple of P are uniformly permuted. There are six, not necessarily distinct, conjugates of P. The *main class*, or *species* of P is the set of all partial latin squares that are isotopic to some conjugate of P.

The following theorem is Andersen and Hilton's solution to the Evans conjecture [1].

Theorem 1.4. *If $P \in \mathrm{PLS}(n)$ with exactly n nonempty cells, then P is completable if and only if P is not a species of $B_{k,n}$ for each $k \in [n-1]$.*

More recently, the following family was studied in [2]. For $n > 0$, we say that $P \in \mathrm{PLS}(n)$ satisfies the *RCS property* if there exists $r, c, s \in [n]$ such that for all $(x, y, z) \in P$, either $x = r$, $y = c$, or $z = s$. Casselgren and Häggkvist conjectured the following in [2].

Conjecture 1.5. *Let $n > 0$, $P \in \mathrm{PLS}(n)$, and r, c, and s defined as above. If P satisfies the RCS property, $(r, c, s) \in P$, and $n \notin \{3, 4, 5\}$, then P is completable.*

Observe that Conjecture 1.5 is trivial for $n = 1$ and $n = 2$. Casselgren and Häggkvist confirmed Conjecture 1.5 when $n \in \{6, 7\}$ and $n = 4k$ for all integers $k \geqslant 2$ [2]. Arrays (a), (b), and (c) in Figure 1 illustrate why the conjecture excludes orders 3, 4, and 5.

1	2	3
2	1	
3		1

(a)

1	3	4	2
2	1		
3		1	
4			1

(b)

1	3	2	4	5
2	1			
3		1		
4			1	
5				1

(c)

2	3		4	5
	1			
3		1		
4			1	
5				1

(d)

Figure 1: Incompletable partial latin squares.

For $n > 0$, define $\mathcal{P}_n \subseteq \mathrm{PLS}(n)$ as the partial latin squares that satisfy the RCS property and do not contain a species of $B_{k,n}$ for each $k \in [n-1]$; these species are called *forbidden configurations*. Clearly all completable partial latin squares satisfying the RCS property belong to $\mathcal{P}_n$. Array (d) in Figure 1 contains a forbidden configuration and thus does not belong to $\mathcal{P}_5$.

In this paper, we prove the following:

Theorem 1.6. *Let $n \geqslant 6$. If $P \in \mathcal{P}_n$, then P is completable.*

In Section 2, we show that for each $P \in \mathcal{P}_n$, row r, column c, and symbol s can be completed. Then, in Section 3, we prove Conjecture 1.5 for $n \geqslant 8$. In Section 4, we show that if $n \geqslant 6$, $P \in \mathcal{P}_n$, and a completion of row r, column c, and symbol s does not include (r, c, s), then P is completable.

Observe that if $P \in \mathrm{PLS}(n)$ with the RCS property, then each of its species also has the RCS property. Thus, we assume that $r = c = s = 1$ throughout this paper.

2 Necessary Conditions

Let $P \in \mathcal{P}_n$ such that an empty cell occurs in row 1 or column 1, or symbol 1 appears fewer than n times. We say that P has an *RCS completion* if row 1, column 1, and symbol 1 can be completed. We show that an RCS completion of P exists.

Observation 2.1. If $P(1, 1) = 1$, then P has an RCS completion by arbitrarily filling row 1 and column 1, and adding symbol 1 as necessary.

Lemma 2.2. *Let $P \in \mathcal{P}_n$ and suppose that cell $(1,1)$ of P is empty. Then P has an RCS completion.*

Proof. Let S be the union of symbols appearing in either row 1 or column 1. Since P contains no forbidden configurations, $S \neq [n]$. If $1 \notin S$, set $P(1,1) = 1$ and the result follows from Observation 2.1. Otherwise, let $x \in [n]\backslash S$ and place x in cell $(1,1)$. Without loss of generality, assume that 1 appears in row 1.

Suppose that 1 does not appear in column 1 and cannot be added to column 1. Suppose that P contains k occurrences of 1. Without loss of generality, assume they occur in the first k rows. It follows that the latter $n-k$ cells in column 1 must be filled which, when combined with the k 1's, yield a forbidden configuration. Thus, symbol 1 can be added to column 1. Since 1 appears in column 1 and row 1, symbol 1 can be completed. It follows that row 1 and column 1 can be completed. □

Lemma 2.3. *Let $P \in \mathcal{P}_n$ and suppose that cell $(1,1)$ of P is nonempty. Then P has an RCS completion.*

Proof. If $P(1,1) = 1$, then the result follows from Observation 2.1, so assume that $P(1,1) \neq 1$. If 1 does not appear in row 1, then the conjugate of P obtained by swapping symbol and column coordinates has an empty $(1,1)$ cell, and so the result follows from Lemma 2.2. A similar result holds if 1 does not appear in column 1. If 1 appears in both row 1 and column 1, then P has an RCS completion as outlined at the end of the proof of Lemma 2.2. □

3 A proof for Conjecture 1.5

For this section, we consider only elements $P \in \mathcal{P}_n$ for which an RCS completion of P includes $(1,1,1)$. Let $F_n \subseteq S_n$ be the set of permutations which fix 1, and for each $\pi \in F_n$, define $P_\pi \in \mathrm{PLS}(n)$ as $\{(a,1,a),(1,a,\pi(a)),(a,a,1) \mid a \in [n]\}$. Observe that if $P \in \mathcal{P}_n$, then each RCS completion of P is isotopic to P_π for some $\pi \in F_n$.

In what follows we show that for each $\pi \in F_n$, P_π is completable when $n \geqslant 8$. This, in conjunction with Lemmas 2.2 and 2.3, completes our proof of Conjecture 1.5. We achieve this using *semi-invariant* permutations.

3.1 Semi-Invariant Permutations

Let $\pi \in F_n$. Define π as *semi-invariant over a set* $I \subseteq [n]$ if $|I| = \lfloor n/2 \rfloor$, $1 \in I$, and π is invariant on I (in other words, fixes I setwise). We say that π is *semi-invariant* if such a set I exists. The *reduction* $\pi(I)$ is the restriction of π to I.

Example 3.1. Let $\pi_1, \pi_2 \in F_8$ and $\pi_3, \pi_4 \in F_9$ be the permutations $\pi_1 = (1)(2345)(678)$, $\pi_2 = (1)(2345678)$, $\pi_3 = (1)(234)(56)(78)(9)$, and $\pi_4 = (1)(2934)(56)(78)$, given in disjoint cycle notation. Observe that π_1 is semi-invariant over $\{1, 6, 7, 8\}$, π_3 is semi-invariant over $\{1, 2, 3, 4\}$, $\{1, 5, 6, 9\}$, and $\{1, 7, 8, 9\}$, and both π_2 and π_4 are not semi-invariant. Reductions of π_1 and π_3 are $\pi_1(\{1, 6, 7, 8\}) = (1)(678)$, $\pi_3(\{1, 2, 3, 4\}) = (1)(234)$, $\pi_3(\{1, 5, 6, 9\}) = (1)(56)(9)$, and $\pi_3(\{1, 7, 8, 9\}) = (1)(78)(9)$.

We use the following observation when a permutation is not semi-invariant.

Observation 3.2. Let $\pi \in F_n$ and suppose π is not semi-invariant. There exists a transposition (ab) so that $(ab)\pi$ is semi-invariant over a set I, with $a \in I$ and $b \notin I$.

Example 3.3. Let $\pi_2 = (1)(2345678)$ and $\pi_4 = (1)(2934)(56)(78)$, as given in Example 3.1. Both π_2 and π_4 are not semi-invariant, but $(25)\pi_2$ is semi-invariant over $\{1, 2, 3, 4\}$ and $(39)\pi_4 = \pi_3$ is semi-invariant over $\{1, 7, 8, 9\}$.

3.2 Standard Forms

Let $\pi \in F_n$ be semi-invariant over a set I. After appropriate row and column permutations, we may assume that P_π has the form given in Figure 2 (a). If π is not semi-invariant, there exists $I \subset [n]$ and $a, b \in [n]$ so that $(ab)\pi$ is semi-invariant as in Observation 3.2. Let $c, d \in [n]$ such that $a = \pi(c)$ and $b = \pi(d)$. After appropriate row and column permutations, we may assume that P_π has the form given in Figure 2 (b). We say that these forms are *standard forms* with respect to I. The rows and columns moved to be the first $\lfloor n/2 \rfloor$ rows and columns in Figure 2 (a) and (b) are those indexed by I. When we assume standard forms, we keep I and $[n]\backslash I$ as the indices.

Example 3.4. Let π_3 and π_4 be as given in Example 3.1. Recall that $\pi_3 = (39)\pi_4$ is semi-invariant over $\{1, 7, 8, 9\}$. Standard forms for π_3 and π_4 over $\{1, 7, 8, 9\}$ are given in Figure 3 (a) and (b).

The next observation shows the relationship between P_π and $P_{(ab)\pi}$ when π is not semi-invariant and $(ab)\pi$ is semi-invariant. Assume that P_π is in standard form (see Figure 2 (b)).

Observation 3.5. Let $\pi \in F_n$ and $a, b, c, d \in [n]\backslash\{1\}$ with $\pi(c) = a$ and $\pi(d) = b$, then

$$P_\pi \cup \{(c, c, 1), (d, d, 1)\}\backslash\{(d, c, 1), (c, d, 1)\} = P_{(ab)\pi}.$$

In what follows, if a standard form is needed for P_π, we will state it.

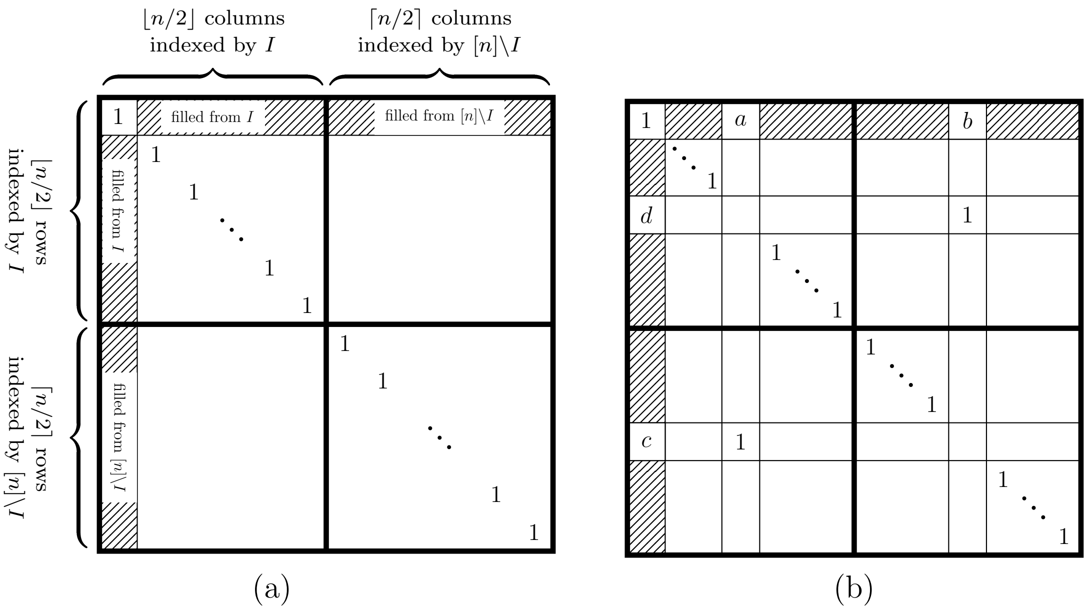


Figure 2: Standard forms of P_π with respect to I if π is (a) semi-invariant or (b) not semi-invariant.

3.3 Adding a Partial Transversal

The following lemma will be used to prove Conjecture 1.5. A *partial transversal of length* $n-1$ is an element of PLS(n) consisting of exactly $n-1$ triples, no two of which agree in a row, column, or symbol coordinate.

Lemma 3.6. *Let $n \geqslant 6$ and $\pi \in F_n$. There is a partial latin square of order n containing P_π and, disjoint from P_π, a partial transversal of length $n-1$.*

Proof. We begin by identifying the cells of the partial transversal to be added to P_π. These cells are $\{(2,3),(3,4),\dots,(n-1,n)(n,2)\}$.

Let G be the bipartite graph with vertex parts $\{r_2,\dots,r_n\}$ and $\{2,\dots,n\}$. We may think of the vertex parts as the rows and symbols of P_π respectively. Edge r_ij is included in G if and only if $j \neq i$ and $j \neq \pi(i+1)$. It follows that the degree of each vertex is at least $(n-1)-2 \geqslant \frac{n}{2}$ since $n \geqslant 6$. Thus, a perfect matching $M = \{(r_2,j_2),(r_3,j_3),\dots,(r_n,j_n)\}$ exists in G by Hall's Marriage Theorem.

The partial latin square $P_\pi \cup \{(k,k+1,j_k) \mid 2 \leqslant k \leqslant n-1\} \cup \{(n,2,j_n)\}$ contains P_π and a partial tranversal of length $n-1$ disjoint from P_π. □

Example 3.7. Let $\pi = (1)(234)(56) \in F_6$. Following the procedure in Lemma 3.6, a partial transversal of length 5 can be added to P_π. See Figure 3 (c).

1	8	7	9	3	4	2	6	5
7	1							
8		1						
9			1					
2				1				
3					1			
4						1		
5							1	
6								1

(a)

1	8	7	9	3	4	2	6	5
7	1							
8		1						
9				1				
2			1					
3					1			
4						1		
5							1	
6								1

(b)

1	3	4	2	6	5
2	1	6			
3		1	4		
4			1	5	
5				1	3
6	2				1

(c)

Figure 3: Standard forms for (a) P_{π_3} and (b) P_{π_4} from Example 3.4.
(c) The addition of a partial transversal in P_{134265} from Example 3.7.

3.4 Main Result

Lemma 3.8. *Let $n \geqslant 8$ and suppose that $\pi \in F_n$ is semi-invariant with a reduction $\overline{\pi}$. If $P_{\overline{\pi}}$ is completable, then P_π is completable.*

Proof. Let $\pi \in F_n$ over $[n]$ be semi-invariant with respect to S_1. Without loss of generality, we assume that P_π is in standard form, and define $S_2 = [n] \backslash S_1$. We begin by partitioning P_π into four subarrays $P_{k\ell}$ $(k, \ell \in \{1,2\})$ defined as $P_{k\ell} = \{(i,j,s) \mid i \in S_k, j \in S_\ell\}$. See Figure 4. Observe that $P_{11} = P_{\overline{\pi}}$, and let Q_{11} be a completion of $P_{\overline{\pi}}$ over S_1.

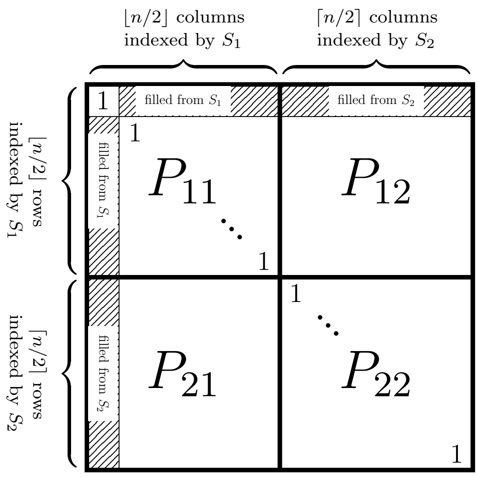


Figure 4: The subarrays of P_π in Lemma 3.8.

First, suppose that n is even. Observe that $|S_1| = |S_2| = n/2$, and so each $P_{k\ell}$ $(k, \ell \in \{1,2\})$ is a partial latin square. Since P_{12}, P_{21}, and P_{22} have only a completed row, column, and symbol, respectively, by Theorem 1.1 there exists completions of each over S_2, S_2, and S_1, respectively. Let Q_{12}, Q_{21}, and Q_{22} denote completions of P_{12}, P_{21}, and P_{22}, respectively. Therefore $Q_{11} \cup Q_{12} \cup Q_{21} \cup Q_{22}$ is a completion of P_π.

Now suppose that n is odd. Then $|S_1| = (n-1)/2$ and $|S_2| = (n+1)/2$. Let ϕ be the permutation of order $(n+3)/2$ induced from π over $S_2 \cup \{1\}$. Let P_ϕ be the partial latin square of order $(n+3)/2$ over $\{1\} \cup S_2$ with triples:

$$\{(i,i,1) \mid i \in \{1\} \cup S_2\} \cup \{(i,1,s) \mid i \in S_2, (i,1,s) \in P_\pi\} \cup \{(1,i,s) \mid i \in S_2, (1,i,s) \in P_\pi\}.$$

From Lemma 3.6, since $(n+3)/2 \geqslant 6$, there exists a partial transversal T on symbol set S_2 of length $(n+1)/2$ which can be added to P_ϕ.

Let m be a new symbol and define the following partial latin squares of order $(n+1)/2$:

$$\begin{aligned} P'_{12} &= P_{12} \cup \{(m,j,s) \mid (i,j,s) \in T\} \subseteq (S_1 \cup \{m\}) \times S_2 \times S_2, \\ P'_{21} &= P_{21} \cup \{(i,m,s) \mid (i,j,s) \in T\} \subseteq S_2 \times (S_1 \cup \{m\}) \times S_2, \text{ and} \\ P'_{22} &= P_{22} \cup \{(i,j,m) \mid (i,j,s) \in T\} \subseteq S_2 \times S_2 \times (S_1 \cup \{m\}). \end{aligned}$$

In other words, P'_{12}, P'_{21}, and P'_{22} are each obtained by adding a new filled row to P_{12}, a new filled column to P_{21}, and by adding $(n+1)/2$ copies of m to the cells of P_{22}. By Theorem 1.1, P'_{12}, P'_{21}, and P'_{22} each have completions over S_2, S_2, and $S_1 \cup \{m\}$ respectively, which we name Q_{12}, Q_{21}, and Q_{22}.

By removing the row and column indexed by m in Q_{12} and Q_{21}, replacing the occurrences of symbol m in Q_{22} with T, and combining these with Q_{11}, we construct a completion of P_π. Formally,

$$Q_{11} \cup \{(i,j,k) \in Q_{12} \mid i \neq m\} \cup \{(i,j,k) \in Q_{21} \mid j \neq m\} \cup \{(i,j,k) \in Q_{22} \mid k \neq m\} \cup T$$

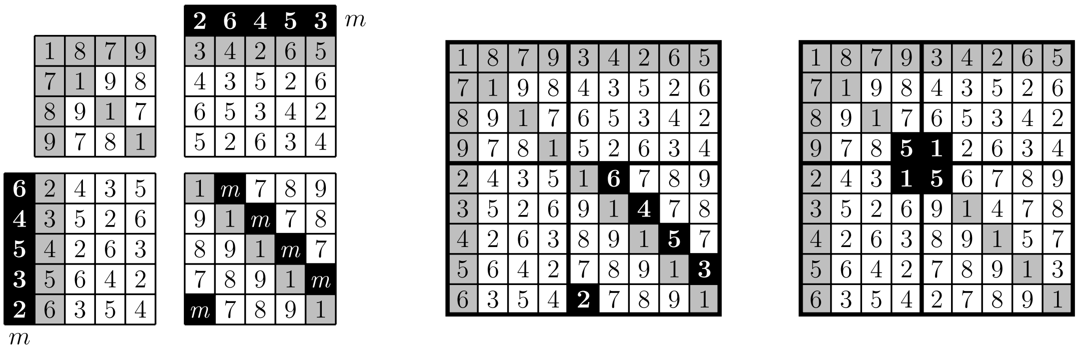
is a completion of P_π. □

Example 3.9. Let $\pi = (1)(78)(9)(234)(56)$. In Figure 5, we illustrate the arrays needed to complete P_π. Observe that π is semi-invariant over $\{1,7,8,9\}$ with reduction $\overline{\pi} = (1)(78)(9)$ and $P_{\overline{\pi}}$ has a completion Q_{11}. Completions of P'_{12}, P'_{21}, and P'_{22}, as outlined in Lemma 3.8, are given in Figure 5 (a). A completion of P_π is given in Figure 5 (b). The ϕ and partial transversal used to complete P_π are the ones in Figure 3 (c).

In what follows, we consider permutations $\pi \in F_n$ which are not semi-invariant. We first find a transposition (ab) so that $(ab)\pi$ is semi-invariant, then find a particular completion of $P_{(ab)\pi}$, and finally perform a slight modification that involves intercalates to produce a completion of P_π.

Definition 3.10. Suppose that $L \in \mathrm{LS}(n)$ and L contains an *intercalate* M – a subset of L of the form $M = \{(i,j,k), (i',j,k'), (i,j',k'), (i',j',k)\}$, for some $i,j,k,i',j',k' \in [n]$ where $i \neq i'$, $j \neq j'$, and $k \neq k'$. The *switch of* M is $\{(i,j,k'), (i',j,k), (i,j',k), (i',j',k')\}$, and the *switch of* L *with respect to* M is obtained by removing M from L and adding the switch of M. Observe that the switch of L with respect to M is an element of $\mathrm{LS}(n)$.

Lemma 3.11. *Let $n \geqslant 8$ and suppose that $\pi \in F_n$ is not semi-invariant, but $(ab)\pi$ is semi-invariant with a reduction $\overline{\pi}$ for some $a, b \in [n]$. If $P_{\overline{\pi}}$ is completable, then P_π is completable.*



(a) (b) (c)

Figure 5: (a) The latin squares $Q_{k\ell}$ ($P'_{k\ell}$ is shaded).
(b) A completion of P_{π_3} from Example 3.9.
(c) A completion of P_{π_4} from Example 3.12.

Proof. Assume that P_π is in standard form with respect to S_1 and let $S_2 = [n]\backslash S_1$, where $a, d \in S_1$, $b, c \in S_2$, and $\pi(d) = b$ and $\pi(c) = a$ (see Figure 2 (b)). From Observation 3.5, $P_\pi = P_{(ab)\pi} \cup \{(d, c, 1), (c, d, 1)\}\backslash\{(d, d, 1), (c, c, 1)\}$. We first complete $P_{(ab)\pi}$ using a completion of $P_{\overline{\pi}}$ (similar to what we did in Lemma 3.8). Define $P_{11} = P_{\overline{\pi}}$, P_{12}, P_{21}, P_{22} as in Lemma 3.8.

Suppose that n is even, and observe that $|S_1| = |S_2| = n/2$. Let Q_{11} be a completion of P_{11} and Q_{22} be a completion of P_{22}, both over S_1. Choose symbol $x \in S_2$ such that $x \neq b$ and $x \neq c$. Such a symbol can be chosen since $|S_2| \geqslant 4$.

Define $P'_{12} = P_{12} \cup \{(d, c, x)\}$ and $P'_{21} = P_{21} \cup \{(c, d, x)\}$. Both P'_{12} and P'_{21} have completions over S_2 by Corollary 1.2; Q_{12} and Q_{21} respectively. Thus $Q = Q_{11} \cup Q_{12} \cup Q_{21} \cup Q_{22}$ is a completion of $P_{(ab)\pi}$.

Observe that Q contains the intercalate

$$M = \{(c, c, 1), (d, d, 1), (c, d, x), (d, c, x)\}. \tag{1}$$

The switch of Q with respect to M is a completion of P_π.

Now suppose that n is odd with $|S_1| = (n - 1)/2$ and $|S_2| = (n + 1)/2$. Let Q_{11} be a completion of P_{11}. Let ϕ be the permutation of order $(n + 3)/2$ induced from $(ab)\pi$ over $S_2 \cup \{1\}$. Let P_ϕ be the partial latin square of order $(n + 3)/2$ over $\{1\} \cup S_2$ with triples:

$$\big\{(i, i, 1) \mid i \in \{1\} \cup S_2\big\} \cup \big\{(i, 1, s) \mid i \in S_2, (i, 1, s) \in P_{(ab)\pi}\big\} \\ \cup\big\{(1, i, s) \mid i \in S_2, (1, i, s) \in P_{(ab)\pi}\big\}.$$

From Lemma 3.6, since $(n + 3)/2 \geqslant 6$, there exists a partial transversal T on symbol set S_2 of length $(n + 1)/2$ which can be added to P_ϕ.

Choose $x \in S_2$ such that $x \neq b$, $x \neq c$, $x \neq T(i,c)$ and $x \neq T(c,j)$ for appropriate $i, j \in S_2$. Such an x can be chosen since $|S_2| \geqslant 5$.

Let m be a new symbol and define the following partial latin squares of order $m = (n+1)/2$.

$$\begin{aligned} P'_{12} &= P_{12} \cup \{(m,j,s) \mid (i,j,s) \in T\} \cup \{(d,c,x)\} \subseteq (S_1 \cup \{m\}) \times S_2 \times S_2, \\ P'_{21} &= P_{21} \cup \{(i,m,s) \mid (i,j,s) \in T\} \cup \{(c,d,x)\} \subseteq S_2 \times (S_1 \cup \{m\}) \times S_2, \text{ and} \\ P'_{22} &= P_{22} \cup \{(i,j,m) \mid (i,j,s) \in T\} \subseteq S_2 \times S_2 \times (S_1 \cup \{m\}). \end{aligned}$$

Each of the partial latin squares P'_{12}, P'_{21}, and P'_{22} have completions over S_2, S_2, and $S_1 \cup \{m\}$ respectively; Q_{12}, Q_{21}, and Q_{22}. The latin square

$$Q_{11} \cup \{(i,j,k) \in Q_{12} \mid i \neq m\} \cup \{(i,j,k) \in Q_{21} \mid j \neq m\} \cup \{(i,j,k) \in Q_{22} \mid k \neq m\} \cup T$$

is a completion of $P_{(ab)\pi}$ containing the intercalate M in (1). Thus, the switch of Q with respect to M is a completion of P_π. □

Example 3.12. Let $\pi_4 = (1)(2934)(56)(78)$ as given in Example 3.1. A completion of $P_{(39)\pi_4}$ can be constructed from Lemma 3.11 with $x = 5$ (see Figure 5 (b)). Switching the intercalate in rows and columns 4 and 5 gives a completion of P_{π_4} (see Figure 5 (c)).

In [2], Casselgren and Häggkvist state that P_π is completable for each $\pi \in F_i$, where $i \in \{6,7,8,9,10\}$. They list all completions when $i \in \{6,7\}$, use their main construction to deduce the conclusion when $i = 8$, and state that they use an exhaustive computer search when $i \in \{9,10\}$. To complete our work, it is only necessary to show that P_π is completable for each $\pi \in F_{11}$. However, our argument can be simply manipulated to prove P_π is completable for each $\pi \in F_i$, where $i \in \{8,9,10\}$ as well. This alleviates the need for a computer search and requires only the methods outlined in this paper, so we give them below as well. For the following arguments, we use the completions given in Figure 6.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

P_{1234}

1	2	4	3
2	1	3	4
3	4	1	2
4	3	2	1

P_{1243}

1	2	3	4	5
2	1	4	5	3
3	5	1	2	4
4	3	5	1	2
5	4	2	3	1

P_{12345}

1	3	2	5	4
2	1	5	4	3
3	4	1	2	5
4	5	3	1	2
5	2	4	3	1

P_{13254}

1	3	4	2	5
2	1	3	5	4
3	5	1	4	2
4	2	5	1	3
5	4	2	3	1

P_{13425}

1	3	4	5	2
2	1	5	4	3
3	5	1	2	4
4	2	3	1	5
5	4	2	3	1

P_{13452}

Figure 6: Completions of selected elements of $\mathcal{P}_4$ and $\mathcal{P}_5$.

Note that the proof of Lemma 3.11 only requires that $(ab)\pi$ be semi-invariant. In the following small cases, it may be that π and $(ab)\pi$ are both semi-invariant, however, any reduction of π may not be completable, whereas a reduction of $(ab)\pi$ is completable. In what follows, we say a *prefix* of a permutation π is a sequence of the form $\pi(1)\pi(2)\cdots\pi(k)$, or in other words, the first k terms of the single-word representation of π for some $k \in [n]$. Before proving our base cases, we make the following observation.

Observation 3.13. Let $\pi, \pi' \in F_n$ for some positive n. If P_π is completable and π and π' are conjugates, then $P_{\pi'}$ is also completable. Hence we only need to show that P_π is completable for a permutation π from each conjugacy class in F_n.

Lemma 3.14. *If $\pi \in F_8 \cup F_9$, then P_π is completable.*

Proof. By Lemmas 3.8 and 3.11, the completions in Figure 6, and Observation 3.13, it is sufficient to show that for each $\pi \in F_8 \cup F_9$, there is a permutation π' which is conjugate to π such that either

(i) the prefix of π' is 1234 or 1243,
(ii) there exist symbols a, b for which $(ab)\pi'$ has a prefix of 1234 or 1243, or
(iii) a completion of π' can be found by brute force.

Let $\pi \in F_i$ where $i \in \{8, 9\}$, and let π_1 be the restriction of π to $\{2, 3, \ldots, i\}$. We consider the following five cases.

(a) Suppose π_1 consists of at least three 1-cycles. Then π has a conjugate whose prefix is 1234, and hence P_π is completable.
(b) Suppose that π_1 consists of at least one 1-cycle and at least one 2-cycle. Then π has a conjugate whose prefix is 1243, and hence P_π is completable.
(c) Suppose that π_1 has no 1-cycle and consists of at least one 2-cycle. Then π has a conjugate π' whose prefix is 1543. Then $(25)\pi'$ has a prefix of 1243, and hence $P_{\pi'}$ and P_π are completable.
(d) Suppose that π_1 consists of no 2-cycles and either one or two 1-cycles. Then π_1 contains a cycle of length at least 3. Then π has a conjugate π' whose prefix is 1245. Then $(35)\pi'$ has a prefix of 1243, and hence $P_{\pi'}$ and P_π are completable.
(e) Suppose that π_1 does not meet any of the above conditions. Then π is a conjugate of (1)(2345678), (1)(234)(6789), (1)(23456789), (1)(2345)(6789), or (1)(234)(56789). These have their completions given in Figure 7.

1	3	4	5	6	7	8	2
2	1	3	4	5	6	7	8
3	4	1	2	7	8	5	6
4	7	8	1	2	3	6	5
5	8	7	6	1	4	2	3
6	5	2	8	4	1	3	7
7	2	6	3	8	5	1	4
8	6	5	7	3	2	4	1

1	3	4	2	6	7	8	5
2	1	3	4	5	6	7	8
3	4	1	7	8	5	6	2
4	2	7	1	3	8	5	6
5	8	6	3	1	2	4	7
6	5	2	8	7	1	3	4
7	6	8	5	2	4	1	3
8	7	5	6	4	3	2	1

1	3	4	5	6	2	8	9	7
2	1	9	3	8	5	4	7	6
3	5	1	2	9	7	6	4	8
4	8	6	1	7	9	2	3	5
5	2	8	7	1	3	9	6	4
6	9	7	8	4	1	5	2	3
7	6	3	9	5	4	1	8	2
8	7	5	4	2	6	3	1	9
9	4	2	6	3	8	7	5	1

1	3	4	5	2	7	8	9	6
2	1	5	9	4	8	6	3	7
3	2	1	4	7	5	9	6	8
4	5	8	1	3	6	2	7	9
5	6	3	2	1	9	7	8	4
6	4	9	7	8	1	3	5	2
7	9	6	8	5	4	1	2	3
8	7	2	6	9	3	4	1	5
9	8	7	3	6	2	5	4	1

1	3	4	5	6	7	8	9	2
2	1	8	3	9	4	6	7	5
3	5	1	2	8	9	4	6	7
4	9	5	1	7	6	2	3	8
5	2	6	7	1	8	9	4	3
6	4	3	8	5	1	7	2	9
7	6	2	9	3	5	1	8	4
8	7	9	4	2	3	5	1	6
9	8	7	6	4	2	3	5	1

Figure 7: Completions of $P_{13456782}$, $P_{13426785}$, $P_{13452897}$, $P_{134527896}$, and $P_{134567892}$.

Hence P_π is completable for every $\pi \in F_8 \cup F_9$. □

Lemma 3.15. *If $\pi \in F_{10} \cup F_{11}$, then P_π is completable.*

Proof. With an argument similar to the proof in Lemma 3.14, we can apply Lemmas 3.8 and 3.11, along with Observation 3.13 the completions in Figure 6. We need only show that in each $\pi \in F_{10} \cup F_{11}$, there is a permutation π' which is conjugate to π for which either

(i) the prefix of π' is 12345, 13254, 13425, or 13452, or
(ii) there exist symbols a, b for which $(ab)\pi'$ has a prefix of 12345, 13254, 13425, or 13452.

Let $\pi \in F_i$ where $i \in \{10, 11\}$, and let π_1 be the restriction of π to $\{2, 3, \ldots, i\}$. We consider the following cases.

(a) Suppose that π_1 has a cycle of length at least 5. Then π has a conjugate π' whose prefix is 13456. Then $(26)\pi$ has the prefix 13452, and hence $P_{\pi'}$ and P_π are completable.
(b) Suppose that π_1 has a cycle of length 4 and none larger. Then π has a conjugate whose prefix is 13452, and hence P_π is completable.
(c) Suppose that π_1 has a cycle of length 3 and none larger. If π_1 also has a 1-cycle, then π has a conjugate whose prefix is 13425, and hence P_π is completable. Otherwise, π has a conjugate π' whose prefix is 13426. Then $(56)\pi'$ has a prefix of 13425, and hence $P_{\pi'}$ and P_π are completable.
(d) Suppose that π_1 has at least two 2-cycles and none larger. Then π has a conjugate whose prefix is 13254, and hence P_π is completable.
(e) If π_1 does not satisfy any of the previous conditions, then π_1 has at least four 1-cycles. So π has a conjugate with a prefix of 12345, and hence P_π is completable.

Therefore P_π is completable for every $\pi \in F_{10} \cup F_{11}$. □

Thus, we have proved the following theorem, confirming Conjecture 1.5.

Theorem 3.16. *Let $P \in \mathcal{P}_n$ where $(r, c, s) \in P$. If $n \notin \{3, 4, 5\}$, then P is completable.*

4 The cases for which $(r, c, s) \notin P$

Let $A \in \mathrm{LS}(n)$. The *back diagonal* of an $n \times n$ array is the set of cells $\{(i, n-i+1) \mid i \in [n]\}$. We build a partial latin square $T(A)$ of order $n + 1$ by setting

- $T(A)(i, j) = n + 1$ for each (i, j) on the back diagonal,
- $T(A)(i, j) = A(i, j)$ for each (i, j) above the back diagonal, and
- cell (i, j) of $T(A)$ is empty for each (i, j) below the back diagonal.

Smetaniuk's proof of Theorem 1.3 [7] uses the following completion result.

Theorem 4.1. *Let $A \in \mathrm{LS}(n)$. Then $T(A) \in \mathrm{PLS}(n + 1)$ is completable.*

We can now prove Theorem 1.6.

Proof of Theorem 1.6. By Lemmas 2.2 and 2.3, we may assume that the first row and column of P are filled and that symbol 1 appears n times. If $P(1,1) = 1$, then the result follows from Theorem 3.16. So assume $P(1,1) \neq 1$.

Without loss of generality, assume that $(1,n,1),(n,1,1) \in P$ and each symbol 1 occurs on the back diagonal of P. Let $P' \in \text{PLS}(n)$ be the array formed from P by removing each occurrence of symbol 1. Define $Q \in \text{PLS}(n-1)$ over $[n]\backslash\{1\}$ such that $Q(i,j) = P'(i,j)$ for all $i,j \in [n-1]$. By Corollary 1.2, Q can be completed to a latin square $L \in \text{LS}(n-1)$ over $[n]\backslash\{1\}$. By Theorem 4.1, $T(L)$ has a completion, and thus P is completable. □

References

[1] L. D. Andersen and A. J. W. Hilton. Thank Evans! *Proc. London Math. Soc. (3)*, 47(3):507–522, 1983.

[2] Carl Johan Casselgren and Roland Häggkvist. Completing partial latin squares with one filled row, column and symbol. *Discrete Math.*, 313(9):1011–1017, 2013.

[3] Charles J. Colbourn. The complexity of completing partial Latin squares. *Discrete Appl. Math.*, 8(1):25–30, 1984.

[4] T. Evans. Embedding incomplete latin squares. *Amer. Math. Monthly*, 67:958–961, 1960.

[5] R. Häggkvist. A solution of the Evans conjecture for Latin squares of large size. *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, 18:495–513, 1978.

[6] Marshall Hall. An existence theorem for Latin squares. *Bull. Amer. Math. Soc.*, 51:387–388, 1945.

[7] Bohdan Smetaniuk. A new construction on Latin squares. I. A proof of the Evans conjecture. *Ars Combin.*, 11:155–172, 1981.