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Steps towards the Well-posedness of the Characteristic Evolution for the Einstein Equations

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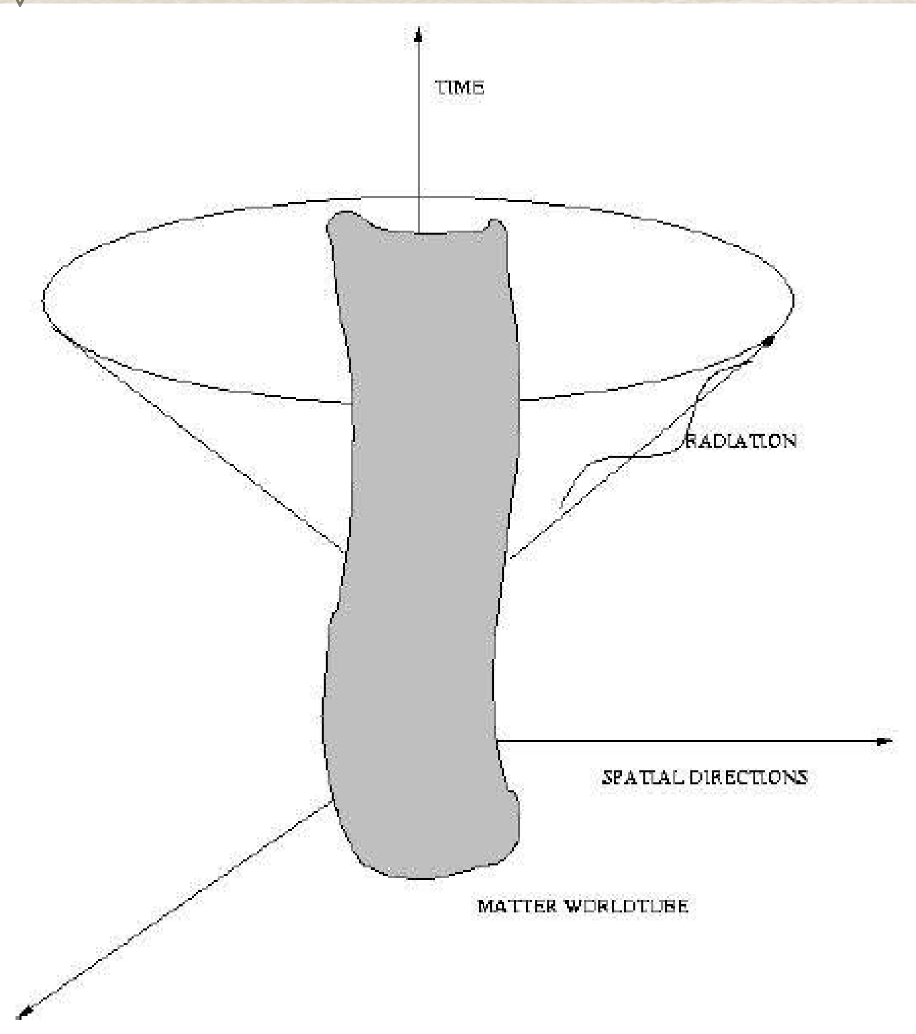
arXiv:1106.4841, Phys.Rev.D84:044057,2011

$Variable$	Re	Im
$\mathcal{E}_2(\Psi)_{R=20}$	1.14×10^{-3}	1.17×10^{-3}
$\mathcal{E}_2(\Psi)_{R=50}$	4.04×10^{-4}	3.53×10^{-4}
$\mathcal{E}_2(\Psi)_{R=100}$	2.81×10^{-4}	2.09×10^{-4}
$\mathcal{E}_2(\delta\psi)_{R=50}$	5.09×10^{-3}	5.08×10^{-3}
$\mathcal{E}_2(\delta\psi)_{R=100}$	6.81×10^{-3}	6.32×10^{-3}
$\mathcal{E}_2(\Psi_{\Delta R(50,100)})$	1.94×10^{-2}	1.91×10^{-2}
$\mathcal{E}_2(\psi_{4,\Delta R(50,100)})$	3.13×10^{-2}	3.14×10^{-2}

* The new Characteristic Extraction Module satisfies the criteria required by Advanced LIGO for detection and measurement.
* Publicly available as part of Einstein Toolkit

Background



- Overall *1.5* accuracy ought to be improved
- Independent error sources
- The only way to ensure both the stability and the accuracy of the simulation is a well-posed algorithm.
- A new characteristic code would be of great value.

Idea

$$g^{ab}\nabla_a \nabla_b \Phi = S,$$

$$\partial_t \partial_x \Phi = S,$$

$$(x = \tilde{t} + \tilde{x},\ t = \tilde{t} - \tilde{x})$$

$$\Phi(x,t) -\!\!> = \Phi(x,t) + g(t)$$

* The well-posedness of the null-timelike problem for the Einstein equations has not yet been established.
* The wave equation in characteristic coordinates allows solution freedom independent of initial data.

Problem

Method

$$\Phi = e^{ax}\Psi \Rightarrow \Psi = e^{-ax}\Phi,\ a > 0$$

H-O Kreiss and J. Winicour, CQG 28, 2011

$$\partial_t(\partial_x\Phi + a\Phi) = \partial_y^2\Phi - 2b\partial_y\Phi + S$$

$$\partial_t(\partial_x\Phi + a\Phi) = (\partial_x^2 + \partial_y^2)\Phi - 2b\partial_y\Phi + S$$

$$\partial_x \to D_{0x},\ \partial_y \to D_{0y},$$

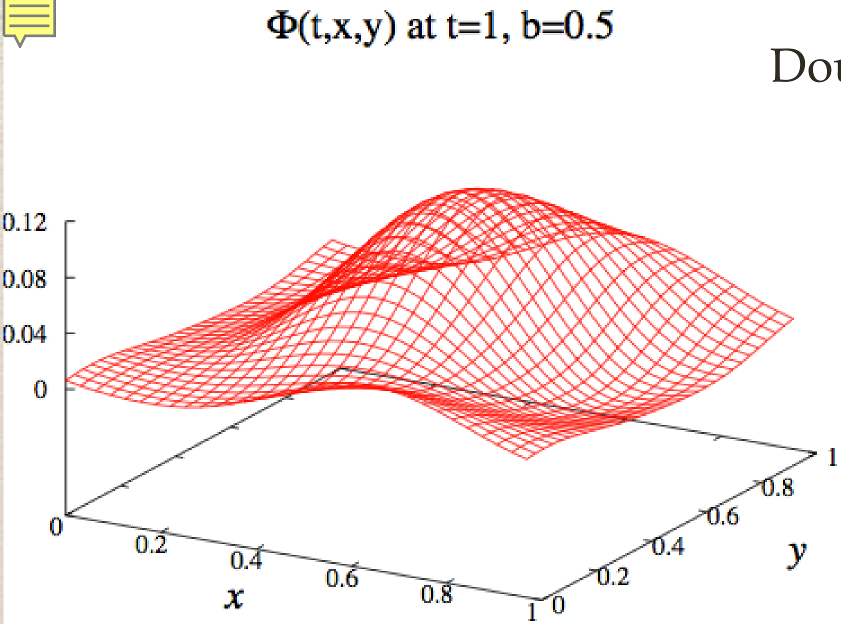
$$\partial_x^2 \to D_{+x}D_{-x},\ \partial_y^2 \to D_{+y}D_{-y}$$

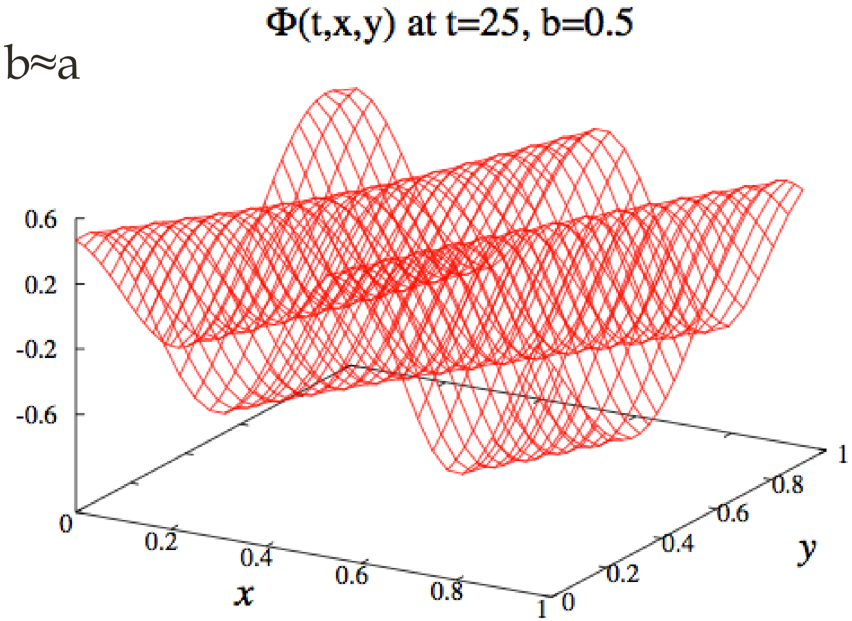
$$\Phi(t, j_1, j_2) = \frac{1}{N}\sum_0^{N-1}\sum_0^{N-1}\hat{\Phi}(t, f_1, f_2)e^{I2\pi h(j_1 f_1 + j_2 f_2)}$$

- Change of variable, introduce the ***a-term***, and bound the growing modes
- Reduce the problem to Cauchy (whole space):
  - Double-null
  - Null-timelike
- Analyze stability against lower-order perturbations for the numeric problem

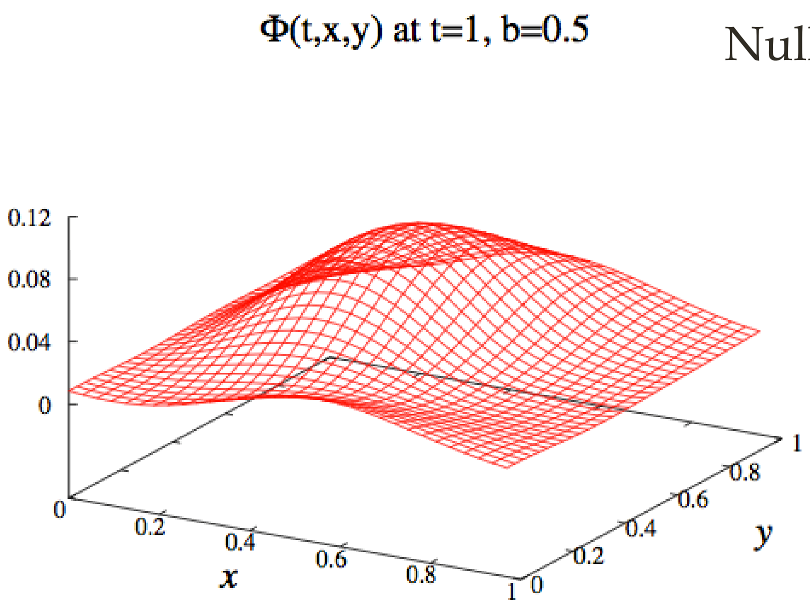
Whole-space

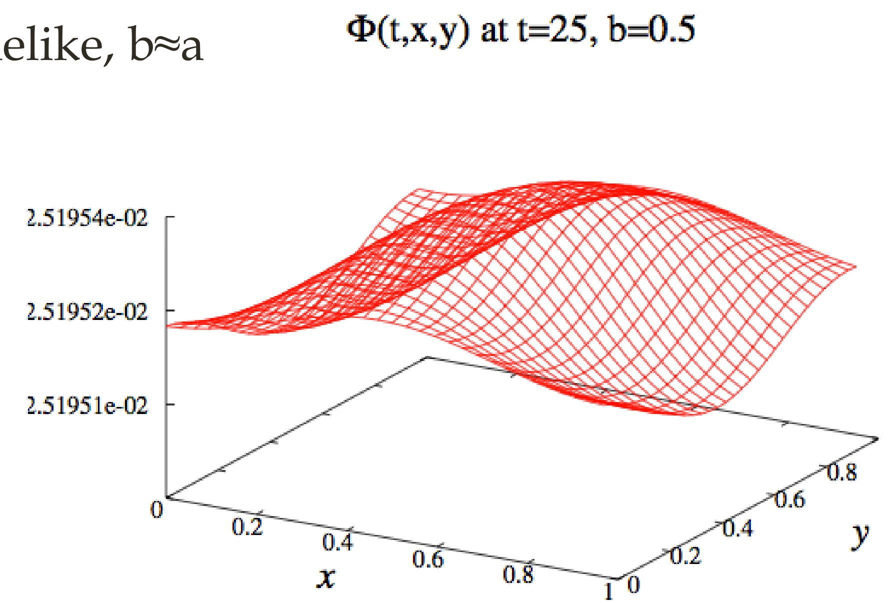
Double-null, b≈a





Null-timelike, b≈a

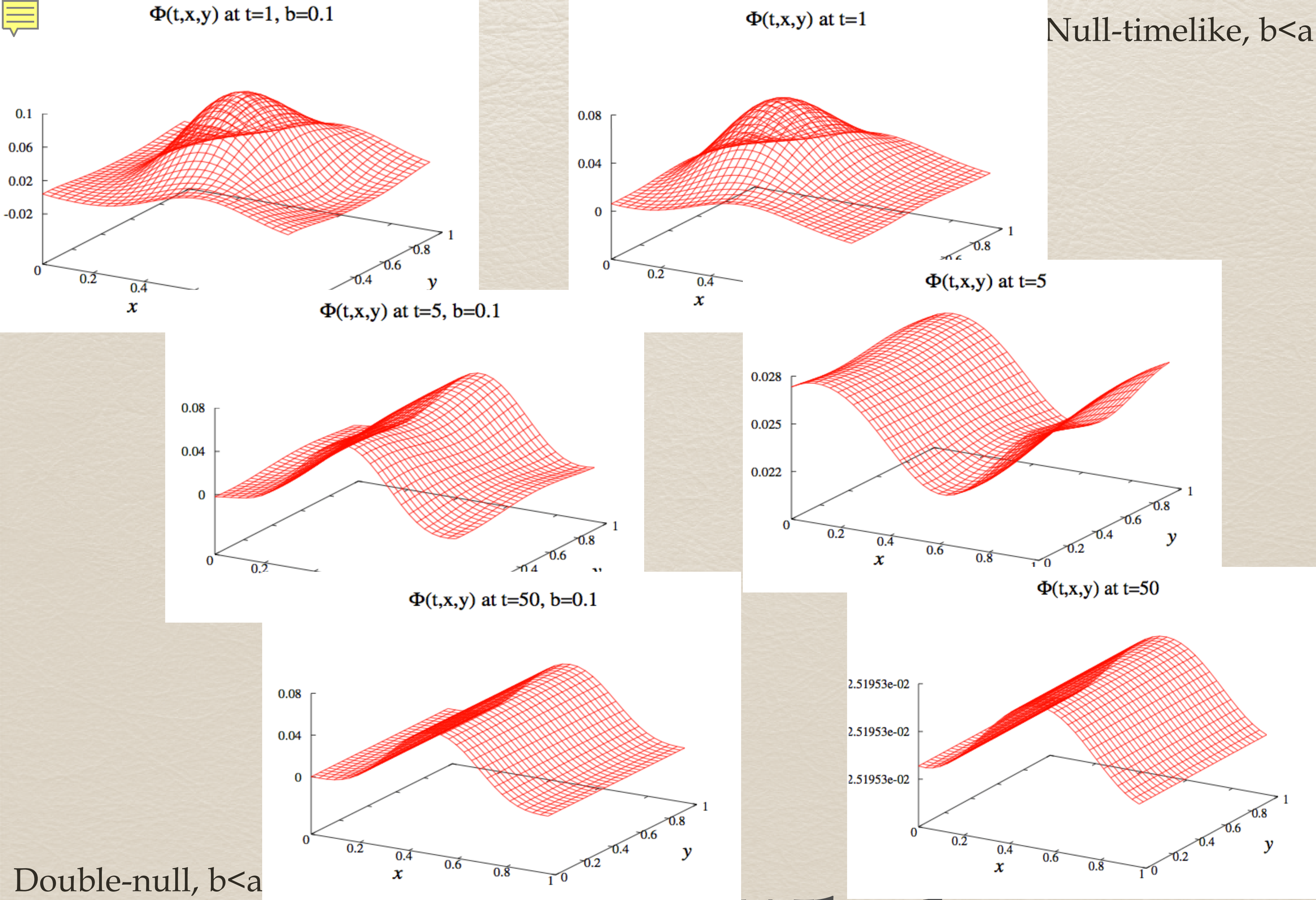


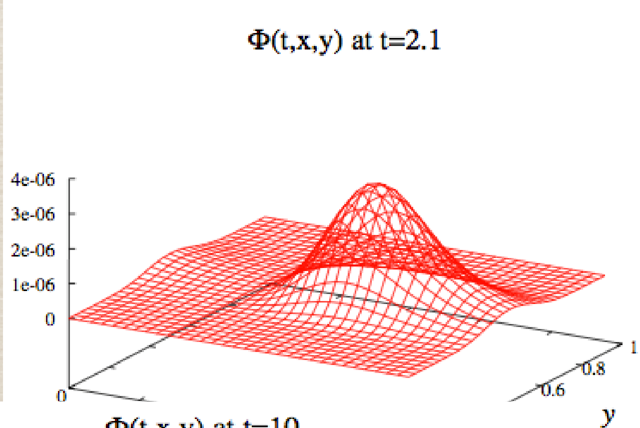


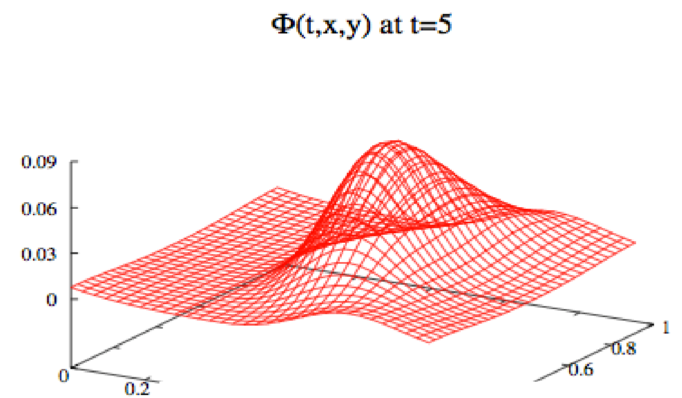
Whole-space

Null-timelike, b<a

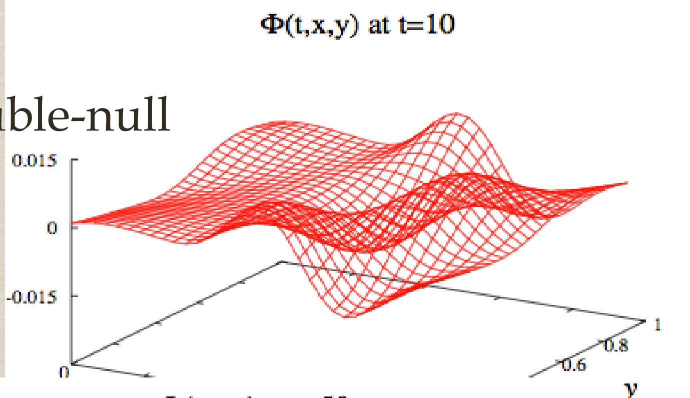
Double-null, b<a



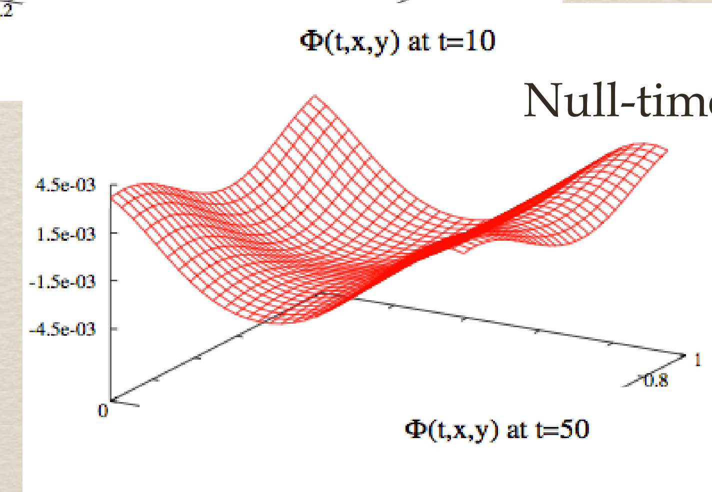


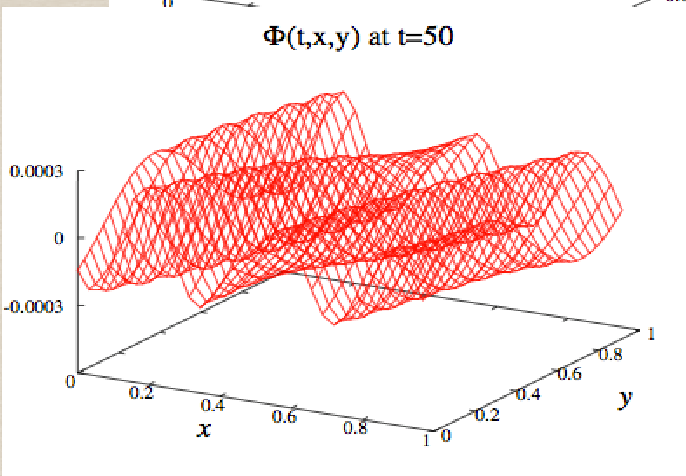


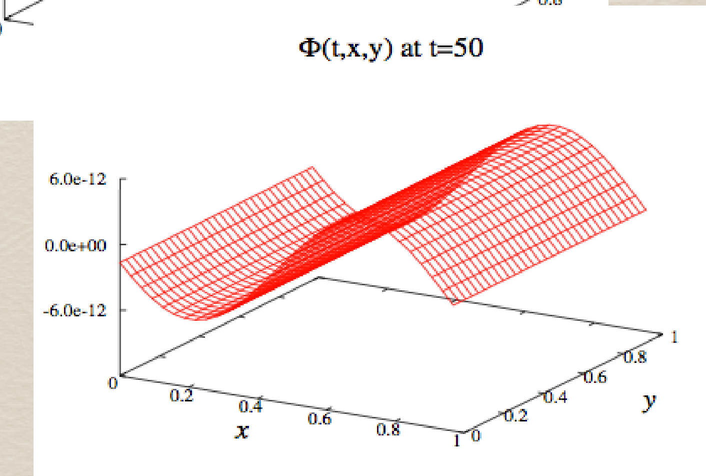
Double-null



Null-timelike







Source

$$ds^2 = -(e^{2\beta}W - r^{-2}h_{AB}W^A W^B)dt^2$$
$$-2e^{2\beta}dtdr - 2h_{AB}W^B dtdx^A + r^2 h_{AB}dx^A dx^B$$

* Bondi-Sachs metric

$$(2\partial_t\partial_r - W\partial_r^2)(r\Phi) = r(\partial_r W)\partial_r\Phi - r^{-1}\partial_r(W^A D_A\Phi)$$
$$+r^{-1}D_A(e^{2\beta}D_A\Phi) - r^{-1}D_A(W^A\partial_r\Phi) + S$$

* Evolution Equation

* Compactification

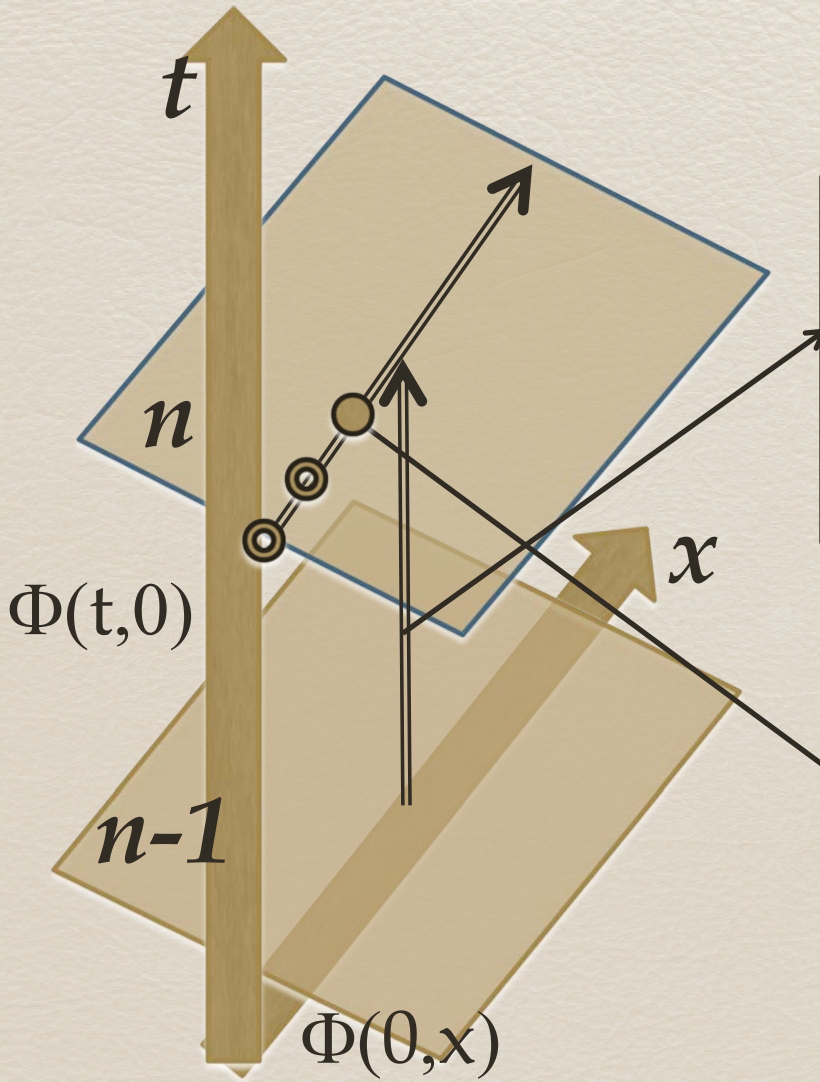
$$r \to x = r/(r+R_E)$$

* Simplified model wave equation, in the null-timelike half-plane case (strip problem)

$$\partial_t(\partial_x\Phi + a\Phi) = \partial_x((1-x)^2\partial_x\Phi)$$
$$+\partial_y^2\Phi - 2b\partial_y\Phi + S$$
$$x = 1 \Rightarrow \partial_t(\partial_x\Phi + a\Phi) = \partial_y^2\Phi - 2b\partial_y\Phi$$
$$x = 0 \Rightarrow \partial_t(\partial_x\Phi + a\Phi) = (\partial_x^2 + \partial_y^2)\Phi$$
$$-2(\partial_x + b\partial_y)\Phi$$

Aim

$$\partial_t(D_{0x}\Phi + a\Phi) = D_{+x}D_{-x}\Phi$$



The only stable 2-levels, marching algorithm

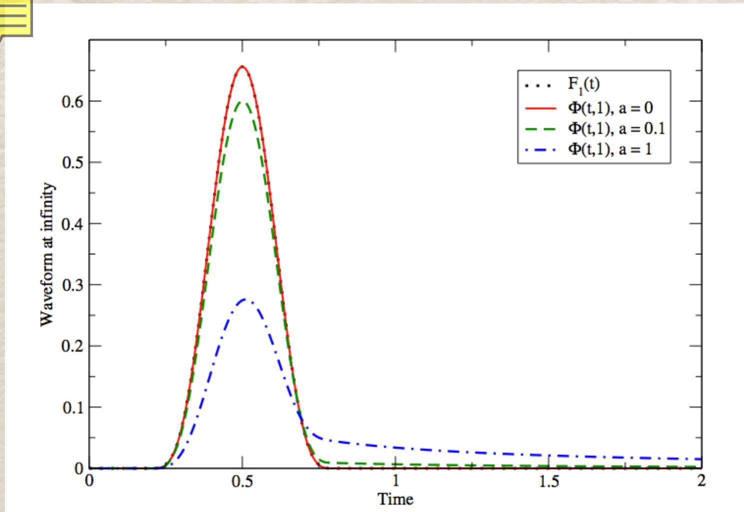
1. Time update

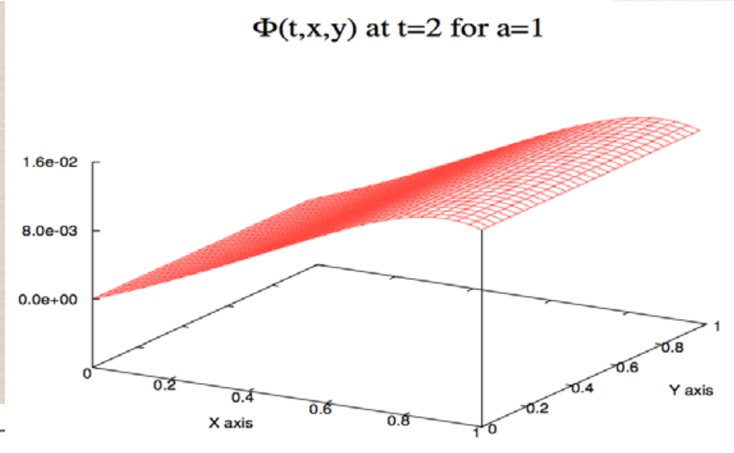
$$\begin{cases} \Upsilon_1^n = f(\Phi_0^{n-1}, \Phi_1^{n-1}, \Phi_2^{n-1}), \\ \Upsilon_{j_1}^n = f(\Phi_{j_1-1}^{n-1}, \Phi_{j_1}^{n-1}, \Phi_{j_1+1}^{n-1}), \\ \Upsilon_N^n = f(\Phi_{N-1}^{n-1}, \Phi_N^{n-1}, 0) \end{cases}$$

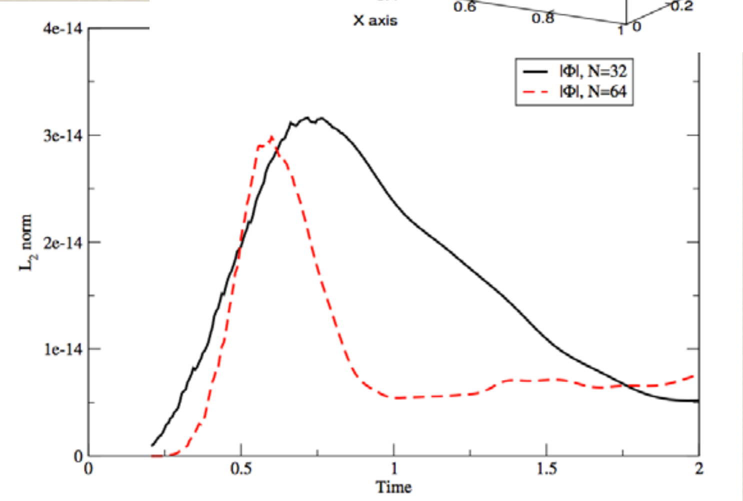
2. Step-up to infinity

$$\begin{cases} \Phi_1^n = f(\Upsilon_1^n, \Phi_0), \\ \Phi_{j_1}^n = f(\Upsilon_{j_1}^n, \Phi_{j_1-1}^n, \Phi_{j_1-2}^n) \end{cases}$$

Algorithm







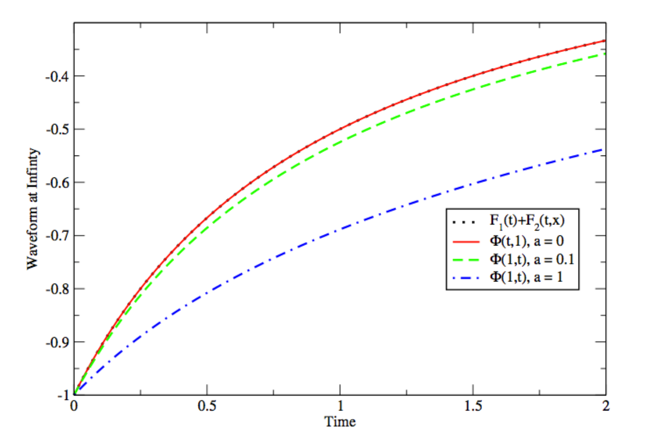
- Purely outgoing wave

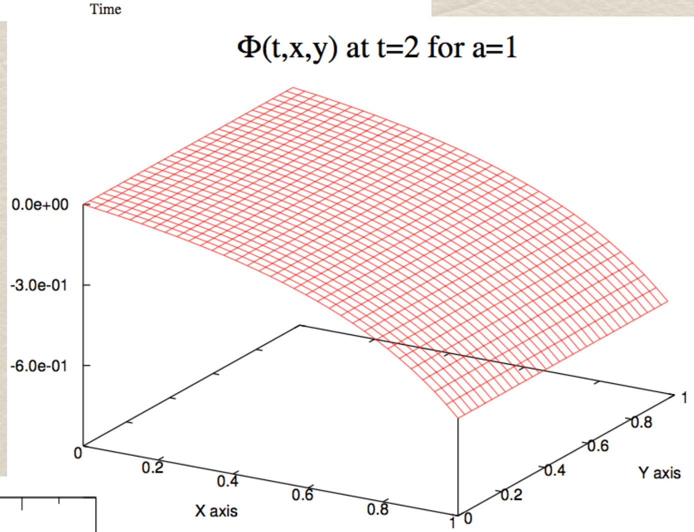
$$\Phi(0,x)=0,\ \Phi(t,0)=F_1(t)$$

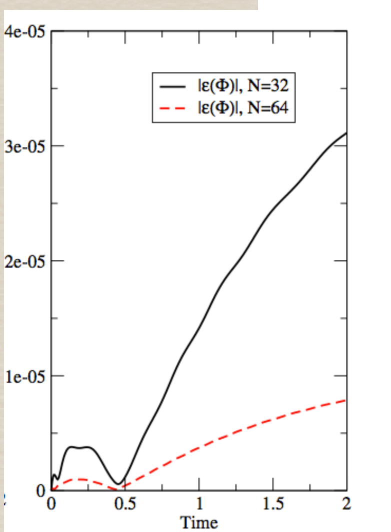
$$F_1(t)=A(t-t_1)^4(t-t_2)^4$$

- Damping due to a-term
- Horizontal "tail" near null infinity boundary
- Time dependence resolved exact, errors ~ machine precision

Boundary







* Purely incoming wave

$$\Phi(0,x) = -x,\ \Phi(t,0) = 0$$

$$F_1(t) = -\frac{1}{1+t},\ F_2(t) = \frac{1-x}{1+(1-x)t}$$

* Damping due to a-term
* Smooth reflection at x=0
* Small 2nd order errors, from discretization on x

Initial Data

The Y-terms

$$D_{+j_2} D_{-j_2} \Phi^{n-1}_{j_1,j_2} - 2bD_{0j_2} \Phi^{n-1}_{j_1,j_2} \Rightarrow \hat{\Phi}^{n-1}_{j_1,j_2} \hat{s}_{j_1,f_2},$$

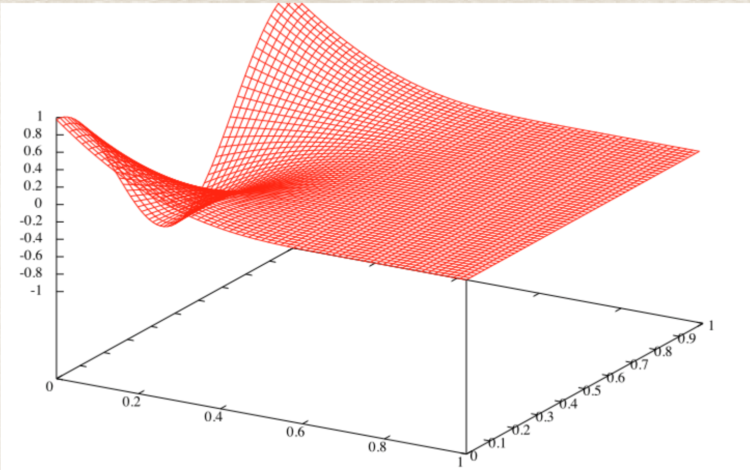
$$\varepsilon (D_{+j_2} D_{-j_2})^2 \Phi^{n-1}_{j_1,j_2} \Rightarrow \hat{\Phi}^{n-1}_{j_1,j_2} \hat{d}_{j_1,f_2},$$

$$D_{+j_2} D_{-j_2} \Phi^{n}_{j_1,j_2} - 2bD_{0j_2} \Phi^{n}_{j_1,j_2} \Rightarrow \hat{\Phi}^{n}_{j_1,j_2} \hat{s}_{j_1,f_2}$$

$$\varepsilon (D_{+j_2} D_{-j_2})^2 \Phi^{n}_{j_1,j_2} \Rightarrow \hat{\Phi}^{n}_{j_1,j_2} \hat{d}_{j_1,f_2}$$

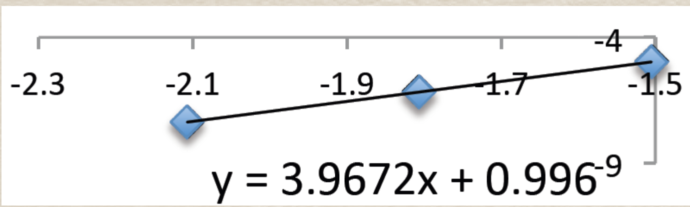
1. Initial data Fourier transformed in y
2. Add y-derivative and y-dissipation
3. Evolve in Fourier space to next level
4. Step in x direction
5. Add y-derivative and y-dissipation

a>0 keeps the problem well posed, for but b≠0 we expect an instability, so dissipation is required

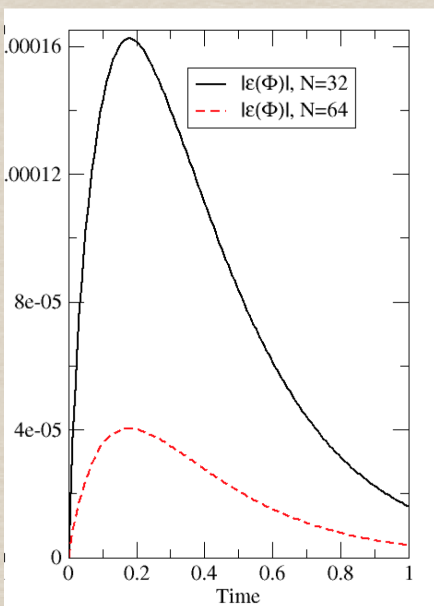


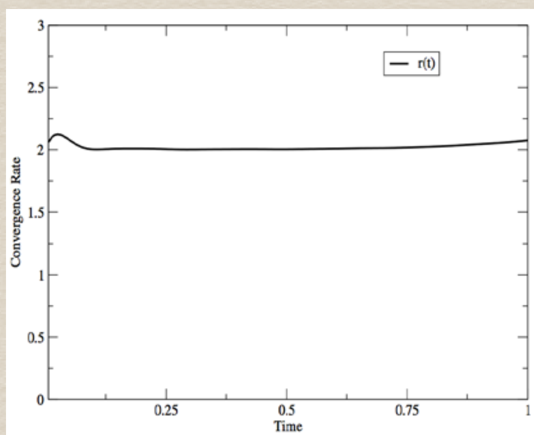
* Exact solution

$$\Phi = e^{st} e^{sx/(1-x)} cos(\omega y),\ s = -\frac{\omega^2}{a}$$



* Local 4^{th}-order convergence
* Global 2^{nd}-order convergence





Convergence

a=-1, b=0

Ill-Posed

* The proof for well-posedness was done analyzing the behavior of continuous and discrete exponential Fourier-Laplace modes as solutions of the wave equations in characteristic coordinates

* Well-posedness is ensured for correct choice of parameters ***(a,b)***

* For the whole space, ***a>0*** renders the runs stable and convergent
    * the double-null system still has exponentially growing modes
    * the null-timelike system has no growing modes for $|b| < a$

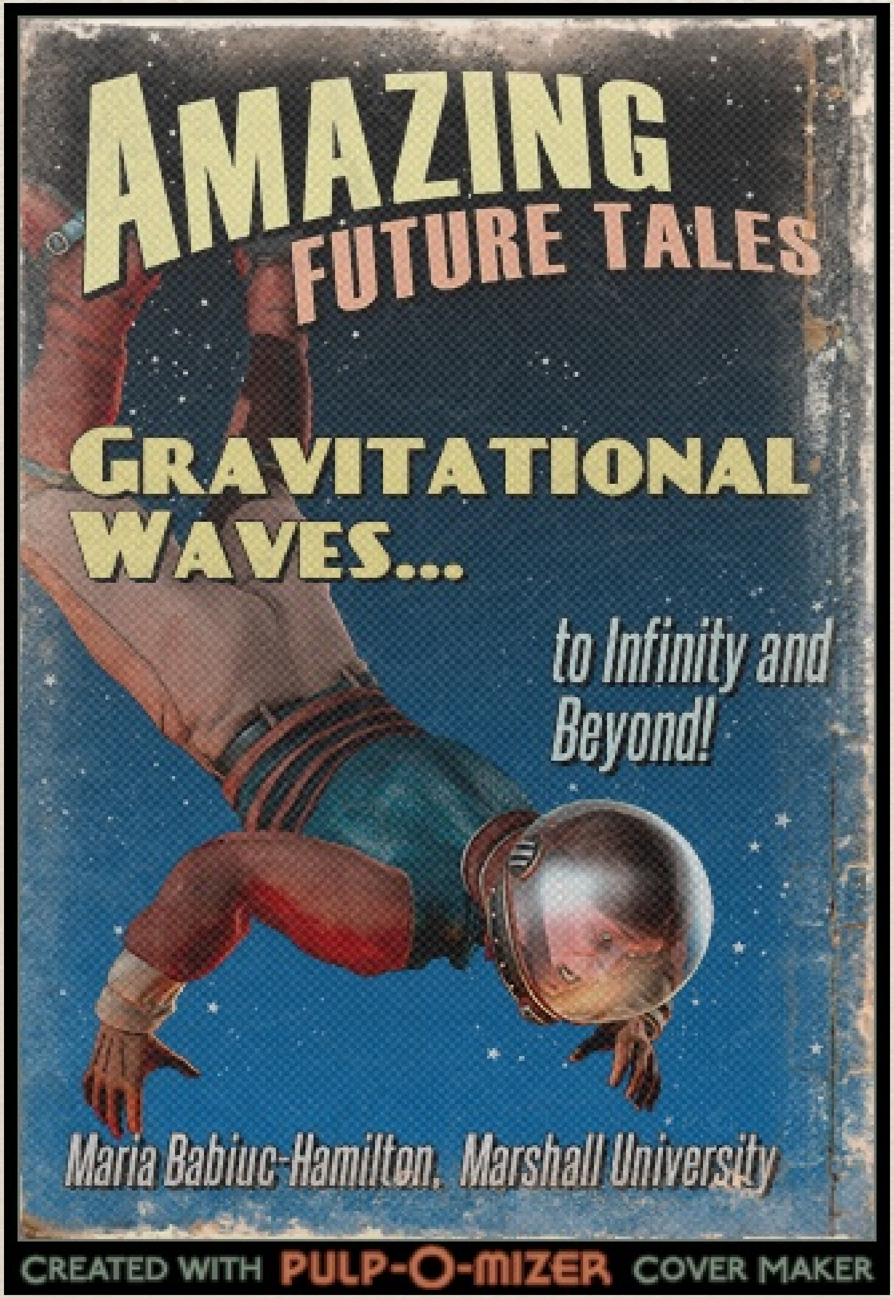
* For the half-plane, due to the term ***(1-x)***, there is a range for ***a***

    * at x=0, weak strong condition for well-posedness for ***a>-2***

    * at x=1, strong strong condition for well-posedness for ***a>0***

* For $b^2 < a(a+2)$ there are no growing modes. There is an inherent instability for $b \neq 0$, and angular dissipation is necessary.

Conclusions



* This proof of well-posedness of the corresponding problem for the quasilinear wave equation is a first step toward treating the fully nonlinear gravitational case.

* Jeff Winicour, Pittsburgh Univ.
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Thank You