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A VARIATIONAL PRINCIPLE FOR DISCONTINUOUS POTENTIALS

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ABSTRACT. Let X be a compact space, $f\colon X \to X$ a continuous map, and $\Lambda \subset X$ be any f-invariant subset. Assume that there exists a nested family of subsets $\{\Lambda_l\}_{l\geq 1}$ that exhaust Λ, that is $\Lambda_l \subset \Lambda_{l+1}$ and $\Lambda = \bigcup_{l\geq 1} \Lambda_l$. Assume that the potential $\varphi\colon X \to \mathbb{R}$ is continuous on the closure of each Λ_l but not necessarily continuous on Λ. We define the topological pressure of φ on Λ. This definition is shown to have a corresponding variational principle. We apply the topological pressure and variational principle to systems with nonzero Lyapunov exponents, countable Markov shifts, and unimodal maps.

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1. Introduction

Let X be a compact metric space, and $f\colon X \to X$ a continuous map. Consider any f-invariant subset $\Lambda \subset X$ possessing a nested family of subsets $\{\Lambda_l\}_{l\geq 1}$ which exhaust Λ, that is $\Lambda_l \subset \Lambda_{l+1}$ for every $l \geq 1$ and $\Lambda = \bigcup_{l\geq 1} \Lambda_l$. The Λ and the Λ_l not are required to be compact; the Λ_l are not required to be f-invariant. Consider a measurable potential $\varphi\colon X \to \mathbb{R}$. We say that φ is *continuous with respect to the family of subsets* $\{\Lambda_l\}$ if φ is continuous on the closure of each Λ_l. The potential function φ is not (necessarily) continuous on Λ. We define the *topological pressure* of φ on Λ, with respect to f, as

$$P_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Lambda_l}(\varphi), \tag{1}$$

where $P_{\Lambda_l}(\varphi)$ is the topological pressure of φ on Λ_l as defined in [10] (see Section 1.1). The topological pressure does depend on the dynamical system f, though the dependence is not reflected in the notation $P_\Lambda(\varphi)$. We show that the topological pressure does not depend on the choice of the family of sets $\{\Lambda_l\}$ (see Section 2 for details).

The classical thermodynamic formalism was developed by Ruelle [11], Sinai [14], and Bowen [4]. The classical topological pressure requires that the potential φ be continuous and the set Λ be compact and f-invariant. (A good exposition is given in Bowen [4].) Pesin extended the classical topological pressure to sets which are not compact or f-invariant using a Carathéodory dimension structure, however, φ is still required to be continuous (see [10], and Section 1.1).

For nonuniformly hyperbolic systems (systems with nonzero Lyapunov exponents) it is natural to consider potentials which are discontinuous. For example, the potential $\varphi(x) = -\log \mathrm{Jac}(df_{|E^u_x})$ is measurable but not continuous for many nonuniformly hyperbolic dynamical systems, such as unimodal maps, Hénon maps, and the time-1 map of geodesic flows on non-positively curved manifolds. Many important classes of nonuniformly hyperbolic systems have a natural family of sets

$\{\Lambda_l\}_{l\geq 1}$, called the regular sets, such that the potential $\varphi(x) = -\log \mathrm{Jac}(df_{|E^u_x})$ is continuous on the closure of each Λ_l, but not on $\Lambda = \bigcup_{l\geq 1}\Lambda_l$. (See [3] for a review of nonuniformly hyperbolic dynamical systems.)

Let $\mathcal{M}_\varphi(\Lambda, f)$ be the set of Borel f-invariant ergodic probability measures on Λ for which φ is integrable. Set $V(x)$ to be the set of weak-$*$ limits of the sequence of measures

$$\mu_{n,x} = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x},$$

where δ_y is the delta measure at the point y. Let

$$\mathcal{L}(\Lambda,\varphi) = \{x \in \Lambda : V(x) \cap \mathcal{M}_\varphi(\Lambda, f) \neq \varnothing\};$$

$\mathcal{L}(\Lambda,\varphi)$ is Borel and f-invariant. The following variational principle holds.

Theorem 1. *Assume that an f-invariant subset $\Lambda \subset X$ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to the family $\{\Lambda_l\}$. Then*

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) = \sup\left\{h_\mu(f) + \int_\Lambda \varphi d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

The proof of Theorem 1 is given in Section 4.

Equilibrium measures are measures which achieve the supremum in the variational principle. Though there are no results in this paper regarding the existence of such measures, in many cases they can be shown to exist.

1.1. **Thermodynamic Formalism for Continuous Potentials.** We give a review of the topological pressure using a Carathéodory dimension structure, see Pesin [10]. The notation in this section will be used in the proof of Theorem 1.

Let (X, ρ) be a compact metric space, $f : X \to X$ a continuous map, and $\varphi : X \to \mathbb{R}$ a continuous function. Fix a finite open cover $\mathcal{U}$ of X. Denote by $S_m(\mathcal{U})$ the set of all strings $\underline{U} = \{U_{i_0} \dots U_{i_{m-1}} : U_{i_k} \in \mathcal{U}\}$ of length $m = m(\underline{U})$. Set $S(\mathcal{U}) = \cup_{m\geq 0} S_m(\mathcal{U})$. To any string $\underline{U} = U_{i_0} \dots U_{i_{m-1}}$ we associate the set

$$X(\underline{U}) = \{x \in X : f^k(x) \in U_{i_k} \text{ for } k = 0, \dots, m-1\}.$$

Consider the following collection of subsets of X

$$\mathcal{F} = \mathcal{F}(\underline{U}) = \{X(\underline{U}) : \underline{U} \in S(\mathcal{U})\}$$

and the following three set functions $\xi, \eta, \psi : S(\mathcal{U}) \to \mathbb{R}$ defined as

$$\xi(\underline{U}) = \exp\left(\sup_{x\in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x))\right),$$

$$\eta(\underline{U}) = \exp(-m(\underline{U})), \quad \psi(\underline{U}) = m(\underline{U})^{-1}.$$

The set S, the collection of subsets $\mathcal{F}$, and the three functions ξ, η, ψ generate a Carathéodory dimension structure $\tau = (S, \mathcal{F}, \xi, \eta, \psi)$ on X. Thus for any set $Z \subset X$ (Z not necessarily compact or f-invariant) and real number α there is a corresponding Carathéodory function

$$m_C(Z, \alpha) = \lim_{N\to\infty} M(Z, \alpha, \varphi, \mathcal{U}, N),$$

where

$$M(Z,\alpha,\varphi,\mathcal{U},N) = \inf_G \left\{ \sum_{\underline{U}\in G} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x)) \right) \right\}$$

with the infimum taken over all finite or countable collections of strings $G \subset S(\mathcal{U})$ such that G covers Z and $m(\underline{U}) \geq N$ for all $\underline{U} \in G$. The structure τ also generates the Carathéodory dimension of Z, which depends on the cover $\mathcal{U}$ and the map f,

$$P_Z(\varphi,\mathcal{U}) = \inf\{\alpha : m_C(Z,\alpha) = 0\} = \sup\{\alpha : m_C(Z,\alpha) = \infty\}.$$

Let $|\mathcal{U}| = \max\{|U_i| : U_i \in \mathcal{U}\}$ be the diameter of the cover $\mathcal{U}$.

The *topological pressure* of the function φ on the set Z, with respect to f, is

$$P_Z(\varphi) = \lim_{|\mathcal{U}|\to 0} P_Z(\varphi,\mathcal{U}).$$

2. Properties

In this section we give several properties of the topological pressure for discontinuous potentials and the corresponding variational principle.

The topological pressure of φ on Λ satisfies many of the properties of the classical topological pressure. In particular, each of the properties in Theorem 3 below are properties of the classical topological pressure.

The topological pressure for discontinuous potentials relies on the family of set $\{\Lambda_l\}$, however, the value of $P_\Lambda(\varphi)$ does not depend on the choice of the family $\{\Lambda_l\}$ in the following sense.

Theorem 2. *Assume that an f-invariant subset $\Lambda \subset X$ has two nested families of subsets $\{\Lambda_l\}$ and $\{\Gamma_l\}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to both $\{\Lambda_l\}$ and $\{\Gamma_l\}$. Then $P_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Lambda_l}(\varphi) = \sup_{l\geq 1} P_{\Gamma_l}(\varphi)$.*

Proof. Set $P'_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Lambda_l}(\varphi)$ and $P''_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Gamma_l}(\varphi)$.

For every $\epsilon > 0$ there exists an n such that $P_{\Lambda_n}(\varphi) \geq P'_\Lambda(\varphi) - \epsilon$. As the Γ_l exhaust Λ, we can write $\Lambda_n = \bigcup_{m\geq 1}(\Lambda_n \cap \Gamma_m)$. As φ is continuous on the closure of Λ_n and each Γ_m,

$$P_{\Lambda_n}(\varphi) = \sup_{m\geq 1} P_{\Lambda_n\cap\Gamma_m}(\varphi) \leq \sup_{m\geq 1} P_{\Gamma_m}(\varphi) = P''_\Lambda(\varphi).$$

The above inequality uses the properties of the topological pressure for continuous potentials, see Theorem 3(c) below. Thus $P''_\Lambda(\varphi) \geq P'_\Lambda(\varphi) - \epsilon$ for every ϵ. Reversing the roles of $P'_\Lambda(\varphi)$ and $P''_\Lambda(\varphi)$ gives the result. □

The theorem below follows from the definition of the topological pressure (as a supremum) and the corresponding results for continuous potentials.

We say that φ is *cohomologous* to ψ if there exists an h continuous on X such that $\varphi - \psi = h - h \circ T$.

Theorem 3. *Assume that an f-invariant subset Λ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to $\{\Lambda_l\}$. Let $Z \subset \Lambda$. The following properties hold:*

(a) $P_\varnothing(\varphi) \leq 0$,
(b) $P_{Z_1}(\varphi) \leq P_{Z_2}(\varphi)$, *if* $Z_1 \subset Z_2 \subset \Lambda$,
(c) $P_Z(\varphi) = \sup_{i\geq 1} P_{Z_i}(\varphi)$, *where* $Z = \bigcup_{i\geq 1} Z_i$,
(d) *If the original map f on X is a homeomorphism then* $P_Z(\varphi) = P_{f(Z)}(\varphi)$,

(e) *If* $h\colon X \to X$ *is a homeomorphism which commutes with* f $(f \circ h = h \circ f)$ *then* $P_Z(\varphi) = P_{h(Z)}(\varphi \circ h^{-1})$,
(f) $|P_Z(\varphi) - P_Z(\psi)| \leq ||\varphi - \psi||$, *where* $||\cdot||$ *denotes the supremum norm,*
(g) $P_Z(\varphi + c) = P_Z(\varphi) + c$,
(h) $P_Z(t\varphi + (1-t)\psi) \leq tP_Z(\varphi) + (1-t)P_Z(\psi)$, *and*
(i) *If* φ *is cohomologous to* ψ, *then* $P_Z(\varphi) = P_Z(\psi)$,

Assume that an f-invariant subset Λ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to $\{\Lambda_l\}$. The function $\varphi_t = t\varphi\colon X \to \mathbb{R}$ is also continuous with respect to $\{\Lambda_l\}$. Define the *pressure function* $\Psi\colon \mathbb{R} \to \mathbb{R}$ associated to φ as

$$\Psi(t) = P_{\mathcal{L}(\Lambda,\varphi_t)}(\varphi_t) = \sup\left\{h_\mu(f) - t\int_\Lambda \varphi d\mu : \mu \in M_{\varphi_t}(\Lambda, f)\right\}.$$

The potential φ_1 is integrable with respect to the measure μ if and only if the potential φ_t is integrable with respect to the measure μ. Thus $\mathcal{M}_{\varphi_t}(\Lambda, f) = \mathcal{M}_{\varphi_1}(\Lambda, f)$, for every $t \in \mathbb{R}$, which implies that $\mathcal{L}(\Lambda, \varphi_t) = \mathcal{L}(\Lambda, \varphi_1)$ for every $t \in \mathbb{R}$. In particular, the pressure function $\Psi(t)$ is well-defined for all t.

Theorem 4. *Assume that* $\Lambda \subset X$ *has a nested family of subsets* $\{\Lambda_l\}$ *which exhaust* Λ. *Let* φ *be continuous with respect to the family* $\{\Lambda_l\}$. *The following properties hold for the pressure function* $\Psi(t)$.

(a) *Monotone Decreasing: If* $t < s$, *then* $\Psi(t) \geq \Psi(s)$*;*
(b) *Subadditive:* $\Psi(s+t) \leq \Psi(s) + \Psi(t)$.

3. Examples

The thermodynamic formalism for discontinuous potentials applies to several interesting examples. In each of the examples considered, the potential has a discontinuity on the closure of the set Λ. The family of subsets $\{\Lambda_k\}$ is chosen such that the closure of each Λ_k is strictly contained in Λ.

3.1. **Systems with Nonzero Lyapunov Exponents.** Let $f\colon \mathcal{R} \to \mathcal{R}$ be a $C^{1+\epsilon}$ diffeomorphism of a compact smooth Riemannian manifold $\mathcal{R}$. Given $x \in \mathcal{R}$ and $v \in T_x\mathcal{R}$, recall that the Lyapunov exponent of v at x is given by

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{\log ||df_x^n v||}{n}.$$

If x is fixed then the function $\lambda(x, \cdot)$ can achieve only finitely many distinct values. $\lambda^{(1)}(x) > \cdots > \lambda^{(q(x))}(x)$. The functions $\lambda^{(i)}(x)$ and $q(x)$ are measurable and f-invariant.

Let μ be an ergodic Borel f-invariant measure on $\mathcal{R}$. Thus the functions $\lambda^{(i)}(x) = \lambda_\mu^{(i)}$ and $q(x) = q$ are constant μ almost everywhere. Assume that μ is hyperbolic, that is there exists k, $1 \leq k < q$ such that

$$\lambda_\mu^{(1)} > \cdots > \lambda_\mu^{(k)} > 0 > \lambda_\mu^{(k+1)} > \cdots > \lambda_\mu^{(q)},$$

and for μ almost every point $x \in \mathcal{R}$ there exist df-invariant stable and unstable subspaces $E^{(s)}(x), E^{(u)}(x) \subset T_x\mathcal{R}$ such that

(a) $E^{(s)}(x) \oplus E^{(u)}(x) = T_x\mathcal{R}$

(b) for any $n \geq 0$

$$||df_x^n v|| \leq C_1(x)\gamma^n ||v|| \text{ if } v \in E^s(x),$$
$$||df_x^{-n} v|| \leq C_1(x)\gamma^n ||v|| \text{ if } v \in E^u(x),$$

where $0 < \gamma < 1$ is a constant and $C_1(x) > 0$ is a measurable function

(c) $\angle(E^s(x), E^u(x)) \geq C_2(x) > 0$, where $C_2(x)$ is a measurable function and $\angle$ denotes the angle between the two subspaces

(d) $C_1(f^n(x)) \leq C_1(x)e^{n\delta}$, $C_2(f^n(x)) \geq C_2(x)e^{-n\delta}$ for any $n \geq 0$, where $\delta > 0$ is fixed, but can be chosen to be arbitrarily small.

Denote by Λ_l, $l \geq 1$ the set of regular points $x \in \mathcal{R}$,

$$\Lambda_l = \{x \in \mathcal{R} : C_1(x) \leq l \text{ and } C_2(x) \geq 1/l\}.$$

These sets satisfy $\Lambda_l \subset \Lambda_{l+1}$, and the set $\Lambda = \bigcup_{l \geq 1} \Lambda_l$ is f-invariant. In addition, Λ coincides with $\mathcal{R}$ up to a set of μ measure zero. (See [3] for a review of systems with nonzero Lyapunov exponents.)

Thus for any potential function φ which is continuous with respect to $\{\Lambda_l\}$, the topological pressure can be defined as (1), and the variational principle, Theorem 1, holds. In particular, the topological pressure can be defined for the family of functions $\varphi_t(x) = -t \log \mathrm{Jac}(df_{|E^u_x})$, each of which is continuous on the closure each Λ_l, but can be unbounded on Λ. We have the following variational principle.

Theorem 5. *Let $f\colon \mathcal{R} \to \mathcal{R}$ be a $C^{1+\epsilon}$ diffeomorphism of a compact Riemannian manifold. Assume that there exists a hyperbolic measure ν on $\mathcal{R}$. Then*

$$P_{\mathcal{L}(\Lambda,\varphi_1)}(\varphi_t) = \sup\left\{h_\mu(f) - t\int_\Lambda \log Jac(df_{|E^u_x})d\mu : \mu \in \mathcal{M}_{\varphi_1}(\Lambda, f)\right\}.$$

3.2. **Countable Markov shifts.** Various authors have studied the thermodynamic formalism of countable Markov shifts. A suitable definition of the pressure has been developed with a corresponding variational principle (see [12, 8]). The existence and uniqueness of equilibrium and Gibbs measures for countable Markov shifts has been widely studied. (In addition to the above, see for example [1, 9, 13, 15, 2].)

Let $S = \mathbb{N}$ and $A = (a_{ij})_{S\times S}$ a matrix of zeros and ones with no columns or rows which are all zeros. Let Y be the set

$$Y = \{x = x_0x_1\cdots \in S^{\mathbb{N}_0} : a_{x_ix_{i+1}} = 1 \text{ for all } i \geq 0\}.$$

Define the shift map $\sigma\colon Y \to Y$ as $(\sigma(x))_i = x_{i+1}$. Assume that (Y, σ) is topologically mixing, that is, there exists an N such that for every $n \geq N$ and every $s_1, s_2 \in S$ there exists an admissible sequence of length n from s_1 to s_2.

Define the variations of ϕ as

$$V_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x, y \in X,\ x_i = y_i,\ 0 \leq i \leq n-1\}.$$

Assume that $\phi\colon Y \to \mathbb{R}$ satisfies

$$\sum_{n=1}^{\infty} V_n(\phi) < \infty, \tag{2}$$

then the *Gurevich pressure* of ϕ is

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \phi(\sigma^k(x))\right) 1_{[s]}(x),$$

where $1_{[s]}$ is the indicator function on the cylinder set $[s]$; the limit exists and is independent of $s \in S$.

If M is a topologically mixing countable Markov shift and ϕ satisfies (2), then

$$P_G(\phi) = \sup\{P_Z(\phi) : Z \subset X \text{ topologically mixing finite Markov shift}\}. \tag{3}$$

Starting with the set $\Lambda_l = \{\omega = \omega_0\omega_1\omega_2 \cdots \in X : \forall i,\, 0 \le a_i < l\}$ one can add a finite number of states and construct a topologically mixing finite Markov shift Z_l, with $\Lambda_l \subset Z_l \subset Y$. These sets Z_l are compact σ-invariant subsets of Y. The family of sets $\{Z_l\}$ has a subfamily Z_{l_k} such that $Z_{l_k} \subset Z_{l_{k+1}}$. The set $\Lambda = \bigcup_{k\ge 1} Z_{l_k}$ is σ-invariant and strictly contained in Y. If ϕ satisfies (2) then ϕ is continuous though (possibly) unbounded on Y; in particular, ϕ is continuous on each Z_{l_k}. Thus the topological pressure of ϕ on Λ can be defined as $P_\Lambda(\phi) = \sup_{k\ge 1} P_{Z_{l_k}}(\phi)$. Theorem 2 and Equation 3 give the following theorem.

Theorem 6. *Assume Y is a topologically mixing countable Markov shift. Let $\phi\colon Y \to \mathbb{R}$ satisfy (2). Then*

$$P_\Lambda(\phi) = P_G(\phi).$$

Theorem 2 and Theorem 6 together imply that the Gurevich pressure can be computed (as in (1)) by taking the supremum over sets which are neither compact nor σ-invariant, for example over the sets Λ_l.

3.3. **Unimodal Maps.** Let $f\colon \mathcal{I} = [0,1] \to \mathcal{I}$ be a unimodal map with critical point c.

Let $\mathcal{C} = \bigcup_{k\ge 0} f^{-k}(c)$, and let $\Lambda_l = ([0, c-1/l] \bigcup [c+1/l, 1]) \bigcap \mathcal{C}$. Then the family $\{\Lambda_l\}$ is nested and exhausts $\Lambda = \mathcal{I} \setminus \mathcal{C}$. Thus the topological pressure of any φ continuous with respect to the family $\{\Lambda_l\}$ can be defined as (1).

The function $\varphi_t(x) = -t \log |f'|$ is bounded, hence continuous, on the closure of each Λ_l. Thus the topological pressure and pressure function can be defined for φ_t for every t.

For a unimodal map f which has the following properties:

(a) f is C^3 on $\mathcal{I} - \{c\}$

(b) there exists an $l > 1$ and a continuous strictly positive $M\colon \mathcal{I} \to \mathbb{R}$ such that $|f'(x)| = M(x)|x-c|^{l-1}$ for every $x \in \mathcal{I}$,

work of Bruin and Keller [6] shows that φ_1 is integrable with respect to every measure $\mu \in \mathcal{M}(\mathcal{I}, f)$. (In [6] Bruin and Keller also require that f have negative Swartzian derivative. Recently it has been shown that this condition can be dropped; see [5, 7]). Thus $\mathcal{M}_{\varphi_1}(\Lambda, f) = \mathcal{M}(\Lambda, f)$, and $\mathcal{L}(\Lambda, \varphi_1) = \mathcal{L}(\Lambda)$. The following variational principle holds.

Theorem 7. *Let f be a unimodal map with critical point c satisfying (a) and (b). Then*

$$P_{\mathcal{L}(\Lambda)}(\varphi_t) = \sup\left\{h_\mu(f) - t\int \log|f'|d\mu : \mu \in \mathcal{M}_{\varphi_1}(\Lambda, f)\right\}.$$

4. Proof of Theorem 1

The notation used in the proof follows that given in Section 1.1 (see also Pesin [10]).

Proposition 1.

$$P_\Lambda(\varphi) \ge \sup\left\{h_\mu(f) + \int_\Lambda \varphi d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Proof. Fix $\mu \in \mathcal{M}_\varphi(\Lambda, f)$. We will show that there exists an l so that

$$P_{\Lambda_l}(\varphi) \geq h_\mu(f) + \int_\Lambda \varphi d\mu. \tag{4}$$

Lemma 1. *For any $\epsilon > 0$ there exits δ, $0 < \delta \leq \epsilon$, a finite Borel partition $\zeta = \{C_1, \dots, C_m\}$ of Λ, and a finite open cover $\mathcal{U} = \{U_1, \dots, U_k\}$, $k \geq m$, of X such that*

(a) *diam $U_i \leq \epsilon$, diam $C_j \leq \epsilon$, $i = 1, \dots, k$, $j = 1, \dots, m$;*
(b) *$\overline{U}_i \subset C_i$, $i = 1, \dots, m$;*
(c) *$\mu(C_i \setminus U_i) \leq \delta$, $i = 1, \dots, m$ and $\mu(\bigcup_{i=m+1}^k U_i) \leq \delta$;*
(d) *$2\delta \log m \leq \epsilon$.*

Given $y \in \Lambda$, let $t_n(y)$ denote the number of those p, $0 \leq p < n$ for which $f^p(y) \in U_i$, for some $i = m+1, \dots, k$. Using Lemma 1 (c) and the Birkhoff ergodic theorem, there exists $N_1 > 0$ and a set $A_1 \subset \Lambda$ such that $\mu(A_1) \geq 1 - \delta$ and for any $y \in A_1$ and $n \geq N_1$,

$$n^{-1} t_n(y) \leq 2\delta. \tag{5}$$

Set $\zeta_n = \zeta \vee f^{-1}\zeta \vee \dots \vee f^{-n+1}\zeta$. As f is a continuous map on a compact metric space we have that $h_\mu(f) < \infty$. Using the Shannon-McMillan-Breiman theorem, there exists $N_2 > 0$ and a set $A_2 \subset \Lambda$ so that $\mu(A_2) \geq 1 - \delta$ and for any $y \in A_2$ and $n \geq N_2$,

$$\mu(C_{\zeta_n}(y)) \leq \exp(-(h_\mu(f, \zeta) - \delta)n), \tag{6}$$

where $C_{\zeta_n}(y)$ denotes the element of the partition ζ_n containing y and $h_\mu(f, \zeta)$ denotes the measure-theoretic entropy of the partition ζ with respect to f.

Using the Birkhoff ergodic theorem, there exists $N_3 > 0$ and a set $A_3 \subset \Lambda$ so that $\mu(A_3) \geq 1 - \delta$ and for any $y \in A_3$ and $n \geq N_3$,

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(y)) - \int_\Lambda \varphi d\mu \right| \leq \delta. \tag{7}$$

Set $N = \max\{N_1, N_2, N_3\}$ and $A = A_1 \bigcap A_2 \bigcap A_3$. Note that $\mu(A) \geq 1 - 3\delta$. Choose any $n \geq N$ and any

$$\lambda < h_\mu(f, \zeta) + \int_\Lambda \varphi d\mu - \epsilon. \tag{8}$$

Since the Λ_i are nested and exhaust Λ we can choose l so that $\mu(\Lambda_l) > 1 - \delta$. We have that

$$\mu(\Lambda_l \bigcap A) > 1 - 4\delta. \tag{9}$$

Choose a cover $G \subset S(\mathcal{U})$ of Λ_l with $m(\underline{U}) \geq N$, for every $\underline{U} \in G$, such that

$$\left| \sum_{\underline{U} \in G} \exp\left(-\lambda m(\underline{U}) + \sup_{x \in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x)) \right) - M(\Lambda_l, \lambda, \varphi, \mathcal{U}, N) \right| \leq \delta. \tag{10}$$

Let $G_p \subset G$ be the subcollection of strings from the cover G for which $m(\underline{U}) = p$ and $X(\underline{U}) \bigcap A \neq \varnothing$. Denote by P_p the cardinality of G_p. Set $Y_p = \bigcup_{\underline{U} \in G_p} X(\underline{U})$.

Lemma 2. *We have $P_p \geq \mu(Y_p \bigcap A) \exp\left((h_\mu(f, \zeta) - \delta - 2\delta \log m)p\right)$.*

Proof. Let L_p be the number of those elements C_{ζ_p} of the partition ζ_p for which

$$C_{\zeta_p} \bigcap Y_p \bigcap A \neq \varnothing. \tag{11}$$

The collection of such partition elements cover the set $Y_p \bigcap A$. Thus we have that

$$\sum \mu(C_{\zeta_p}) \geq \mu(Y_p \bigcap A), \tag{12}$$

where the sum is taken over all elements of the partition ζ_p for which condition (11) holds. Since $C_{\zeta_p} \bigcap A_2 \neq \varnothing$ we have that

$$L_p \geq \mu(Y_p \bigcap A) \exp((h_\mu(f,\zeta) - \delta)p), \tag{13}$$

using inequalities (6) and (12). Fix a string $\underline{U} \in G_p$. Set $S_p(\underline{U})$ to be the number of those elements $C_{\zeta_p} = C_{i_0} \bigcap \cdots \bigcap f^{-(p-1)} C_{i_{p-1}}$ of the partition ζ_p, for which $C_{\zeta_p} \bigcap X(\underline{U}) \bigcap A \neq \varnothing$. Since $X(\underline{U}) \bigcap A_1 \neq \varnothing$ we have that there is only one choice for the partition element C_{i_j} for every j except at most $2\delta p$ times when there are no more than m possibilities. Thus we have the following estimate.

$$S_p(\underline{U}) \leq m^{2\delta p} = \exp(2\delta p \log m). \tag{14}$$

Now, since $P_p S_p(\underline{U}) = L_p$, the inequality follows using (13) and (14). □

We have the following estimate.

$$\begin{aligned}
&\sum_{\underline{U} \in G} \exp\left(-\lambda m(\underline{U}) + \sup_{x \in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x))\right) \\
&\geq \sum_{p=N}^{\infty} \sum_{\underline{U} \in G_p} \exp\left(-\lambda p + \sup_{x \in A \bigcap X(\underline{U})} \sum_{k=0}^{p-1} \varphi(f^k(x))\right) \\
&\geq \sum_{p=N}^{\infty} P_p \exp\left(\left(-\lambda + \int_\Lambda \varphi d\mu - \delta\right) p\right) \\
&\geq \sum_{p=N}^{\infty} \mu(Y_p \bigcap A) \exp\left(\left(h_\mu(f,\zeta) + \int_\Lambda \varphi d\mu - 2\delta - 2\delta \log m - \lambda\right) p\right) \\
&\geq \sum_{p=N}^{\infty} \mu(Y_p \bigcap A) \geq \mu(\Lambda_l \bigcap A) \geq 1 - 4\delta.
\end{aligned}$$

Above we used that for sufficiently small ϵ inequality (8) implies

$$h_\mu(f,\zeta) + \int_\Lambda \varphi d\mu - 2\delta - 2\delta \log m - \lambda > 0.$$

Thus $M(\Lambda_l, \lambda, \varphi, \mathcal{U}, N) \geq 1 - 5\delta \geq 1/2$ for ϵ, and thus δ, small enough. Hence, $P_{\Lambda_l}(\varphi, \mathcal{U}) \geq \lambda$, which implies

$$P_{\Lambda_l}(\varphi) \geq P_{\Lambda_l}(\varphi, \mathcal{U}) \geq h_\mu(f,\zeta) + \int_\Lambda \varphi d\mu - \epsilon.$$

Letting ϵ go to zero gives that $h_\mu(f,\zeta) \to h_\mu(f)$; thus we have (4). □

Corollary 1 (of Proposition 1)**.**

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) \geq \sup\left\{h_\mu(f) + \int_\Lambda \varphi d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Proof. Given a measure $\mu \in \mathcal{M}_\varphi(\Lambda, f)$ denote by $Z_\mu = \{x \in \Lambda : V(x) = \{\mu\}\}$. We have that $\mu(Z_\mu) = 1$ and $Z_\mu \subset \mathcal{L}(\Lambda, \varphi)$. Therefore, by Proposition 1,

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) \geq P_{Z_\mu}(\varphi) \geq h_\mu(f) + \int_\Lambda \varphi d\mu. \qquad \square$$

Proposition 2.

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) \leq \sup \left\{ h_\mu(f) + \int_\Lambda \varphi d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f) \right\}.$$

Proof. As $\mathcal{L}(\Lambda, \varphi) = \bigcup_{l \geq 1} (\mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l)$, we have that

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) = \sup_{l \geq 1} P_{\mathcal{L}(\Lambda,\varphi) \bigcap \Lambda_l}(\varphi).$$

We show that for every $l \geq 1$

$$P_{\mathcal{L}(\Lambda,\varphi) \bigcap \Lambda_l}(\varphi) \leq \sup_{\mu \in \mathcal{M}_\varphi(\Lambda, f)} \left\{ h_\mu(f) + \int_\Lambda \varphi d\mu \right\}.$$

Let E be a finite set and $\underline{a} = (a_0, \ldots, a_{k-1}) \in E^k$. Define the measure $\mu_{\underline{a}}$ on E by

$$\mu_{\underline{a}}(e) = \frac{1}{k}(\text{the number of those } j \text{ for which } a_j = e).$$

Set

$$H(\underline{a}) = -\sum_{e \in E} \mu_{\underline{a}}(e) \log \mu_{\underline{a}}(e).$$

Consider the set

$$R(k, h, E) = \{\underline{a} \in E^k : H(\underline{a}) \leq h\}.$$

The following statement describes the asymptotic growth in k of the number of elements in the set $R(k, h, E)$, see Bowen [4].

Lemma 3. *The following inequality holds.*

$$\limsup_{k \to \infty} \frac{1}{k} \log |R(k, h, E)| \leq h.$$

Let $\mathcal{U} = \{U_1, \ldots U_r\}$ be an open cover of X and $\epsilon > 0$. Set

$$\gamma_l(\mathcal{U}) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in U_i \bigcap \Lambda_l \text{ for some } U_i \in \mathcal{U}\}.$$

Notice that $\gamma_l(\mathcal{U}) \to 0$ as $|\mathcal{U}| \to 0$.

Lemma 4. *Given $x \in \mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l$ and $\mu \in V(x) \bigcap \mathcal{M}_\varphi(\Lambda, f)$, there exists a number $m > 0$ such that for any $n > 0$ one can find $N > n$ and a string $\underline{U} \in S(\mathcal{U})$ with $m(\underline{U}) = N$ satisfying:*

(a) $x \in X(\underline{U})$;

(b)

$$\sup_{x \in X(\underline{U}) \bigcap \Lambda_l} \sum_{k=0}^{N-1} \varphi(f^k(x)) \leq N \left(\gamma_l(\mathcal{U}) + \int_\Lambda \varphi d\mu + \epsilon \right);$$

(c) *the string $\underline{U}$ contains a substring $\underline{U}'$ of length $m(\underline{U}') = km \geq N - m$ which, being written as $\underline{a} = (a_0, \ldots, a_{k-1})$, satisfies the inequality*

$$\frac{1}{m} H(\underline{a}) \leq h_\mu(f) + \epsilon. \qquad (15)$$

Proof. (a) and (c) are found in Pesin [10]. Since μ_{x,n_j} converges to the measure μ and φ is integrable with respect to μ we obtain for sufficiently large N that

$$\left|\sup_{y\in X(\underline{U})\bigcap\Lambda_l}\sum_{k=0}^{N-1}\varphi(f^k(y)) - N\int_\Lambda \varphi d\mu\right| \leq$$

$$\left|\sup_{y\in X(\underline{U})\bigcap\Lambda_l}\sum_{k=0}^{N-1}\varphi(f^k(y)) - \sum_{k=0}^{N-1}\varphi(f^k(x))\right| + \left|\sum_{k=0}^{N-1}\varphi(f^k(x)) - N\int_\Lambda \varphi d\mu\right| \leq$$

$$N\gamma_l(\mathcal{U}) + N\epsilon. \qquad \square$$

Given a number $m > 0$, denote by Y_m the set of points $y \in \mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l$ for which Lemma 4 holds for this m and some measure $\mu \in V(x)\bigcap\mathcal{M}_\varphi(\Lambda, f)$. We have that $\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l = \bigcup_{m>0} Y_m$. Denote also by $Y_{m,u}$ the set of points $y \in Y_m$ for which Lemma 4 holds for some measure μ satisfying $\int_\Lambda \varphi d\mu \in [u-\epsilon, u+\epsilon]$. Set

$$c = \sup_{\mu\in\mathcal{M}_\varphi(\Lambda,f)}\left\{h_\mu(f) + \int_\Lambda \varphi d\mu\right\}.$$

Note that if $x \in Y_{m,u}$ then the corresponding measure μ satisfies $h_\mu(f) \leq c-u+\epsilon$.

Let $G_{m,u}$ be the collection of all strings $\underline{U}$ described in Lemma 4 that correspond to all $x \in Y_{m,u}$ and all N exceeding some number N_0. It follows from (15) that for any $x \in Y_{m,u}$, the substring constructed in Lemma 4 is contained in $R(k, m(h+\epsilon), \mathcal{U}^m)$, where $h = c-u+\epsilon$. Therefore, the total number of the strings constructed in Lemma 4 does not exceed $b(N) = |\mathcal{U}|^m|R(k, m(h+\epsilon), \mathcal{U}^m)|$. By Lemma 3 we obtain that

$$\limsup_{N\to\infty}\frac{\log b(N)}{N} \leq h+\epsilon. \tag{16}$$

Since the collection of strings $G_{m,u}$ covers the set $Y_{m,u}$ we conclude using Lemma 4 and (16) that

$$\begin{aligned} M'(Y_{m,u},\lambda,\varphi,\mathcal{U},N_0) &\leq \sum_{N=N_0}^{\infty} b(N)\exp\left(-\lambda m(\underline{U}) + \sup_{x\in X(\underline{U})\bigcap\Lambda_l}\sum_{k=0}^{m(\underline{U})-1}\varphi(f^k(x))\right) \\ &\leq \sum_{N=N_0}^{\infty} b(N)\exp\left(-\lambda m(\underline{U}) + N\left(\int_\Lambda \varphi d\mu + \gamma_l(\mathcal{U}) + \epsilon\right)\right). \end{aligned}$$

If N_0 is sufficiently large, we have that $b(N) \leq \exp(N(h+2\epsilon))$. Hence,

$$M'(Y_{m,u},\lambda,\varphi,\mathcal{U},N_0) \leq \frac{\beta^{N_0}}{1-\beta}, \tag{17}$$

where

$$\beta = \exp\left(-\lambda + h + \int_\Lambda \varphi d\mu + \gamma_l(\mathcal{U}) + 3\epsilon\right).$$

It follows from (17) that if $\lambda > c + \gamma_l(\mathcal{U}) + 4\epsilon$ then $m_C(Y_{m,u},\lambda) = 0$. Hence, $\lambda \geq P_{Y_{m,u}}(\varphi,\mathcal{U})$. Set $A = |\int_\Lambda \varphi d\mu|$. Assume that points $u_1,\dots,u_r$ form an ϵ-dense set of the interval $[-A, A]$. Then

$$\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l = \bigcup_{m=1}^{\infty}\bigcup_{i=1}^{r} Y_{m,u_i}.$$

We have that $\lambda \geq P_{Y_{m,u_i}}(\varphi,\mathcal{U})$ for any m and i. Therefore,

$$\lambda \geq \sup_{m,i} P_{Y_{m,u_i}}(\varphi,\mathcal{U}) = P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi,\mathcal{U}).$$

This implies that $c+\gamma_l(\mathcal{U})+4\epsilon \geq P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi,\mathcal{U})$. Since ϵ can be chosen arbitrarily small it follows that $c+\gamma_l(\mathcal{U}) \geq P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi,\mathcal{U})$. Taking the limit as $|\mathcal{U}| \to 0$ yields $c \geq P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi)$ and the desired result follows. □

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