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THE THERMODYNAMIC FORMALISM FOR ALMOST-ADDITIVE SEQUENCES

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Abstract. We study the nonadditive thermodynamic formalism for the class of almost-additive sequences of potentials. We define the topological pressure $P_Z(\Phi)$ of an almost-additive sequence Φ, on a set Z. We give conditions which allow us to establish a variational principle for the topological pressure. We state conditions for the existence and uniqueness of equilibrium measures, and for subshifts of finite type the existence and uniqueness of Gibbs measures. Finally, we compare the results for almost-additive sequences to the thermodynamic formalism for the classical (additive) case [10] [11] [3], the sequences studied by Barreira [1], Falconer [5], and that of Feng and Lau [7], [6].

1. Introduction.

In [7] and [6], Feng and Lau developed a thermodynamic formalism for some sequences of functions on subshifts of finite type in the context of multifractal formalism associated to certain iterated function systems with overlaps. These sequences of functions are not additive and fall into the category of sequences introduced in this paper, called almost-additive. We give conditions under which their thermodynamic formalism can be generalized to this class of sequences on an arbitrary compact metric space, with respect to a continuous map f.

Let (X, ρ) be a compact metric space with metric ρ, $f : X \to X$ a continuous measure preserving transformation on X, and $\Phi = \{\varphi_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ a sequence of continuous functions.

Definition 1. We call the sequence Φ almost-additive if there exist functions $c_1, c_2 : \mathbb{N} \to \mathbb{R}$ such that for every $m, n \geq 1$, and $x \in X$

a. $c_1(n) \leq c_2(n)$,
b. there exist constants C_1, C_2 such that $C_1 \leq c_1(n)$ and $c_2(n) \leq C_2$,
c. $$c_1(n+m) + \varphi_n(x) + \varphi_m(f^n(x)) \leq \varphi_{n+m}(x) \leq \varphi_n(x) + \varphi_m(f^n(x)) + c_2(n+m).$$

In section 3, we examine some properties of almost-additive sequences. We note that an almost-additive sequence may not be close to any additive sequence (see Section 2, Examples 2 and 4).

In this paper we study the nonadditive thermodynamic formalism for the class of almost-additive sequences. Several examples of almost-additive sequences are given

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in Section 2, among which is the sequence studied by Feng and Lau. The thermodynamic formalism of the example sequences has been studied (see Ruelle [10], Sinai [11], Bowen [3], Barreira [1], Falconer [5], Feng and Lau [7], [6]). We compare these previous results to our thermodynamic formalism.

1.1. **Topological Pressure and the Variational Principle.**

Let $\mathcal{U}$ be a finite open cover of the compact metric space X. Define an *m-string* $\underline{U}$ to be an ordered collection of m open sets from the cover $\mathcal{U}$,

$$\underline{U} = U_{i_0} U_{i_1} \dots U_{i_{m-1}}.$$

Let $W_m(\mathcal{U})$ be the set of all m-strings. For $\underline{U} \in W_m(\mathcal{U})$ denote $m(\underline{U}) = m$ and define the set

$$X(\underline{U}) = \{x \in X : f^k(x) \in U_{i_k} \text{ for } k = 0, \dots, m-1\}.$$

A set $\Gamma \subset \cup_{m \geq 1} W_m(\mathcal{U})$ is said to be a cover of Z if $Z \subset \bigcup_{\underline{U} \in \Gamma} X(\underline{U})$.

For every $n \geq 1$ define

$$\gamma_n(\Phi, \mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x, y \in X(\underline{U}) \text{ for some } \underline{U} \in W_n(\mathcal{U})\}.$$

Set

$$Z_n(\Phi, Z, \mathcal{U}) = \inf_{\Gamma} \sum_{\underline{U} \in \Gamma} \exp \sup_{x \in X(\underline{U})} \varphi_n(x),$$

where the infimum is taken over all $\Gamma \subset W_n(\mathcal{U})$ covering Z.

Definition 2. The *topological pressure* of an almost-additive sequence Φ, on a compact f-invariant set $Z \subset X$, is given by

$$P_Z(\Phi) = \lim_{\text{diam}(\mathcal{U}) \to 0} \lim_{n \to \infty} \frac{1}{n} Z_n(\Phi, Z, \mathcal{U}).$$

When necessary, we write $P_{f,Z}(\Phi) = P_Z(\Phi)$ for the topological pressure of Φ with respect to the function f.

If the almost-additive sequence of functions Φ satisfies

$$\lim_{\text{diam}(\mathcal{U}) \to 0} \overline{\lim}_{n \to \infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n} = 0\,, \tag{1}$$

then the limits in the definition of topological pressure exist. In Section 4, we show that this is the definition given by Barreira [1].

We study the variational principle in Section 5. Let $M(X, f)$ be the set of f-invariant Borel probability measures on X. We obtain the following general estimate of the topological pressure for any almost-additive sequence of functions.

Theorem 1. *Suppose that Φ is an almost-additive sequence of functions satisfying property (1). Then*

$$P(\Phi) \geq \sup\left\{h_\mu(f) + \lim_{n \to \infty} \frac{1}{n} \int \varphi_n d\mu(x) : \mu \in M(X, f)\right\}.$$

To obtain the inverse inequality, and in particular the variational principle, the sequence of functions must satisfy an additional condition.

Theorem 2. *Suppose that Φ is an almost-additive sequence of functions satisfying property (1), and the functions $c_1(n)$ and $c_2(n)$ satisfy*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} c_1(n-k) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} c_2(n-k). \tag{2}$$

Then

$$P(\Phi) = \sup\left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu(x) : \mu \in M(X,f) \right\}.$$

Condition (2) is a strong requirement on the sequence Φ. It is satisfied by some almost-additive sequences but not by the sequence studied by Feng and Lau in [7] and [6]. For these sequences we provide another condition which guarantees the variational principle. This requirement works well in the case when the system is a subshift of finite type. The extra structure provided by the subshift allows us to develop a thermodynamic formalism with conditions different than (2).

Let (X, σ) be a subshift of finite type and Φ an almost-additive sequence satisfying (1). The *topological pressure* on a compact σ-invariant set $Z \subset X$ is given by the following formula

$$P_Z(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\underline{U}\in\Gamma} \exp \sup_{x\in X(\underline{U})} \varphi_n(x),$$

where Γ is the unique cover of Z by n-cylinders.

Theorem 3. *Suppose that (X, σ) is a mixing subshift of finite type and Φ an almost-additive sequence satisfying (1). Also, suppose that there exists γ such that for every $n \geq 1$, $x \in X(\underline{U})$*

$$\varphi_n(x) \leq \sup_{x\in X(\underline{U})} \varphi_n(x) \leq \gamma + \varphi_n(x). \tag{3}$$

Then

$$P(\Phi) = \sup\left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu(x) : \mu \in M(X,\sigma) \right\}.$$

Remark. After this paper was finished, I became aware of a preprint by Barreira *Nonadditive Thermodynamic Formalism: Equilibrium and Gibbs Measures* [2]. Building on some ideas of his work one can actually drop requirement (2) and prove the following result.

Theorem 4. *Suppose that Φ is an almost-additive sequence of functions satisfying property (1). Then*

$$P(\Phi) = \sup\left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu(x) : \mu \in M(X,f) \right\}.$$

At the end of Section 5, we show how to modify the proof of Theorem 2 to obtain this result.

1.2. **Equilibrium Measures.**

Definition 3. We call a measure $\mu_\Phi \in M(X,f)$ an *equilibrium measure* associated with Φ if

$$h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu_\Phi = \sup\left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu : \mu \in M(X,f) \right\}.$$

We explore the existence and uniqueness of equilibrium measures in Section 6. The existence of equilibrium measures is, in itself, an interesting problem. We obtain the following result on existence of such measures without requiring the topological pressure to exist.

A map f is called *expansive* if there exists an $\epsilon > 0$ so that for any points $x, y \in X$ with $\rho(f^k(x), f^k(y)) < \epsilon$ for all $k \in \mathbb{Z}$ then $x = y$.

Theorem 5. *Suppose that f is an expansive homeomorphism of X. Then for any almost-additive sequence of functions Φ there exists an equilibrium measure μ_Φ on X.*

Corollary 1. *Suppose that Φ is an almost-additive sequence satisfying (1) and (2), and that f is an expansive homeomorphism of X. Then there exists a measure μ_Φ on X such that*

$$P(\Phi) = h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu_\Phi. \tag{4}$$

We note that some authors would define an equilibrium measure to be a measure μ_Φ satisfying (4).

Let C_n be an n-cylinder in a subshift of finite type X.

Definition 4. A probability measure μ on X is a *Gibbs measure* if there exist constants $A_1, A_2 > 0$ such that

$$A_1 \leq \frac{\mu(C_n)}{\exp(-nP(\Phi) + \varphi_n(x))} \leq A_2 \tag{5}$$

for any $n > 0, C_n \subset \Sigma_A$ and $x \in C_n$.

In Section 6.1 we obtain the following results on existence and uniqueness of equilibrium measures for a mixing subshift of finite type (X, σ).

Theorem 6. *Suppose that Φ is an almost-additive sequences of functions on a mixing subshift of finite type (X, σ) satisfying (1) and (3). Then there exists a unique Gibbs measure on X. Moreover, this measure is the unique equilibrium measure for Φ.*

For an additive sequence, if φ_1 is Hölder continuous, then condition (3) holds, and Theorem 6 gives the classical result on the existence of equilibrium measures for subshifts of finite type.

For almost-additive sequences, which are not additive, condition (3) holds for sequences with $\gamma_n(\Phi, \mathcal{U}) \leq b\alpha^n$ for every n, for some positive constants b and $\alpha \in (0, 1)$. We note that this condition is satisfied by sequences for which each φ_n is α-Hölder continuous with common constant C, as required for the sequences studied by Feng and Lau [7], [6].

A diffeomorphism $f : X \to X$ is called *Axiom A* if the set $\Omega(f)$ of non-wandering points is hyperbolic and is the closure of the periodic points. The spectral decomposition theorem gives that $\Omega(f) = \Omega_1 \bigcup \cdots \bigcup \Omega_s$, where the Ω_i are pairwise disjoint closed sets with

a. $f(\Omega_i) = \Omega_i$ and $f_{|\Omega_i}$ is topologically transitive,
b. $\Omega_i = Z_{i,1} \bigcup \cdots \bigcup Z_{i,n_i}$, with the $Z_{i,j}$ pairwise disjoint closed sets with $f(Z_{i,j}) = Z_{i,j+1}$ and $f_{|Z_{i,j}}$ topologically mixing.

The map f is expansive on its hyperbolic set; thus by Theorem 5 there exists at least one equilibrium measure for Φ. We show the following theorem in Section 6.2.

Theorem 7. *Suppose that Ω is a basic set for an Axiom A diffeomorphism f and Φ an almost-additive sequence of functions satisfying conditions (1) and (3). Then there exists a unique equilibrium measure μ_Φ for Φ.*

In the course of proving Theorem 7, we show the variational principle without requiring Φ to satisfy (2).

Corollary 2. *Suppose that (X, f) is an Axiom A diffeomorphism, Φ is an almost-additive sequence of functions satisfying conditions (1) and (3). Then*

$$P(\Phi) = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu : \mu \in M(X, f) \right\}.$$

2. Examples of Almost-Additive Sequences of Functions.

2.1. **Additive Sequences.** Let $\phi : X \to \mathbb{R}$ be a continuous function. The sequence

$$\varphi_n(x) = \sum_{k=0}^{n-1} \phi(f^k(x)).$$

is an additive sequence and thus is almost-additive with $c_1(n)$ and $c_2(n)$ identically equal to zero. Such sequences appear in the classical thermodynamic formalism studied by Ruelle [10], Sinai [11], and Bowen [3].

These sequences trivially satisfy properties (1) and (2). Thus we can apply Definition 2 to obtain the classical topological pressure of ϕ on a compact f-invariant set Z,

$$P_Z(\phi) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} \lim_{n\to\infty} \frac{1}{n} \log \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp \sup \sum_{k=0}^{n-1} \phi \circ f^k(x),$$

where the infimum is taken over all $\Gamma \subset W_n(\mathcal{U})$ covering Z. Theorem 2 gives the classical variational principle

$$P_Z(\phi) = \sup \left\{ h_\mu(f) + \int \phi d\mu : M(Z, f). \right\}$$

If ϕ is Hölder continuous then condition (3) holds, and we recover the existence and uniqueness of Gibbs and equilibrium measures under the same conditions as in the classical case.

2.2. **Nonadditive Sequences.** Consider a sequence satisfying (1) for which there is a continuous function $\psi : X \to \mathbb{R}$ so that

$$\varphi_n - \varphi_{n-1} \circ f \to \psi \tag{6}$$

uniformly on X as $n \to \infty$. Set $\gamma_n = ||\varphi_n - \varphi_{n-1} \circ f - \psi||_\infty$. Then for every $m \geq 1$ and every $x \in X$, we have that

$$-\gamma_m + \psi(x) \leq \varphi_m(x) - \varphi_{m-1}(f(x)) \leq \psi(x) + \gamma_m.$$

Fix two numbers $m, n \geq 1$. One can show the following inequality.

$$\sum_{k=0}^{n-1} \left[\psi\left(f^k(x)\right) - \gamma_{(m+n)-k} \right] + \varphi_m\left(f^n(x)\right) \leq \varphi_{n+m}(x) \leq \\ \sum_{k=0}^{n-1} \left[\psi\left(f^k(x)\right) + \gamma_{(m+n)-k} \right] + \varphi_m(f^n(x)). \tag{7}$$

Without loss of generality set $\varphi_0 \equiv 0$. The above inequality with $m = 0$ implies that

$$\sum_{k=0}^{n-1} \left[\psi(f^k(x)) - \gamma_{n-k} \right] \leq \varphi_n(x) \leq \sum_{k=0}^{n-1} \left[\psi(f^k(x)) + \gamma_{n-k} \right]. \tag{8}$$

Combining (7) and (8) we have that

$$-\left(\sum_{k=0}^{n-1}\gamma_{(m+n)-k}+\sum_{k=0}^{n-1}\gamma_{n-k}\right)+\varphi_n(x)+\varphi_m(f^n(x))\le\varphi_{n+m}(x)\le$$
$$\varphi_n(x)+\varphi_m(f^n(x))+\left(\sum_{k=0}^{n-1}\gamma_{(m+n)-k}+\sum_{k=0}^{n-1}\gamma_{n-k}\right).$$

The above inequality shows that a sequence satisfying (6) is almost-additive if the function

$$c_2(n+m)=\sum_{k=0}^{n-1}\gamma_{(m+n)-1}+\sum_{k=0}^{n-1}\gamma_{n-k}$$

is bounded by a constant C_2 for every $n\geq 1$; since $c_1(n)=-c_2(n)$, for every n, we would also have the bound $-C_2\leq c_1(n)$, for every n. This condition on the functions c_1, c_2 is satisfied if the sequence $\varphi_n-\varphi_{n-1}\circ f$ converges fast enough to the function ψ.

On the other hand, there are sequences which are almost-additive but do not satisfy (6). For example, let $X=S^1, f(x)=x$. The sequence $\Phi=\{\varphi_n(x)=\sin(x+n\alpha)\}$ is almost-additive with $c_1=-2, c_2=2$, but there are α for which

$$\begin{aligned}\varphi_n(x)-\varphi_{n-1}\circ f(x)&=\sin(x+n\alpha)-\sin(x+(n-1)\alpha)\\&=k_1\cos(k_2n+k_3),\end{aligned}$$

which does not converge as $n\to\infty$, where k_i are constants depending on α.

Sequences satisfying (6) were shown by Barreira to have a variational principle corresponding to the nonadditive topological pressure (see Barreira [1], and Section 3 below). An almost-additive sequence satisfying (1) and (2) need not satisfy (6). Thus the set of sequences admitting a variational principle as in Barreira [1] is not disjoint from those as in this paper, but neither is either class contained in the other.

For sequences satisfying (1) and (6), which are also almost-additive satisfying (2), we can apply Theorems 2 to obtain the variational principle. Since these sequences satisfy (6), the variational principle can be expressed as

$$P_Z(\Phi)=\sup\left\{h_\mu(f)+\int\psi d\mu:\mu\in M(Z,f)\right\}.$$

For a compact f-invariant set Z, we note that $\mathcal{L}(Z)=Z$.

Suppose that f is an expansive homeomorphism of X. Theorem 5 implies that for sequences satisfying (6) which are also almost-additive there exists an equilibrium measure. Uniqueness of equilibrium measures follows from Theorem 7 for f which are Axiom A, and almost-additive sequences satisfying conditions (1) and (3).

2.3. **Subadditive Sequences.** Consider $(X,f)=(\Sigma_A,\sigma)$, where Σ_A is a mixing subshift of finite type with transition matrix A. Let Φ be an almost-additive sequence with $c_2(n)\equiv 0$. Assume that the following conditions hold

a. a uniform bound $|(1/n)\varphi_n(x)|\leq M$,
b. a Lipschitz condition $|(1/n)\varphi_n(x)-(1/n)\varphi_n(y)|\leq a|x-y|$, and
c. bounded variation, i.e. there exists a constant b independent of n such that $|\varphi_n(x)-\varphi_n(y)|\leq b$ whenever $x,y\in C_n$, for some n-cylinder C_n.

Conditions (a)-(c) imply that these sequences satisfy conditions (1) and (3). Thus we can define the topological pressure of the sequence Φ on the set Z (see Definition 2). Theorem 3 gives a corresponding variational principle, and Theorem 6 gives the existence and uniqueness of a Gibbs (and equilibrium) measure.

The thermodynamic formalism for these sequences was first introduced by Falconer [5] while studying mixing repellers. They are a particular case of the non-additive thermodynamic formalism of Barreira [1].

2.4. **The Almost-Additive Sequence of Feng and Lau.** Consider $(X, f) = (\Sigma_A, \sigma)$, where Σ_A is a mixing subshift of finite type with transition matrix A. Let M be a Hölder continuous function on Σ_A taking values in the set of all positive $d \times d$ matrices. Define the matrix norm $||M|| = 1^t M 1$. The sequence

$$\varphi_n(x) = \log ||M(x) \dots M(\sigma^{n-1}(x))||$$

is almost-additive with $c_1(n) = c_1 < 0$ and $c_2 = 0$. To show this we follow the argument in Feng and Lau [7]. Since $||\cdot||$ is a norm, we have that $||AB|| \leq ||A|| \cdot ||B||$. Thus

$$\log ||M(x) \dots M(\sigma^{n+m-1}(x))|| \leq \log||M(x) \dots M(\sigma^{n-1}(x))|| + \log ||M(\sigma^n(x)) \dots M(\sigma^{n+m-1}(x))||.$$

For the opposite inequality, Feng and Lau [7] show that there is a constant $C > 0$ such that

$$\frac{\min_{i,j} M_{i,j}(x)}{\max_{i,j} M_{i,j}(x)} > C \quad \text{for all } x \in \Sigma_A,$$

which implies that $M(x) \geq \frac{C}{d}(1 \cdot 1^t)M(x)$. Thus

$$\log ||M(x) \dots M(\sigma^{n+m-1}(x))|| \geq \log||M(x) \dots M(\sigma^{n-1}(x))|| + \log ||M(\sigma^n(x)) \dots M(\sigma^{n+m-1}(x))|| + \log \tfrac{C}{d}.$$

As $C \leq 1$, we have that $c_1 = \log \frac{C}{d} < 0$.

We can apply Definition 2 to obtain the topological pressure

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{C_n \in \Sigma_A} \sup_{x \in C_n} ||M(X)M(\sigma x) \dots M(\sigma^{n-1}(x))||.$$

Applying Theorem 3 gives the variational principle:

$$P(\Phi) = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log ||M(y)M(\sigma y) \dots M(\sigma^{n-1} y)|| d\mu(y) \right\}.$$

The above pressure and variational principle are those given by Feng and Lau [7],[6]. Applying Theorem 6 gives the existence of a unique Gibbs and equilibrium measure, which recovers the results of Feng and Lau.

As mentioned in the introduction, the thermodynamic formalism of the sequence $\varphi_n(x) = \log ||M(x) \dots M(\sigma^{n-1}(x))||$ was studied by Feng and Lau, [7] [6], in the context of multifractal analysis for iterated function systems with overlaps.

3. **Properties of Almost-Additive Sequences of Functions.** We first note that if $c_2(n) = 0$ for every n, then Φ is a sub-additive sequence. If in addition, $c_1(n) = 0$ for all n, then the almost-additive sequence of functions is additive.

Lemma 1. *Let Φ be an almost-additive sequence of functions. Then the limit $\lim_{n\to\infty} \varphi_n/n$ exists almost everywhere; it is possibly $-\infty$ or ∞.*

Proof. The sequence $(\varphi_n + C_2)/n$ is sub-additive since for every m, n we have

$$\varphi_{m+n}(x) + C_2 \leq (\varphi_m(x) + C_2) + (\varphi_n(f^m(x)) + C_2).$$

By the subadditive ergodic theorem (see [12]) $\lim_{n\to\infty}(\varphi_n + C_2)/n$ exists almost everywhere. As we have

$$\lim_{n\to\infty} \frac{\varphi_n}{n} = \lim_{n\to\infty} \frac{\varphi_n + c_2(n)}{n} \leq \lim_{n\to\infty} \frac{\varphi_n + C_2}{n} = \lim_{n\to\infty} \frac{\varphi_n}{n},$$

the result is shown. □

The following two lemmas are easily shown from Definition 1 (c).

Lemma 2. *Let Φ be an almost-additive sequence of functions. Then $\int \varphi_n d\mu$ is an almost-additive sequence of numbers, i.e. for $a_n = \int \varphi_n d\mu$ we have that*

$$c_1(n+m) + a_n + a_m \leq a_{n+m} \leq a_n + a_m + c_2(n+m)$$

Lemma 3. *Let Φ be an almost-additive sequence of functions. Then for a fixed $m \geq 0$, the sequence*

$$S_m\Phi = \left\{ \phi_n^m = \sum_{k=0}^{n-1} \varphi_m \circ f^k \right\}_{i=0}^{\infty}$$

is additive.

4. **Topological Pressure.** We begin with the definition of the nonadditive topological pressure, see [1] or [9].

For every $Z \subset X$ define

$$M(Z, \alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(\underline{U})} \varphi_n(x)\right), \tag{9}$$

where the infimum is taken over all $\Gamma \subset \cup_{k\geq n} W_k(\mathcal{U})$ that cover Z. Similarly, define

$$\underline{M}(Z, \alpha, \Phi, \mathcal{U}) = \underline{\lim}_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(\underline{U})} \varphi_n(x)\right), \tag{10}$$

$$\overline{M}(Z, \alpha, \Phi, \mathcal{U}) = \overline{\lim}_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(\underline{U})} \varphi_n(x)\right), \tag{11}$$

where the infimum are each taken over all $\Gamma \subset W_n(\mathcal{U})$ that cover Z. In equations (9), (10), and (11), if $X(\underline{U}) = \emptyset$, then set $\sup_{x\in X(\underline{U})} \varphi_n(x) = -\infty$.

The general theory of Carathéodory dimension characteristics gives that each of the three equations above jumps from $+\infty$ to 0 at a unique critical value; it is possibly $-\infty$ or $+\infty$. Define the critical values as

$$P_Z(\Phi, \mathcal{U}) = \inf\{\alpha : M(Z, \alpha, \Phi, \mathcal{U}) = 0\} = \sup\{\alpha : M(Z, \alpha, \Phi, \mathcal{U}) = +\infty\},$$
$$\underline{CP}_Z(\Phi, \mathcal{U}) = \inf\{\alpha : \underline{M}(Z, \alpha, \Phi, \mathcal{U}) = 0\} = \sup\{\alpha : \underline{M}(Z, \alpha, \Phi, \mathcal{U}) = +\infty\},$$
$$\overline{CP}_Z(\Phi, \mathcal{U}) = \inf\{\alpha : \overline{M}(Z, \alpha, \Phi, \mathcal{U}) = 0\} = \sup\{\alpha : \overline{M}(Z, \alpha, \Phi, \mathcal{U}) = +\infty\}.$$

Theorem 8 (Barreira [1]). *Let Φ be a sequence of functions satisfying (1). The following limits exists*

$$\begin{aligned} P_Z(\Phi) &= \lim_{diam(\mathcal{U})\to 0} P_Z(\Phi,\mathcal{U}),\\ \underline{CP}_Z(\Phi) &= \lim_{diam(\mathcal{U})\to 0} \underline{CP}_Z(\Phi,\mathcal{U}),\\ \overline{CP}_Z(\Phi) &= \lim_{diam(\mathcal{U})\to 0} \overline{CP}_Z(\Phi,\mathcal{U}). \end{aligned}$$

Definition 5. The *topological pressure*, on any set $Z \subset X$, of a sequence Φ satisfying (1) is given by $P_Z(\Phi)$. The values $\underline{CP}_Z(\Phi)$ and $\overline{CP}_Z(\Phi)$ are the *lower* and *upper capacity pressure* of Φ on Z, respectively.

The following theorem shows that Definition 2 is the topological pressure for an almost-additive sequence satisfying (1) on a compact, f-invariant set.

Theorem 9. *Let Φ be an almost-additive sequence of functions satisfying (1) and let $Z \subset X$ be f-invariant.*

a. $CP_Z(\Phi) = \underline{CP}_Z(\Phi) = \overline{CP}_Z(\Phi) =$

$$\lim_{diam(\mathcal{U})\to 0} \lim_{n\to\infty} \frac{1}{n} \log Z_n(\Phi, Z, \mathcal{U}).$$

b. *If in addition Z is compact, then $P_Z(\Phi) = CP_Z(\Phi)$.*

Proof. (a) Given sets $\Gamma_m \subset W_m(\mathcal{U})$ and $\Gamma_n \subset W_n(\mathcal{U})$, define the set

$$\Gamma_{m,n} = \{\underline{UV} : \underline{U} \in \Gamma_m, \underline{V} \in \Gamma_n\} \subset W_{m+n}(\mathcal{U}).$$

Since Z is f-invariant, if the two collections Γ_m and Γ_n cover Z, then their concatenation $\Gamma_{m,n}$ also covers Z. As Φ is almost-additive we have that

$$\sup_{x\in X(\underline{UV})} \varphi_{m+n}(x) \le \sup_{x\in X(\underline{U})} \varphi_m(x) + \sup_{x\in X(\underline{V})} \varphi_n(x) + C_2,$$

for every $\underline{UV} \in \Gamma_{m,n}$. Thus we have that

$$Z_{m+n}(\Phi,\mathcal{U}) \le (\exp C_2) Z_m(\Phi,\mathcal{U}) Z_n(\Phi,\mathcal{U}).$$

As $\mathcal{U}$ is a finite cover we have that

$$\inf_{n\ge 1} \frac{\log Z_n(\Phi,\mathcal{U})}{n} > -\infty$$

Almost-additivity implies that the limit as n goes to infinity of $\log Z_n(\Phi,\mathcal{U})/n$ exists and is finite.

Thus, as a result of Theorem 2.2 in [9] and Theorem 8, we have that

$$\underline{CP}_Z(\Phi) = \overline{CP}_Z(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\Phi, Z, \mathcal{U}).$$

(b) For an almost-additive sequence of functions, we have that $\varphi_n < \varphi_{n+1} + K$, where $K = \sup_{x\in X} |\varphi_1(x)| + \max\{|C_1|, |C_2|\} + 1$. Let $\Gamma \subset \cup_{n\ge 1} W_n(\mathcal{U})$ be a cover of Z. As Z is compact, we can assume that Γ is finite. Thus there exists an M such that $\Gamma \subset \cup_{n\le M} W_n(\mathcal{U})$.

Set $\Gamma^n = \{\underline{U}_1 \dots \underline{U}_n : \underline{U}_i \in \Gamma\}$ for each $n \ge 1$. As Z is f-invariant and Γ covers Z, for all n the collection Γ^n covers Z. Since Φ is almost-additive,

$$\sup_{x\in X(\underline{U}_1\cdots\underline{U}_n)} \varphi_{m(\underline{U}_1)+\cdots+m(\underline{U}_n)}(x) \le \sum_{i=1}^{n} \sup_{x\in X(\underline{U}_i)} \varphi_{m(\underline{U}_i)}(x) + (n-1)C_2.$$

Set

$$N(\Gamma) = \sum_{U \in \Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x \in X(U)} \varphi_{m(U)}(x)\right).$$

We have that $N(\Gamma^n) \le [(\exp C_2) N(\Gamma)]^n$.

If $\alpha = P_Z(\Phi, \mathcal{U})$ then there exists an $m \ge 1$ and a cover $\Gamma \subset \cup_{n \ge m} W_n(\mathcal{U})$ of Z such that $(\exp C_2) N(\Gamma) < 1$. Set $\Gamma^\infty = \{\underline{U} : \underline{U} \in \Gamma^n \text{ for some } n\}$. Then

$$N(\Gamma^\infty) = \sum_{n=1}^{\infty} N(\Gamma^n) < \infty.$$

As Γ covers Z, for any $N \ge 1$ and $x \in Z$ there exists $\underline{U} \in \Gamma^\infty$ such that $x \in X(\underline{U})$ and $N \le m(\underline{U}) \le N + M$. Let $\Gamma^* \subset W_N(\mathcal{U})$ be the collection of strings $\underline{U}^*$ that consist of the first N elements of some string $\underline{U} \in \Gamma^\infty$. We have that

$$\sup \varphi_{m(\underline{U}^*)} \le \sup \varphi_{m(\underline{U})} + KM$$

and

$$N(\Gamma^*) \le N(\Gamma^\infty) \max\{1, \exp(\alpha M)\} \exp(KM).$$

Thus $\overline{M}(Z, \alpha, \Phi, \mathcal{U}) < \infty$, which implies that $\alpha > \overline{CP}_Z(\Phi, \mathcal{U})$. Thus $P_Z(\Phi, \mathcal{U}) \ge \overline{CP}_Z(\Phi, \mathcal{U})$. Then using part (a) above, the result is shown. □

Let $\sigma : \Sigma_A \to \Sigma_A$ be the shift map on one-sided sequences of p symbols, with adjacency matrix A. Write $\mathcal{U}_n$ for the open cover of Σ_p formed by the n-cylinder sets. We note that $\operatorname{diam}(\mathcal{U}_n)$ goes to zero as n goes to infinity.

Consider any set $Z \subset \Sigma_p$. There is a unique $\Gamma \subset W_n(\mathcal{U}_l)$ covering Z. For each $\underline{U} \in W_n(\mathcal{U}_l)$ the set $X(\underline{U})$ is an $(n+l-1)$-cylinder, and $X(\underline{U}) \cap X(\underline{V}) = \emptyset$ if $\underline{U} \ne \underline{V}$. Thus for each $l \ge 1$ we have that

$$p^{l+1} \sum_{\underline{U} \in \Gamma_{l+n-1}} \exp \sup_{x \in X(\underline{U})} \varphi_n(x) \le \sum_{\underline{V} \in \Gamma_n} \exp \sup_{x \in X(\underline{V})} \varphi_n(x) \le$$
$$\sum_{\underline{U} \in \Gamma_{l+n-1}} \exp \sup_{x \in X(\underline{U})} \varphi_n(x).$$

For an almost-additive sequence satisfying (1) this implies that

$$P_Z(\Phi, \mathcal{U}_l) = P_Z(\Phi, \mathcal{U}_1).$$

Thus

$$P_Z(\Phi) = \lim_{l \to \infty} P_Z(\Phi, \mathcal{U}_l) = P_Z(\Phi, \mathcal{U}_1) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\underline{U} \in \Gamma} \exp \sup_{x \in X(\underline{U})} \varphi_n(x).$$

An almost-additive sequence satisfying (1) on a subshift of finite type the topological pressure can be computed without the limit as the diameter of the cover goes to zero.

Also, we note that it is possible to prove directly that

$$\lim_{\operatorname{diam}(\mathcal{U}) \to 0} \lim_{n \to \infty} \frac{1}{n} \log Z_n(\Phi, Z, \mathcal{U})$$

exists for an almost-additive sequence of functions on a compact f-invariant set Z. This direct proof shows that the condition (1) is required for the limit as the diameter of the cover goes to zero. Thus for an almost-additive sequence on a subshift of finite type, we define the topological pressure without this condition, as in Definition 2.

5. **Variational Principle.** For the rest of the paper, we will be considering Z a compact f-invariant subset of X; thus without loss of generality we assume that $X = Z$, and write $P(\Phi) = P_Z(\Phi)$.

We first show that for Φ an almost-additive sequence satisfying condition (1)

$$P(\Phi) \geq \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu(x) : \mu \in M(X,f) \right\}. \tag{12}$$

We note that condition (2) is not needed for the proof of this inequality. The proof of the inequality follows the line of argument in Bowen [3] (see also [9]). We only indicate necessary modifications for the almost-additive case.

Let $\Gamma \in W_n(\mathcal{U})$ be a cover of X. For $\lambda > 0$ denote

$$Z(\Gamma, \lambda) = \sum_{\underline{U}\in\Gamma} \lambda^n \exp \sup_{x\in X(\underline{U})} \varphi_n(x).$$

Fix $m \geq 1$ and let $\Gamma^m = \{\underline{U}_1 \ldots \underline{U}_m : \underline{U}_i \in \Gamma\}$.

Lemma 4.

$$Z(\Gamma^m, \lambda) \leq \Big[(\exp C_2) Z(\Gamma, \lambda)\Big]^m. \tag{13}$$

Proof. Almost-additivity implies

$$\sup_{x\in X(\underline{U}_1\ldots\underline{U}_m)} \varphi_{mn}(x) \leq \sup_{X(\underline{U}_1)} \varphi_n(x) + \cdots + \sup_{X(\underline{U}_m)} \varphi_n(x) + (m-1)C_2.$$

Hence

$$\sum_{\underline{U}_1\ldots\underline{U}_m\in\Gamma^m} \lambda^{mn} \exp \sup_{x\in X(\underline{U}_1\ldots\underline{U}_m)} \varphi_{mn}(x) \leq$$

$$\exp\Big[(m-1)(C_2)\Big] \sum_{\underline{U}_1\ldots\underline{U}_m\in\Gamma^m} \left[\lambda^n \exp \sup_{x\in X(\underline{U}_1)} \varphi_n(x)\right] \ldots \left[\lambda^n \exp \sup_{x\in X(\underline{U}_m)} \varphi_n(x)\right]$$

$$\leq \left[(\exp C_2)\left(\sum_{\underline{U}\in\Gamma} \lambda^n \exp \sup_{x\in X(\underline{U})} \varphi_n(x)\right)\right]^m. \qquad \square$$

Lemma 5. $\lim_{n\to\infty}(\log Z_{nm}(\Phi,\mathcal{U}))/n \leq mP(\Phi,\mathcal{U})$.

Proof. Beginning with Equation (13) of Lemma 4 we have

$$\lim_{n\to\infty} \frac{1}{n} \log \inf_{\Gamma\in W_n(\mathcal{U})} Z(\Gamma^m, 1) \leq m \lim_{n\to\infty} \frac{1}{n}\left[\log \inf_{\Gamma\in W_n(\mathcal{U})} Z(\Gamma,1) + C_2\right]$$
$$= m\, P(\Phi,\mathcal{U}) \qquad \square$$

Lemma 6. *For any fixed $m \geq 1$ we have $C_1 + P_{f^m}(S_m\Phi) \leq mP_f(\Phi)$.*

Proof. Let $\mathcal{V} = \mathcal{U} \vee \cdots \vee f^{-m+1}\mathcal{U}$. Then $W_n(\mathcal{V})$ and $W_{mn}(\mathcal{U})$ are in one-to-one correspondence. Namely, for $\underline{U} = U_{i_0} \ldots U_{i_{mn-1}}$ let $\underline{V} = V_{i_0} \ldots V_{i_{m-1}}$ where $V_{i_k} = U_{i_{km}} \cap \cdots \cap f^{-m+1} U_{i_{km+m-1}}$. Then

$$X_f(\underline{U}) = X_{f^m}(\underline{V}), \tag{14}$$

since

$$\{x \in X : f^k \in U_k, k = 0, \ldots, mn-1\} = \{x \in X : f^{mk} \in V_k, k = 0, \ldots, n-1\}.$$

By almost-additivity and Equality (14),

$$\sup_{x\in X(\underline{V})}\sum_{k=0}^{n-1}\varphi_m(f^{mk}(x))+(n-1)C_1\le \sup_{x\in X(\underline{U})}\varphi_{mn}(x),$$

which implies that

$$C_1+P_{f^m}(S_m\Phi,\mathcal{V})\le \lim_{n\to\infty}\frac{1}{n}\log \inf_{\Gamma\in W_{mn}(\mathcal{U})}\sum_{\underline{U}\in\Gamma}\exp\sup_{x\in X(\underline{U})}\varphi_{mn}$$

The desired result follows from Lemma 5. □

Lemma 7. *Let $\lambda>0$. Suppose that $(\exp C_2)Z(\Gamma,\lambda)<1$ for some Γ covering X. Then $\lambda\le\exp(-P(\Phi,\mathcal{U}))$.*

Lemma 8 (Bowen [3])**.** *Let real numbers $a_1,\dots a_n$ be given. Then the quantity*

$$F(p_1,\dots p_n)=\sum_{i=1}^{n}-p_i\log p_i+\sum_{i=1}^{n}p_ia_i$$

has maximum value $\log\sum_{i=1}^{n}\exp a_i$ for $(p_1,\dots p_n)$ with $p_i\ge 0$ and $p_1+\dots+p_n=1$, and the maximum is assumed only at $p_j=e^{a_i}(\sum_{i=0}^{n}e^{a_i})^{-1}$.

Set $M(X)$ to be the set of all Borel probability measures on X. We set $H_\mu(C)$ as the entropy of μ with respect to the partition C, and $H_\mu(C|D)$ as the conditional entropy of the partition C with respect to the partition D.

Lemma 9 (Bowen [3])**.** *Let X be a compact metric space, $\mu\in M(X)$, $\epsilon>0$, and $\mathcal{C}$ a finite Borel partition. There is a $\delta>0$ so that $H_\mu(\mathcal{C}|\mathcal{D})<\epsilon$ whenever $\mathcal{D}$ is a partition with diameter less than δ.*

Lemma 10 (Bowen [3])**.** *Let $\mathcal{A}$ be a finite open cover of X. For each $n>0$ there is a Borel partition $\mathcal{D}(n)$ of X so that*

a. *$D\in\mathcal{D}(n)$ lies inside some member of $T^{-k}\mathcal{A}$ for each $k=0,\dots,n-1$.*
b. *at most $n|\mathcal{A}|$ sets in $\mathcal{D}(n)$ can have a point in all their closures.*

Lemma 11. *Suppose $\mathcal{D}$ is a Borel partition of X such that each $x\in X$ is in the closure of at most M members of $\mathcal{D}$. Then*

$$h_\mu(f,\mathcal{D})+\lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu\le P(\Phi)+\log M.$$

Proof. Let $\mathcal{U}$ be a finite open cover of X each member of which intersects at most M members of $\mathcal{D}$. For each $B\in\mathcal{D}_m=\mathcal{D}\vee\dots\vee f^{-m+1}\mathcal{D}$ pick x_B with $\int_B\varphi_m d\mu\le\mu(B)\varphi_m(x_B)$. Now, using properties of entropy (see for example Katok and Hasselblatt [8]), and Lemma 8 we have

$$\begin{aligned}h_\mu(f,\mathcal{D})+\lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu &\le \frac{1}{m}\left(H_\mu(\mathcal{D}_m)+\int\varphi_m d\mu+C_2\right)\\ &\le\frac{1}{m}\left(\sum_{B\in D_m}\mu(B)\left(-\log\mu(B)+\varphi_m(x_B)\right)+C_2\right)\\ &\le\frac{1}{m}\log\sum_{B\in D_m}\exp\varphi_m(x_B)+\frac{C_2}{m}.\end{aligned}\tag{15}$$

Fix $C > 0$. Let $\Gamma_m \subset W_m(\mathcal{U})$ be a cover of X so that

$$\sum_{\underline{U}\in\Gamma_m} \exp \sup_{x\in X(\underline{U})} \varphi_m(x) \leq C + \inf_{\Gamma\in W_m(\mathcal{U})} \sum_{\underline{U}\in\Gamma} \exp \sup_{x\in X(\underline{U})} \varphi_m(x). \tag{16}$$

For each x_B pick $\underline{U}_B \in \Gamma_m$ with $x_B \in X(\underline{U}_B)$. This map $B \to \Gamma_m$ is at most M^m to one. As $\varphi_m(x_B) \leq \sup_{x\in X(\underline{U}_B)} \varphi_m(x)$, one has by inequalities (15) and (16)

$$\begin{aligned} h_\mu(f,\mathcal{D}) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu &\leq \frac{1}{m} \log \sum_{\underline{U}\in\Gamma_m} M^m \exp \sup_{x\in X(\underline{U})} \varphi_m(x) + \frac{C_2}{m} \\ &\leq \log M + \frac{1}{m} \log Z_m(\Phi,\mathcal{U}) + \frac{C_2}{m} + \frac{\log C}{m}. \end{aligned}$$

By letting $m \to \infty$ and then diameter $\mathcal{U} \to 0$, we obtain the desired inequality. □

Proof of Inequality (12). Let $\mathcal{C}$ be a Borel partition and $\epsilon > 0$. By Lemma 9 we can find a finite open cover $\mathcal{A}$ of X so that $H_\mu(\mathcal{C}|\mathcal{D}) \leq \epsilon$ whenever $\mathcal{D}$ is a partition every member of which is contained in some member of $\mathcal{A}$.

Fix $n > 0$. Let $\mathcal{C}_n = \mathcal{C} \vee \cdots \vee f^{-n+1}\mathcal{C}$, and $\mathcal{D}(n)$ be as in Lemma 10. Then by Lemma 11 we have

$$\begin{aligned} h_\mu(f,\mathcal{C}_n) + \lim_{m\to\infty} \frac{1}{m} \int \varphi_m d\mu &\leq \frac{1}{n} \left(h_\mu(f^n,\mathcal{C}_n) + \lim_{m\to\infty} \frac{1}{m} \int \phi^n_{m,f^n} d\mu \right) \\ &\leq \frac{1}{n} \left(h_\mu(f^n,\mathcal{D}(n)) + \lim_{m\to\infty} \frac{1}{m} \int \phi^n_{m,f^n} d\mu \right) + \\ &\qquad \frac{1}{n} H_\mu(\mathcal{C}_n|\mathcal{D}(n)) \\ &\leq \frac{1}{n} \left(P_{f^n}(S_n\Phi) + \log n|\mathcal{A}| \right) + \frac{1}{n} H_\mu(\mathcal{C}_n|\mathcal{D}(n)). \end{aligned}$$

We have $H_\mu(\mathcal{C}_n|\mathcal{D}(n)) \leq \sum_{k=0}^{n-1} H_\mu(f^{-k}\mathcal{D}|\mathcal{D}(n))$. Since $\mathcal{D}(n)$ refines $f^{-k}\mathcal{A}$ for each k, one has $H_\mu(f^{-k}\mathcal{C}|\mathcal{D}(n)) \leq \epsilon$ (since μ is f-invariant, $f^{-k}\mathcal{A}$ bears the same relation to $f^{-k}\mathcal{C}$ as $\mathcal{A}$ to $\mathcal{C}$). Hence, using Lemma 6,

$$h_\mu(f,\mathcal{C}) + \lim_{m\to\infty} \frac{1}{m} \int \varphi_m d\mu \leq P_f(\Phi) + \frac{\log n|\mathcal{A}|}{n} + \epsilon + \frac{C_1}{n}.$$

Letting $n \to \infty$ and then $\epsilon \to 0$ shows the desired inequality. □

Next we show that for an almost-additive sequence Φ satisfying conditions (1) and (2)

$$P(\Phi) \leq \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi d\mu : \mu \in M(X,f) \right\}. \tag{17}$$

The following lemmas contain several steps needed in the proof the inequality.

Lemma 12 (Bowen [3]). *Fix a finite set E and $h \geq 0$. Let*

$$R(k,h) = \left\{ \underline{a} \in E^k : H(\underline{a}) \leq h \right\}.$$

Then $\overline{\lim}_{k\to\infty} \frac{1}{k} \log |R(k,h)| \leq h$.

Let δ_x be the unit measure on $x \in X$, and $\delta_{x,n} = \frac{1}{n}(\delta_x + \cdots + \delta_{f^{n-1}x})$. Define the set $V(x) = \{\mu \in M(X) : \delta_{x,n_k} \to \mu \text{ for some } n_k \to \infty\}$. As X is a compact metric space $V(x)$ is not empty. Also, every measure in $V(x)$ is f-invariant; so $V(x) \subset M(X,f)$.

Lemma 13. *Let $x \in X$, $\mu \in V(x)$. Fix $\delta > 0$, and $\epsilon > 0$. Then there exists $m, N \in \mathbb{N}$ for which one can find a string $\underline{U} \in W_N(\mathcal{U})$ satisfying the following*

a. $x \in X(\underline{U})$
b. $\underline{U}$ *contains a substring of length $km \geq N - m$ which, when viewed as $\underline{a} = a_0, \dots, a_{k-1} \in (\mathcal{U}^m)^k$, satisfies $\frac{1}{m} H(\underline{a}) \leq h_\mu(f) + \delta$.*
c. $\sup_{x \in X(\underline{U})} \varphi_N(x) \leq \varphi_N(x) + \gamma_N(\Phi, \mathcal{U})$
$\leq N \left(\lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu + 3\epsilon\right) + \gamma_N(\Phi, \mathcal{U})$

Proof. Parts (a) and (b) can be found in Bowen [3]. Let N be large enough so that

$$\left|\frac{1}{N}\int \varphi_N d\mu - \lim_{n\to\infty} \frac{1}{n}\int \varphi_n d\mu\right| \leq \epsilon, \quad \text{and}$$
$$\left|\delta_{x,N}(\varphi_1) - \int \varphi_1 d\mu\right| \leq \epsilon.$$

Thus

$$\begin{aligned}
&\left|\frac{1}{N}\ \varphi_N(x) - \lim_{n\to\infty}\frac{1}{n}\int \varphi_n d\mu\right| \leq \\
&\quad \left|\frac{1}{N}\varphi_N(x) - \frac{1}{N}\int \varphi_N d\mu\right| + \left|\frac{1}{N}\int \varphi_N d\mu - \lim_{n\to\infty}\frac{1}{n}\int \varphi_n d\mu\right| \leq \\
&\quad \left|\frac{1}{N}\sum_{k=0}^{N-1} \varphi_1\left(f^k(x)\right) - \int \varphi_1 d\mu\right| + \\
&\qquad \left|\frac{1}{N}\sum_{k=0}^{N-1} c_2(N-k) - \frac{1}{N}\sum_{k=0}^{N-1} c_1(N-k)\right| + \epsilon \leq B(N) + 2\epsilon,
\end{aligned}$$

where

$$B(n) = \left|\frac{1}{n}\sum_{k=0}^{n-1} c_2(n-k) - \frac{1}{n}\sum_{k=0}^{n-1} c_1(n-k)\right|.$$

As (2) is satisfied we have for N large enough that $B(N) < \epsilon$, and the result is shown. □

Proof of Inequality (17). Let $\mathcal{U}$ be a finite open cover of X and $\epsilon > 0$. Cover X with countably many sets X_m, where each X_m is the set of points for which Lemma 13 holds with this m and some $\mu \in V(X)$.

Choose an ϵ-dense set $u_1, \dots, u_r$ in the closed set

$$\left[-||\lim_{n\to\infty} \varphi_n/n||_\infty, ||\lim_{n\to\infty} \varphi_n/n||_\infty\right].$$

Cover each X_m with the sets $Y_m(u_i)$, where each $Y_m(u_i)$ is the set of $x \in X_m$ for which Lemma 13 holds for this m and some $\mu \in V(x)$ with $\lim_{n\to\infty} \int(\varphi_n d\mu)/n \in [u_i - \epsilon, u_i + \epsilon]$.

Next we cover each $Y_m(u)$ by $\Gamma_{m,u}$ and show that we can make

$$(\exp C_2) Z(\Gamma_{m,u}, \lambda)$$

as small as desired. Taking the unions of such $\Gamma_{m,u}$ we obtain a Γ covering X with $(\exp C_2) Z(\Gamma, \lambda) < 1$.

Let $S(N)$ be the number of all possible strings $\underline{U}$ satisfying Lemma 13 for some $x \in Y_m(u)$. By Lemma 13 part (2), and Lemma 12

$$\begin{aligned} S(N) &\leq |\mathcal{U}|^m \left|\left\{\underline{V} \in (\mathcal{U}^m)^k : H(\underline{V}) \leq m(h_\mu(f)+\epsilon)\right\}\right| \\ &\leq \exp\left[N\left(h_\mu(f)+\epsilon\right)\right] \end{aligned}$$

for all sufficiently large N.

For every integer N_0, the strings satisfying Lemma 13, for some $x \in Y_m(u)$ and $N \geq N_0$, cover the set $Y_m(u)$. Let $\Gamma_{m,u}$ be the collection of $\underline{U}$ showing up in the present situation for some sufficiently large $N > N_0$ fixed. One can show

$$\begin{aligned} Z(\Gamma_{m.u},\lambda) &\leq \sum_{N=N_0}^{\infty} \lambda^N S(N) \exp\left[N\left(\lim_{n\to\infty}\frac{1}{n}\int \varphi_n d\mu + 3\epsilon\right) + \gamma_N(\Phi,\mathcal{U})\right] \\ &\leq \sum_{N=N_0}^{\infty} \lambda^N \exp\left[N\left(\lim_{n\to\infty}\frac{1}{n}\int \varphi_n d\mu + h_\mu(f) + 4\epsilon + \lim_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}\right)\right] \\ &\leq \sum_{N=N^0}^{\infty} \beta^N = \frac{\beta^{N_0}}{1-\beta}, \end{aligned}$$

where

$$\beta = \lambda \exp\left(c + 4\epsilon + \lim_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}\right) < 1 \tag{18}$$

and

$$c = \sup_\mu \left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int \varphi_n d\mu\right).$$

From Equation (18) we see that for

$$\lambda < \exp -\left(c + 4\epsilon + \lim_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}\right)$$

any $Y_m(u)$ can be covered by $\Gamma_{m,u} \subset \bigcup_{m\geq 0} W_m(\mathcal{U})$ with $(\exp c_2)Z(\Gamma_{m,u},\lambda)$ as small as desired. Taking the unions of such $\Gamma_{m,u}$ we obtain a Γ covering X with $(\exp c_2)Z(\Gamma,\lambda) < 1$. By Lemma 7, $\lambda \leq \exp(-P(\Phi,\mathcal{U}))$, meaning

$$P(\Phi,\mathcal{U}) \leq c + 4\epsilon + \lim_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}.$$

The result is shown by letting $\epsilon \to 0$ and $\mathrm{diam}(\mathcal{U}) \to 0$. □

To prove Theorem 4, we demonstrate a new proof of Lemma 13 part (c) without condition (2).

Proof of Lemma 13 part(c). Let $x \in V(x)$ and $\epsilon > 0$. For each n there exists a sequence of integers $m_k = m_k(n)$ such that

$$\left|\delta_{x,m_k}(\varphi_n) - \int \varphi_n d\mu\right| < \epsilon$$

for every $k \in \mathbb{N}$. Since the sequence Φ is almost-additive, for each $n \in \mathbb{N}$ we have

$$-Cm + \sum_{k=0}^{m-1} \varphi_n \circ f^n j \leq \varphi_{nm} \leq \sum_{k=0}^{m-1} \varphi_n \circ f^n j + Cm.$$

Thus

$$\left|\frac{\varphi_{mn}}{mn} - \frac{1}{n}\frac{1}{m}\sum_{j=0}^{m-1}\varphi_n(f^{nj})\right| < \frac{C}{n}.$$

Then for N large enough, setting $m = m_k(N)$, we obtain

$$\left|\frac{\varphi_{mN}(x)}{mN} - \lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu\right| \le \left|\frac{\varphi_{mN}(x)}{mN} - \frac{1}{N}\frac{1}{m}\sum_{j=0}^{m-1}\varphi_N(f^{Nj}x)\right|$$

$$+\frac{1}{N}\left|\frac{1}{m}\sum_{j=0}^{m-1}\varphi_N(f^{Nj}x) - \int\varphi_N d\mu\right| + \left|\frac{1}{N}\int\varphi_N d\mu - \lim_{n\to\infty}\frac{1}{n}\int\varphi_N d\mu\right|$$

$$\frac{C}{N} + \frac{\delta}{N} + \left|\frac{1}{N}\int\varphi_N d\mu - \lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu\right| \le 3\epsilon.$$

Thus we have $\varphi_N(x) \le N(\lim n \to \infty\frac{1}{n}\int\varphi_n d\mu + 3\epsilon)$ without condition (2). □

The proof of Theorem 4 now procedes as in the proof of Theorem 2.

6. **Equilibrium Measures.**

Proof of Theorem 5. We show that the function $\mu \mapsto h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu$ is upper semi-continuous. Then use the fact that upper semi-continuous functions achieve their supremum on a compact set.

The map $\mu \mapsto h_\mu(f)$ is known to be upper semi-continuous for an expansive homeomorphism, see for example Bowen [3].

As a consequence of almost-additivity for the sequence of numbers $\int\varphi_n d\mu$ we have

$$\frac{1}{m}\left[\int\varphi_m d\mu + C_1\right] \le \lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu \le \frac{1}{m}\left[\int\varphi_m d\mu + C_2\right]$$

for any $m \ge 1$ and any measure $\mu \in M(X,f)$. Let $\mu_k \in M(X,f)$ converge to μ in the weak-star topology. We have for any measure $\nu \in M(X,f)$

$$\left|\lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\nu - \frac{1}{m}\int\varphi_m d\nu\right| \le \frac{\sup\{|C_1|,|C_2|\}}{m}.$$

Thus

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu_k = \lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu,$$

which shows that $\mu \mapsto h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int\varphi_n d\mu$ is upper semi-continuous. □

6.1. **Subshifts of finite type.** Let $\Phi = \{\varphi_n\}$ be an almost-additive sequence of continuous functions on a mixing subshift of finite type (Σ_A, σ). Let C_n be a cylinder of length n in Σ_A. The pressure function simplifies to

$$P(\Phi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{C_n\subset\Sigma_A}\sup_{x\in C_n}\exp\varphi_n(x).$$

As the system is mixing we have that there is an integer $p > 0$ such that $A^p > 0$. This implies that for any cylinders $C_n \subset \Sigma_A$ and $J_l \subset \Sigma_A$, there exists a p-cylinder $K_p \subset \Sigma_A$ such that $C_nK_pJ_l \subset \Sigma_A$.

We use the notation $A \approx B$ when there exists D_1, D_2 so that $D_1A \le B \le D_2A$.

The following lemma can be shown through a series of computations that follow the work of Feng and Lau [7] [6].

Lemma 14. $\sum_{C_n \in \Sigma_A} \sup_{x \in C_n} \exp \varphi_n(x) \approx \exp(nP(\Phi))$.

For each integer $n > 0$ let $\mathcal{B}_n$ be the σ-algebra generated by the cylinders $C_n \subset \Sigma_A$. We define a sequence of probability measures $\{\nu_{n,\Phi}\}$ on $\mathcal{B}_n$ by

$$\nu_{n,\Phi}(C_n) = \frac{\sup_{x \in C_n} \exp \varphi_n(x)}{\sum_{C_n \subset \Sigma_A} \sup_{x \in C_n} \exp \varphi_n(x)}, \quad \forall C_n \subset \Sigma_A.$$

There exists a subsequence $\{\nu_{n_k,\Phi}\}$ that converges in the weak-star topology to a probability measure ν_Φ.

Lemma 15. *For all* $n > 0, C_n \in \Sigma_A$

$$\sup_{x \in C_n} \exp \varphi_n(x) \exp(-nP(\Phi)) \approx \nu_\Phi(C_n).$$

Lemma 16. *There is a unique* σ*-invariant, ergodic Gibbs measure* μ_Φ *on* Σ_A*.*

Proof. Let μ_Φ be a limit point of a subsequence of

$$\left\{ \frac{1}{m} \left(\nu_\Phi + \nu_\Phi \circ \sigma^{-1} + \cdots + \nu_\Phi \circ \sigma^{-(m-1)} \right) \right\}$$

in the weak-star topology. By definition μ_Φ is a σ-invariant probability measure on Σ_A. For each $C_n \subset \Sigma_A$ and $l > p$ we have that $\sigma^{-l}(C_n)$ is the union of all cylinders $D_l C_n \subset \Sigma_A$. Thus we can show for every $l > p$ that

$$\nu_\Phi \circ \sigma^{-l}(C_n) \approx \sup_{x \in C_n} \exp \varphi_n(x) \exp(-nP(\Phi)).$$

Sum over all $C_n \subset \Sigma_A$, and divide by m. Taking the limit as m goes to infinity yields that μ_Φ satisfies equation (5).

Given any C_n and $D_l \subset \Sigma_A$ and any $i > n + 2p$ we have

$$\mu_\Phi \left(C_n \cap \sigma^{-i}(D_l) \right) \geq C \mu_\Phi(C_n) \mu_\Phi(D_l).$$

As any Borel set can be approximated within ϵ by a finite disjoint union of cylinder sets, the above inequality holds for all Borel sets. Thus for any μ_Φ positive measure Borel sets $A, B \subset \Sigma$ there exists $n > 0$ such that $\mu_\Phi(A \cap \sigma^{-n}(B)) > 0$. Hence μ_Φ is ergodic.

Any two distinct ergodic measures must be singular, but property (5) shows that they must be absolutely continuous to each other. Thus the measure μ_Φ is unique. □

Lemma 17. *The Gibbs measure* μ_Φ *is an equilibrium measure for* Φ*.*

Proof. Since μ_Φ satisfies equation (5), for each $n \in \mathbb{N}, C_n \subset \Sigma_A$, and $x \in C_n$ we have $\log A_1 \leq nP(\Phi) + \log \mu_\Phi(C_n) - \varphi_n(x) \leq \log A_2$. Integrate over $x \in C_n$, sum over all $C_n \subset \Sigma_A$, and divide by n to get

$$\frac{\log C_1}{n} \leq P(\Phi) + \frac{1}{n} \sum_{C_n \subset \Sigma_A} \mu_\Phi(C_n) \log \mu_\Phi(C_n) - \frac{1}{n} \int \varphi_n d\mu_\Phi \leq \frac{\log C_2}{n}.$$

Letting n go to infinity and combining with Inequality (12) yields the result. □

During the proof of the above lemma we also showed the variational principle for subshifts of finite type without requiring (2), i.e. Theorem 3.

Lemma 18. *The Gibbs measure* μ_Φ *is the unique equilibrium measure.*

To prove Lemma 18 we use the following two lemmas.

Lemma 19 (Bowen [3]). *Let X be a compact metric space, $\mu \in M(X)$, and $\mathcal{D} = \{D_1, \ldots, D_n\}$ a Borel parition of X. Suppose $\{\mathcal{C}_m\}_{m=1}^{\infty}$ is a sequence of partitions so that diam$(\mathcal{C}_m) = max_{C \in C_m}$ diam$(C) \to 0$ as $m \to \infty$. Then there is a sequence of partitions $E_m = \{E_1^m, \ldots, E_n^m\}$ so that*

a. *each E_i^m is a union of members of $\mathcal{C}_m$*
b. $\lim_{m\to\infty} \mu(E_i^m \triangle D_i) = 0$ *for each i.*

Lemma 20 (Bowen [3]). *Suppose $0 \le p_i, \ldots, p_m \le 1$, $s = p_1 + \cdots + p_m \le 1$ and $a_1, \ldots, a_m \in \mathbb{R}$. Then $\sum_{i=1}^m p_i(a_i - \log p_i) \le s(\log \sum_{i=1}^m e^{a_i} - \log s)$.*

Proof of Lemma 18. Let $\nu \in M(\Sigma_A, \sigma)$ be a second equilibrium measure, i.e. $P(\Phi) = h_\nu(\sigma) + \lim_{n\to\infty} \int \varphi_n d\nu$.

First suppose that ν is singular with respect to the Gibbs measure μ_Φ. Then there is a Borel set B with $\sigma(B) = B$, $\mu_\Phi(B) = 0$, and $\nu(B) = 1$. Let $\mathcal{C}_m = \sigma^{-[\frac{m}{2}]+1}\mathcal{U} \vee \cdots \vee \mathcal{U}$. Then diam $(\mathcal{C}_m) \to 0$ as $m \to \infty$. Applying Lemma 19 to the partition $\{B, X \setminus B\}$ one finds sets E^m which are unions of elements of $\mathcal{C}_m$ and satisfy $(\mu + \nu)(B \triangle E^m) \to 0$. As $\mu + \nu$ is σ-invariant and $\sigma^{-m+[\frac{m}{2}]}(B) = B$, one has $(\mu+\nu)(B \triangle F^m) \to 0$, where $F^m = \sigma^{-m+[\frac{m}{2}]} E^m$ is a union of members of $\sigma^{-m+1}\mathcal{U} \vee \cdots \vee \mathcal{U}$. For every ϵ there exists an m large enough so that

$$P(\Phi) = h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\nu \le \frac{1}{m}\left(H_\nu(\mathcal{U} \vee \cdots \vee \sigma^{-m+1}\mathcal{U}) + \int \varphi_m d\nu\right) + \epsilon$$

$$\text{or} \quad mP(\Phi) \le \sum_{B \in \mathcal{U} \vee \cdots \vee \sigma^{-m+1}\mathcal{U}} \left[-\nu(B)\log\nu(B) + \int_B \varphi_m d\nu\right].$$

As φ_m is continuous there exists a d so that for each B one can find $X_B \in B$ with $\varphi_m(x) \le \varphi_m(x_B) + d$ on B. Thus

$$mP(\Phi) \le d + \epsilon + \sum_B \nu(B)(\varphi_m(x_B) - \log\nu(B))$$

$$mP(\Phi) \le d + \epsilon + \sum_{B \in F^m} \nu(B)(\varphi_m(x_B) - \log\nu(B)) + \sum_{B \in X \setminus F^m} \nu(B)(\varphi_m(x_B) - \log\nu(B))$$

Applying Lemma 20 gives

$$mP(\Phi) - d \le \nu(F^m)\log \sum_{B \in F^m} \exp\varphi_m(x_B) + \nu(X \setminus F^m) \sum_{B \in X\setminus F^m} \exp\varphi_m(x_B) + 2K,$$

where $K = \sup_{0 \le s \le 1}(-s\log s)$. Using the fact that μ_Φ is a Gibbs measure, we have that

$$\begin{aligned}-2K - d - \epsilon &\le \nu(F^m)\log \sum_{B \in F^m} \exp(\varphi_m(x_B) - mP) + \\ &\qquad \nu(X \setminus F^m)\exp(\varphi_m(x_B) - mP(\Phi)) \\ &\le \nu(F^m)\log \sum_{B \in F^m} A_1^{-1}\mu_\Phi(B) + \nu(X\setminus F^m)\log \sum_{B \in X\setminus F^m} A_1^{-1}\mu_\Phi(B) \\ &\le \log A_1^{-1} + \nu(F^m)\log\mu_\Phi(F^m) + \nu(X\setminus F^m)\log\mu(X\setminus F^m).\end{aligned}$$

Letting $\epsilon \to 0$, $m \to \infty$, $\nu(F^m) \to 1$, and $\mu(F^m) \to 0$, which leads to a contradiction with the above inequality.

In general, for $\nu' \in M(\Sigma_A, \sigma)$, write $\nu' = \beta\nu + (1-\beta)\mu'$, where $\beta \in [0,1]$, $\nu \in M(\Sigma_A, \sigma)$ is singular with respect to μ_Φ and $\mu' \in M(\Sigma_A, \sigma)$ is absolutely continuous with respect to μ_Φ. As ν and μ' are supported on disjoint sets

$$h_{\nu'}(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\nu' = \beta \left(h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\nu \right) + (1-\beta) \left(h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu \right).$$

Suppose that ν' is an equilibrium measure for Φ. As μ_Φ is ergodic, we must have $\beta = 0$, or $\beta = 1$. Above we showed that $h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\nu \neq P(\Phi)$. Thus $\nu' = \mu'$ and write $\nu' = \frac{d\nu'}{d\mu_\Phi} \mu_\Phi$. As ν' and μ_Φ are σ-invariant, $\frac{d\nu'}{d\mu_\Phi}$ must be a constant. Since both measures are probability measures we must have $\nu' = \mu_\Phi$. □

6.2. **Axiom A diffeomorphisms.** We show the existence of a unique equilibrium measure for Axiom A systems using a conjugacy with a mixing subshift of finite type.

Lemma 21. *Let $\pi : \Sigma_A \to X$ be continuous and onto so that $\pi \circ \sigma = f \circ \pi$. Then $P_\sigma(\Phi \circ \pi) \geq P_f(\Phi)$.*

Proof. Let $\mathcal{U}$ be an open cover of X, and $s \in \Sigma_A$. If $s \in \pi^{-1}(X(\underline{U}))$, then $\pi s \in X(\underline{U})$. Therefore, for fixed $n \geq 1$,

$$\sup_{x \in X(\underline{U})} \varphi_n(x) = \sup_{y \in \pi^{-1} X(\underline{U})} \varphi_n \circ \pi(y).$$

Thus $P_f(\Phi, \mathcal{U}) = P_\sigma(\Phi \circ \pi, \pi^{-1}\mathcal{U})$. We have that

$$P_\sigma(\Phi \circ \pi, \pi^{-1}\mathcal{U}) \leq P_\sigma(\Phi \circ \pi) + \gamma(\pi^{-1}\mathcal{U}).$$

Letting $\operatorname{diam}(\mathcal{U}) \to 0$ gives that $\gamma(\pi^{-1}\mathcal{U}) \to 0$. Thus the result is shown. □

Proof of Theorem 7. Let $\mathcal{R}$ be a Markov partition for Ω of diameter less or equal ϵ, A the transition matrix for $\mathcal{R}$, and $\pi : \Sigma_A \to \Omega$. As in Bowen [3], π is one-to-one except on a set of measure zero.

First we assume that $f_{|\Omega}$ is mixing. Then $\sigma_{|\Sigma_A}$ is mixing and there exists an equilibrium measure $\mu_{\Phi\circ\pi}$. Let $\mu_\Phi = \pi * \mu_{\Phi\circ\pi}$, meaning $\mu_\Phi(E) = \mu_{\Phi\circ\pi}(\pi^{-1}E)$. Then μ_Φ is f-invariant. The measure spaces $(\Sigma_A, \sigma, \mu_{\Phi\circ\pi})$ and (X, f, μ_Φ) are conjugate, since π is one-to-one except on a set of $\mu_\phi \circ \pi$ measure zero. In particular, $h_{\mu_\Phi}(f) = h_{\mu_{\Phi\circ\pi}}(\sigma)$ and so by Lemma 21

$$\begin{aligned} P_f(\Phi) \geq h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu_\Phi &= h_{\mu_{\Phi\circ\pi}}(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \circ \pi d\mu_{\Phi\circ\pi} \\ &= P_\sigma(\Phi \circ \pi) \geq P_f(\Phi). \end{aligned}$$

Hence $P_\sigma(\Phi \circ \pi) = P_f(\Phi)$ and μ_Φ is an equilibrium state for Φ.

If $f_{|\Omega}$ is not mixing, then use the spectral decomposition to show that there exists a bijection between $M(\Omega, f)$ and $M(X_1, f^m)$. Thus we have that

$$h_{\mu'}(f^m) = m h_\mu(f) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \int \sum_{k=0}^{m-1} \varphi_n \circ f^k d\mu' = m \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu.$$

Maximizing $h_{\mu'}(f^m) + \lim_{n\to\infty} \frac{1}{n} \int \sum_{k=0}^{m-1} \varphi_n \circ f^k d\mu'$ is equivalent to maximizing

$$h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu.$$

For Φ Hölder on Ω, $S_m\Phi$ will be Hölder on X_1 and therefore we are done since X_1 is a mixing basic set of f^m.

To show that the measure is unique, suppose μ is any equilibrium state of Φ and choose $\nu \in M(\Sigma_A, \sigma)$ with $\pi * \nu = \mu$. Then $h_\nu(\sigma) \geq h_\mu(f)$ and so

$$\begin{aligned} P_\sigma(\Phi \circ \pi) \geq h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \circ \pi d\nu \geq h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu \\ = P_f(\Phi) = P_\sigma(\Phi \circ \pi). \end{aligned}$$

Thus ν is an equilibrium measure for $\Phi \circ \pi$ which implies $\nu = \mu_{\Phi\circ\pi}$. Then $\mu = \pi * \mu_{\Phi\circ\pi} = \mu_\Phi$. □

During the proof of the above lemma we also showed the variational principle for Axiom A diffeomorphisms without requiring (2), i.e. Corollary 2.

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