1-1-2024

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mathematics

Article

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Abstract: We extend existing functional relationships for the discrete generating series associated with a single-variable linear polynomial coefficient difference equation to the multivariable case.

Keywords: forward difference operator; difference equation; generating series; shift operator; characteristic polynomial; Cauchy problem

MSC: 05A15; 39A05; 39A06

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Citation: Akhtamova, S.S.; Cuchta, T.; Lyapin, A.P. An Approach to Multidimensional Discrete Generating Series. *Mathematics* **2024**, *12*, 143. https://doi.org/10.3390/math12010143

Academic Editors: Sergei Sitnik and Patricia J. Y. Wong

Received: 2 December 2023
Revised: 20 December 2023
Accepted: 28 December 2023
Published: 1 January 2024

1. Introduction

An approach to build the general theory of a discrete generating series of one variable and its connection with the linear difference equations was introduced in [1] . We extend those results to the multidimensional case. We define a discrete generating series for $f\colon \mathbb{Z}^n \to \mathbb{C}$ and derive functional relations for such series.

The general theory of linear recurrences with constant coefficients and the Stanley hierarchy [2,3] of its generating functions (rational, algebraic, D-finite) depending on the initial data function was considered in [4]. Difference equations with polynomial coefficients is an effective means to study lattice paths with restriction [5,6]. Some properties of linear difference operators whose coefficients have the form of infinite two-sided sequences over a field of characteristic zero are considered in [7]. An effective method of obtaining explicit formulas for the coefficients of a generating function related to the Aztec diamond and a generating function related to the permutations with cycles was derived in [8,9]. Using the notion of amoeba [10] of the characteristic polynomial of a difference equation, a description for the solution space of a multidimensional difference equation with constant coefficients was obtained in [11]. A generalization to several variables of the classical Poincaré theorem on the asymptotic behavior of solutions of a linear difference equation is presented in [12]. We can also note that the almost periodic and the almost automorphic solutions to the difference equations depending on several variables are not well explored in the existing literature [13].

Let $\mathbb{Z}_{\geqslant}$ denote the non-negative integers, $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ be the n-dimensional integers, and $\mathbb{Z}^n_{\geqslant} = \mathbb{Z}_{\geqslant} \times \cdots \times \mathbb{Z}_{\geqslant}$ for $n \in \mathbb{Z}_{\geqslant}$ be its non-negative orthant. For any $z \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geqslant}$, we define the falling factorial $z^{\underline{n}} = z(z-1)\cdots(z-n+1)$ with $z^{\underline{0}} = 1$ and the Pochhammer symbol (or rising factorial) is defined by $(z)_n = z(z+1)\cdots(z+n-1)$ with $(z)_0 = 1$. Throughout, we will use the multidimensional notation for convience of expressions: $x = (x_1, \dots, x_n) \in \mathbb{Z}^n_{\geqslant}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, $\xi^x = \xi_1^{x_1} \cdots \xi_n^{x_n}$, $z^{\underline{x}} = z_1^{\underline{x_1}} \cdots z_n^{\underline{x_n}}$, $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n_{\geqslant}$, $x! = x_1! \dots x_n!$. We also will use $x \leq y$ for $x, y \in \mathbb{Z}^n$ componentwise, i.e., that $x_i \leq y_i$ for all $i = 1, \dots, n$.

Given a function $f\colon \mathbb{Z}^n_{\geqslant} \to \mathbb{C}$, we define the associated multidimensional discrete generating series of f as

$$F(\xi;\ell;z) = \sum_{x\in\mathbb{Z}^n_{\geqslant}} f(x)\xi^x z^{\underline{\ell x}} = \sum_{x_1=0}^{\infty}\cdots\sum_{x_n=0}^{\infty} f(x_1,\ldots,x_n)\xi_1^{x_1}\cdots\xi_n^{x_n} z_1^{\underline{\ell_1 x_1}}\cdots z_n^{\underline{\ell_n x_n}}.$$

Let $p_\alpha \in \mathbb{C}[z]$ denote polynomials with complex coefficients. The difference equation under consideration in this work is

$$\sum_{\alpha\in A} p_\alpha(x) f(x-\alpha) = 0, \tag{1}$$

where set $A \subset \mathbb{Z}^n_{\geqslant}$ is finite and there is $m \in A$ such that for all $\alpha \in A$, the inequality $\alpha \leqslant m$, which means $\alpha_j \leqslant m_j, j = 1,\ldots,n$, holds. Occasionally we will use an equivalent notation $0 \leqslant \alpha \leqslant m$, assuming that for some α coefficients, $p_\alpha(x)$ vanishes and only $p_m(x) \not\equiv 0$. In Section 2, we will particularly consider a homogeneous difference equation with constant coefficients.

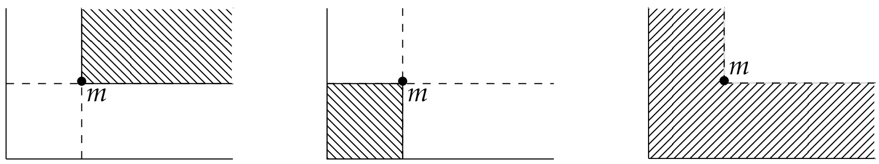
The special case where each $p_\alpha = c_\alpha$ is a constant

$$\sum_{\alpha\in A} c_\alpha f(x-\alpha) = 0 \tag{2}$$

arises in a wide class of combinatorial analysis problems [3], for instance, in lattice path problems [4], the theory of digital recursive filters [14], and the wavelet theory [15]. The question about correctness and well-posedness of (2) was considered in [16–18].

We equip (1) with initial data on a set named X_m, which is used often enough. We introduce the notation $\mathbb{Z}_{\ngeqslant}$ as $X_m = \mathbb{Z}^n_{\geqslant} \setminus \left(m + \mathbb{Z}^n_{\geqslant}\right) = \left\{x \in \mathbb{Z}^n_{\geqslant} : x \ngeqslant m\right\}$ (see Figure 1) and we define the initial data function $\varphi : X_m \to \mathbb{C}$ so that

$$f(x) = \varphi(x), \quad x \in X_m. \tag{3}$$



(a) $x \geqslant m$ **(b)** $x \leqslant m$ **(c)** $x \ngeqslant m$

Figure 1. Illustration of the sets $x \geqslant m$, $x \leqslant m$, and $x \ngeqslant m$.

For convenience, we extend φ to the whole of $\mathbb{Z}^n$ by taking it to be identically zero outside of X_m. The Cauchy problem is to find a solution to difference Equation (1) that coincides with φ on X_m, i.e., $f(x) = \varphi(x)$, for all $x \in X_m$.

In Section 2, functional equations for the discrete generating series are derived for the solution of the difference equations with constant coefficients. In Section 3, a case of difference equations with polynomial coefficients is considered. Section 4 contains two examples that illustrate our approach to discrete generating series.

2. Discrete Generating Series for Linear Difference Equations with Constant Coefficients

In this section, we consider a homogeneous difference equation with constant coefficients (2) and introduce the shift operator by

$$\mathcal{P}(\xi;\ell;z) = \sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}} \rho^{\ell\alpha}. \tag{4}$$

Also useful is its truncation for $\tau \in \mathbb{Z}^n$, defined by the formula

$$\mathcal{P}_\tau(\xi;\ell;z) = \sum_{\substack{0\leqslant\alpha\leqslant m\\ \alpha\ngeqslant\tau}} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}} \rho^{\ell\alpha},$$

and the discrete generating series of the initial data for $\tau \in X_m$ by

$$\Phi_\tau(\xi;\ell;z) = \sum_{x\ngeqslant\tau} \varphi(x)\xi^x z^{\underline{\ell x}}. \tag{5}$$

Let $\delta_j : x \to x + e^j$ be the forward shift operator for $j = 1,\dots,n$ with multidimensional notation $\delta^\alpha = \delta_1^{\alpha_1}\dots\delta_n^{\alpha_n}$ and define the polynomial difference operator

$$P(\delta) = \sum_{0\leqslant\alpha\leqslant m} c_\alpha \delta^\alpha.$$

With this notation, Equation (2) is represented compactly as

$$P(\delta^{-1})f(x) = 0, \quad x \geqslant m.$$

The case of generating series $\sum_x f(x)z^x$ and exponential generating series $\sum_x \frac{f(x)}{x!}z^x$ is well-studied for both one and several variables: one of the first convenient formulas to derive the generating series exploiting the characteristic polynomial and the initial data function was proven in [19]. We will prove analogues of these formulas for the discrete generating series $F(\xi;\ell;z)$.

Theorem 1. *The discrete generating series $F(\xi;\ell;z)$ for the solution to the Cauchy problem for Equation (2) with initial data (3) satisfies the functional equations:*

$$\mathcal{P}(\xi;\ell;z)F(\xi;\ell;z) = \sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}} \rho^{\ell\alpha} \Phi_{m-\alpha}(\xi;\ell;z) \tag{6}$$

$$= \sum_{x\ngeqslant m} P(\delta^{-1})\varphi(x)\xi^x z^{\underline{\ell x}} \tag{7}$$

$$= \sum_{x\ngeqslant m} \mathcal{P}_{m-x}(\xi;\ell;z)\varphi(x) z^{\underline{\ell x}}. \tag{8}$$

Proof. By multiplying (2) by $\xi^x z^{\underline{\ell x}}$ and summing over $x \geqslant m$, we obtain

$$\begin{aligned} 0 &= \sum_{x\geqslant m}\sum_{0\leqslant\alpha\leqslant m} c_\alpha f(x-\alpha)\xi^x z^{\underline{\ell x}} \\ &= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \sum_{x\geqslant m} f(x-\alpha)\xi^x z^{\underline{\ell x}}. \end{aligned}$$

Now, substituting x with $x+\alpha$ yields

$$\begin{aligned} 0 &= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \sum_{x\geqslant m-\alpha} f(x)\xi^{x+\alpha} z^{\underline{\ell(x+\alpha)}} \\ &= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}}\rho^{\ell\alpha} \sum_{x\geqslant m-\alpha} f(x)\xi^x z^{\underline{\ell x}} \\ &= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}}\rho^{\ell\alpha} \left(\sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}} - \sum_{x\ngeqslant m-\alpha} \varphi(x)\xi^x z^{\underline{\ell x}} \right) \\ &= \underbrace{\sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}}\rho^{\ell\alpha}}_{=\mathcal{P}(\xi;\ell;z)} \underbrace{\sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}}}_{=F(\xi;\ell;z)} - \sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}}\rho^{\ell\alpha} \underbrace{\sum_{x\ngeqslant m-\alpha} \varphi(x)\xi^x z^{\underline{\ell x}}}_{=\Phi_{m-\alpha}(\xi;\ell;z)}. \end{aligned}$$

Thus, by (5), we have established (6). Since

$$\begin{aligned}\sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}} \rho^{\ell\alpha} \sum_{x\ngeqslant m-\alpha} \varphi(x)\xi^x z^{\underline{\ell x}} &= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \sum_{x\ngeqslant m-\alpha} \varphi(x)\xi^{x+\alpha} z^{\underline{\ell(x+\alpha)}} \\ &= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \sum_{x\ngeqslant m} \varphi(x-\alpha)\xi^x z^{\underline{\ell x}} \\ &= \sum_{x\ngeqslant m} \underbrace{\left[\sum_{0\leqslant\alpha\leqslant m} c_\alpha \varphi(x-\alpha)\right]}_{=P(\delta^{-1})\varphi(x)} \xi^x z^{\underline{\ell x}},\end{aligned}$$

which yields (7). Finally, collecting (6) by $\varphi(x)$ yields

$$\sum_{0\leqslant\alpha\leqslant m} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}} \rho^{\ell\alpha} \sum_{x\ngeqslant m-\alpha} \varphi(x)\xi^x z^{\underline{\ell x}} = \sum_{x\ngeqslant m} \underbrace{\sum_{\substack{0\leqslant\alpha\leqslant m\\ \alpha\ngeqslant m-x}} c_\alpha \xi^\alpha z^{\underline{\ell\alpha}} \rho^{\ell\alpha}}_{=\mathcal{P}_{m-x}(\xi;\ell;z)} \varphi(x) z^{\underline{\ell x}},$$

completing the proof of (8). □

For $z = (z_1,\ldots,z_n)$, we denote the projection operator $\pi_j z = (z_1,\ldots,z_{j-1},0,z_{j+1},\ldots,z_n)$ and we introduce the notation

$$\pi_j F(\xi;\ell;z) = F(\xi;\ell;\pi_j z) = \sum_{\substack{x\geqslant 0\\ x_j=0}} f(x)\xi^x z^{\underline{\ell x}},$$

and we define the combined projection $\Pi = (1-\pi_1)\circ\cdots\circ(1-\pi_n)$ as the composition of $1-\pi_j$ for all $j = 1,\ldots,n$.

For the next result, we introduce the symbols $I = (1,1,\ldots,1) \in \mathbb{Z}^n$ and the unit vectors $e_j = (0,\ldots,0,1,0,\ldots,0)$ for $j = 1,2,\ldots,n$, which is nonzero only for the jth component. In these two lemmas, we will prove some useful properties of the combined projection Π.

Lemma 1. *The following formula holds:*

$$\Pi \sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}} = \sum_{x\geqslant I} f(x)\xi^x z^{\underline{\ell x}}.$$

Proof. First, compute for any $j = 1,2,\ldots,n$,

$$\begin{aligned}(1-\pi_j)\sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}} &= \sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}} - \pi_j \sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}} \\ &= \sum_{x\geqslant e_j} f(x)\xi^x z^{\underline{\ell x}}.\end{aligned}$$

Thus, we see that applying Π to $\sum_{x\geq 0} f(x)\xi^x z^{\underline{\ell x}}$ yields the desired result. □

We now obtain a similar result as Lemma 1 but for a shifted discrete generating series.

Lemma 2. *The following formula holds:*

$$\Pi \xi_j z_j^{\underline{\ell_j}} \rho^{\ell_j} F(\xi;\ell;z) = \sum_{x\geqslant I} f(x-e^j)\xi^x z^{\underline{\ell x}}.$$

Proof. First, compute

$$
\begin{aligned}
(1-\pi_j)\xi_j z_j^{\underline{\ell_j}}\rho^{\ell_j}F(\xi;\ell;z) &= \xi_j z_j^{\underline{\ell_j}}\rho^{\ell_j}F(\xi;\ell;z) - \pi_j\xi_j z_j^{\underline{\ell_j}}\rho^{\ell_j}F(\xi;\ell;z) \\
&= \xi_j z_j^{\underline{\ell_j}}\rho^{\ell_j}\sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}} \\
&= \sum_{x\geqslant 0} f(x)\xi^{x+e_j} z^{\underline{\ell(x+e^j)}} \\
&= \sum_{x\geqslant e^j} f(x-e^j)\xi^x z^{\underline{\ell x}}.
\end{aligned}
$$

Thus, we see that applying Π to $\xi_j z_j^{\underline{\ell_j}}\rho^{\ell_j}F(\xi;\ell;z)$ completes the proof. □

We introduce the inner product

$$\langle c, \xi z^{\underline{\ell}}\rho^{\ell}\rangle = c_1\xi_1 z_1^{\underline{\ell_1}}\rho_1^{\ell_1} + \cdots + c_n\xi_n z_n^{\underline{\ell_n}}\rho_n^{\ell_n}$$

and

$$\langle c, \delta^{-I}\rangle = c_1\delta_1^{-1} + \cdots + c_n\delta_n^{-1}.$$

We are now prepared to prove an analogue of [20], [Theorem 1.1].

Theorem 2. *The following formula holds:*

$$\Pi\Big[(1-\langle c, \xi z^{\underline{\ell}}\rho^{\ell}\rangle)F(\xi;\ell;z)\Big] = \sum_{x\geqslant I}(1-\langle c,\delta^{-I}\rangle)f(x)\xi^x z^{\underline{\ell x}}. \tag{9}$$

Proof. Applying Π to $(1-\langle c, \xi z^{\underline{\ell}}\rho^{\ell}\rangle)F(\xi;\ell;z)$ yields

$$
\begin{aligned}
\Pi[(1-\langle c, \xi z^{\underline{\ell}}\rho^{\ell}\rangle)F(\xi;\ell;z)\Big] &= \Pi F(\xi;\ell;z) - \langle c, \Pi\xi z^{\underline{\ell}}\rho^{\ell}\rangle F(\xi;\ell;z) \\
&= \Pi F(\xi;\ell;z) - c_1\Pi\xi_1 z_1^{\underline{\ell_1}}\rho_1^{\ell_1}F(\xi;\ell;z) - \cdots - c_n\Pi\xi_n z_n^{\underline{\ell_n}}\rho_n^{\ell_n}F(\xi;\ell;z) \\
&= \sum_{x\geqslant I} f(x)\xi^x z^{\underline{\ell x}} - c_1\sum_{x\geqslant I} f(x-e^1)\xi^x z^{\underline{\ell x}} - \cdots - c_n\sum_{x\geqslant I} f(x-e^n)\xi^x z^{\underline{\ell x}} \\
&= \sum_{x\geqslant I}\Big(f(x) - c_1 f(x-e^1) - \cdots - c_n f(x-e^n)\Big)\xi^x z^{\underline{\ell x}} \\
&= \sum_{x\geqslant I}(1-\langle c,\delta^{-I}\rangle)f(x)\xi^x z^{\underline{\ell x}},
\end{aligned}
$$

thereby completing the proof. □

The following corollary is straightforward.

Corollary 1. *If f solves $(1-\langle c,\delta^{-I}\rangle)f(x) = 0$, then*

$$\Pi[(1-\langle c,\zeta\rangle)F(\xi;\ell;z)] = 0. \tag{10}$$

3. Discrete Generating Series for Linear Difference Equations with Polynomial Coefficients

We define the componentwise forward difference operators Δ_j by

$$\Delta_j F(z) = F(z+e^j) - F(z), \qquad j = 1,\ldots,n.$$

If $z^{\underline{x}} = z_1^{\underline{x_1}} \dots z_n^{\underline{x_n}}$, then $\Delta_j z^{\underline{x}} = x_j z^{\underline{x-e^j}}$. Thus, we can regard Δ_j as a discrete analogue of a partial derivative operator. Now, compute

$$\begin{aligned}\Delta_j F(\xi;\ell;z) &= \Delta_j \sum_{x\geqslant 0} f(x)\xi^x z^{\underline{\ell x}}\\ &= \sum_{x\geqslant 0} f(x)\xi^x \Delta_j z^{\underline{\ell x}}\\ &= \sum_{x\geqslant 0} \ell_j x_j f(x)\xi^x z^{\underline{\ell x - e^j}}.\end{aligned}$$

We denote the componentwise backward jump ρ_j by

$$\rho_j F(z) = F(z - e^j)$$

and we define the componentwise operators $\theta_j = \ell_j^{-1} z_j \rho_j \Delta_j$, which generalizes the single-variable one defined earlier in [21,22]. Now, we prove some useful properties of the operator $\theta^k := \theta_1^k \dots \theta_n^k$.

Lemma 3. *If $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geqslant}^n$, then the following formula holds:*

$$\theta^k F(\xi;\ell;z) = \sum_{x\geqslant 0} x^k f(x)\xi^x z^{\underline{\ell x}}. \tag{11}$$

Proof. We obtain:

$$\begin{aligned}\theta^k F(\xi;\ell;z) &= \theta_1^{k_1}\dots\theta_n^{k_n} F(\xi;\ell;z)\\ &= \theta_1^{fk_1}\dots\theta_{n-1}^{k_{n-1}}(\ell_n^{-1} z_n\rho_n\Delta_n)^{k_n-1}\ell_n^{-1} z_n\rho_n\Delta_n F(\xi;\ell;z)\\ &= \theta_1^{k_1}\dots\theta_{n-1}^{k_{n-1}}(\ell_n^{-1} z_n\rho_n\Delta_n)^{k_n-1}\ell_n^{-1} z_n\rho_n \sum_{x\geqslant 0}\ell_n x_n f(x)\xi^x z^{\underline{\ell x - e^n}}\\ &= \theta_1^{k_1}\dots\theta_{n-1}^{k_{n-1}}(\ell_n^{-1} z_n\rho_n\Delta_n)^{k_n-1}\sum_{x\geqslant 0} x_n f(x)\xi^x z^{\underline{\ell x}}.\end{aligned}$$

Continuing this process $k_n - 1$ times for θ_n and in k_j times in turn for the powers of θ_j, $j = 1, \dots, n-1$ completes the proof. □

The proof of the following lemma resembles the proof of Lemma 3 but for the operator $p(\theta) = \sum\limits_{\alpha\in A\subset\mathbb{Z}_{\geqslant}^n} c_\alpha \theta^\alpha$, so we omit explicitly writing the proof.

Lemma 4. *The following formula holds:*

$$p(\theta)F(\xi;\ell;z) = \sum_{x\geqslant 0} p(x)f(x)\xi^x z^{\underline{\ell x}}. \tag{12}$$

We define an operator $\mathcal{P}_A$ by

$$\mathcal{P}_A(\xi;\ell;z;\theta;\rho) = \sum_{\alpha\in A} p_\alpha(\theta+\alpha)\xi^\alpha z^{\underline{\ell\alpha}}\rho^{\ell\alpha}.$$

Theorem 3. *The discrete generating series $F(\xi;\ell;\cdot)$ of the Cauchy problem for Equation (1) with initial data (3) satisfies the functional equation*

$$\mathcal{P}_A(\xi;\ell;z;\theta;\rho)F(\xi;\ell;z) = \sum_{\alpha\in A}\sum_{x\ngeqslant m-\alpha} p_\alpha(x-\alpha)\varphi(x)\xi^x z^{\underline{\ell(x+\alpha)}}. \tag{13}$$

Proof. Similar to the proof of Theorem 1, we multiply (1) by $\xi^x z^{\underline{\ell x}}$ and sum over $x \geqslant m$ to obtain

$$0 = \sum_{x \geqslant m} \sum_{\alpha \in A} p_\alpha(x) f(x-\alpha) \xi^x z^{\underline{\ell x}} = \sum_{\alpha \in A} \sum_{x \geqslant m-\alpha} p_\alpha(x-\alpha) f(x) \xi^{x+\alpha} z^{\underline{\ell(x+\alpha)}}.$$

Replacing x with $x+\alpha$ then leads to

$$0 = \sum_{\alpha \in A} \xi^\alpha \left(\sum_{x \geqslant 0} p_\alpha(x-\alpha) f(x) \xi^x z^{\underline{\ell(x+\alpha)}} - \sum_{x \ngeqslant m-\alpha} p_\alpha(x-\alpha) \varphi(x) \xi^x z^{\underline{\ell(x+\alpha)}} \right)$$

and routine algebraic manipulation completes the proof. □

4. Examples

Example 1. *We will derive the functional equation for the discrete generating series*

$$F(;;z_1,z_2) = F(\xi_1,\xi_2;\ell_1,\ell_2;z_1,z_2)$$

for the basic combinatorial recurrence

$$f(x_1,x_2) - f(x_1-1,x_2) - f(x_1,x_2-1) = 0. \tag{14}$$

Multiplying both sides of (14) *by* $\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}}$ *and summing over* $(x_1,x_2) \geqslant (1,1)$ *yields*

$$\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} - \sum_{(x_1,x_2)\geqslant(1,1)} f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} - \sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} = 0.$$

We consider each sum separately:

$$\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} = F(;;z_1,z_2) - F(;;0,z_2) - F(;;z_1,0) + F(;;0,0);$$

$$\begin{aligned}\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} &= \sum_{(x_1,x_2)\geqslant(0,1)} f(x_1,x_2)\xi_1^{x_1+1}\xi_2^{x_2} z_1^{\underline{\ell_1 (x_1+1)}} z_2^{\underline{\ell_2 x_2}} = \\ &= \xi_1 z_1^{\underline{\ell_1}} \rho_1^{\ell_1} \sum_{(x_1,x_2)\geqslant(0,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} \\ &= \xi_1 z_1^{\underline{\ell_1}} \rho_1^{\ell_1} \big(F(;;z_1,z_2) - F(;;z_1,0)\big);\end{aligned}$$

$$\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2} z_1^{\underline{\ell_1 x_1}} z_2^{\underline{\ell_2 x_2}} = \xi_2 z_2^{\underline{\ell_2}} \rho_2^{\ell_2} \big(F(;;z_1,z_2) - F(;;0,z_2)\big).$$

Finally, we obtain

$$F(;;z_1,z_2) - F(;;0,z_2) - F(;;z_1,0) + F(;;0,0) - \xi_1 z_1^{\underline{\ell_1}} \rho_1^{\ell_1} \big(F(;;z_1,z_2) - F(;;z_1,0)\big) - \xi_2 z_2^{\underline{\ell_2}} \rho_2^{\ell_2} \big(F(;;z_1,z_2) - F(;;0,z_2)\big) = 0,$$

which yields the functional equation on $F(;;z_1,z_2)$:

$$
\begin{aligned}
&(1-\xi_1 z_1^{\ell_1}\rho_1^{\ell_1}-\xi_2 z_2^{\ell_2}\rho_2^{\ell_2})F(;;z_1,z_2)\\
&\qquad -(1-\xi_2 z_2^{\ell_2}\rho_2^{\ell_2})F(;;0,z_2)-(1-\xi_1 z_1^{\ell_1}\rho_1^{\ell_1})F(;;z_1,0)+F(;;0,0)=0.
\end{aligned}
$$

Example 2. *We consider a difference equation with polynomial coefficients whose solution is a p-recursive series [23]:*

$$
f(x_1,x_2)-(1+x_1x_2)f(x_1-1,x_2)-x_2^2 f(x_1,x_2-1)=0. \tag{15}
$$

Multiplying both sides of (15) by $\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}$ and summing over $(x_1,x_2)\geqslant(1,1)$ yields

$$
\begin{aligned}
&\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}-\sum_{(x_1,x_2)\geqslant(1,1)}(1+x_1x_2)f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}\\
&\qquad -\sum_{(x_1,x_2)\geqslant(1,1)} x_2^2 f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}=0.
\end{aligned}
$$

The first sum is the same as in the previous example. We consider the second and third sum separately:

$$
\begin{aligned}
&\sum_{(x_1,x_2)\geqslant(1,1)}(1+x_1x_2)f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}\\
&\qquad =\sum_{(x_1,x_2)\geqslant(0,1)}(1+(x_1+1)x_2)f(x_1,x_2)\xi_1^{x_1+1}\xi_2^{x_2}z_1^{\ell_1(x_1+1)}z_2^{\ell_2 x_2}\\
&\qquad =(1+(\theta_1+1)\theta_2)\xi_1 z_1^{\ell_1}\rho_1^{\ell_1}\sum_{(x_1,x_2)\geqslant(0,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}\\
&\qquad =(1+(\theta_1+1)\theta_2)\xi_1 z_1^{\ell_1}\rho_1^{\ell_1}\big(F(;;z_1,z_2)-F(;;z_1,0)\big);\\
&\sum_{(x_1,x_2)\geqslant(1,1)} x_2^2 f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1 x_1}z_2^{\ell_2 x_2}\\
&\qquad =(\theta_2+1)^2\xi_2 z_2^{\ell_2}\rho_2^{\ell_2}\big(F(;;z_1,z_2)-F(;;0,z_2)\big),
\end{aligned}
$$

which yields the functional equation

$$
\begin{aligned}
&\big(1-(1+\theta_1\theta_2+\theta_2)\xi_1 z_1^{\ell_1}\rho_1^{\ell_1}-(\theta_2+1)^2\xi_2 z_2^{\ell_2}\rho_2^{\ell_2}\big)F(;;z_1,z_2)\\
&\qquad -\big(1-(1+\theta_1\theta_2+\theta_2)\xi_1 z_1^{\ell_1}\rho_1^{\ell_1}\big)F(;;0,z_2)\\
&\qquad\qquad -\big(1-(\theta_2+1)^2\xi_2 z_2^{\ell_2}\rho_2^{\ell_2}\big)F(;;z_1,0)+F(;;0,0)=0.
\end{aligned}
$$

5. Conclusions

We have initiated the theory of discrete generating series for multidimensional polynomial coefficient difference equations. We introduced a multidimensional polynomial shift operator and established three functional equations that these new discrete generating series obey, revealing some of their structural properties. A strong direction for future research is to generalize to the time scales calculus [24]. The falling factorial functions here are called generalized h_k polynomials in time scales. This suggests some directions for the time scales analogue of this research, which was arguably anticipated with the definition of a moment-generating series for distributions in [25]. One particularly interesting question is what the proper analogue of (1) is for an arbitrary time scale, and perhaps analysis from a generating series perspective would reveal new insights to this problem.

Author Contributions: Conceptualization: T.C. and A.P.L.; methodology: S.S.A. and T.C.; validation: S.S.A.; investigation: A.P.L.; writing—original draft: A.P.L.; writing—review and editing: T.C.; project administration: A.P.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-02-2023-936).

Data Availability Statement: Data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alekseev, V.S.; Akhtamova, S.S.; Lyapin, A.P. Discrete generating functions. *Math. Notes* **2023**, *114*, 5–6.
2. Nekrasova, T.I. On the Cauchy problem for multidimensional difference equations in rational cone. *J. Sib. Fed. Univ. Math. Phys.* **2015**, *8*, 184–191. [CrossRef]
3. Stanley, R. *Enumerative Combinatorics*; Cambridge Univ. Press: Cambridge, MA, USA , 1999; Volume 1.
4. Bousquet-Mélou, M.; Petkovšek, M. Linear recurrences with constant coefficients: The multivariate case. *Discret. Math.* **2000**, *225*, 51–75.
5. Chandragiri, S. Difference equations and generating functions for some lattice path problems. *J. Sib. Fed. Univ. Math. Phys.* **2019**, *12*, 551–559. [CrossRef]
6. Chandragiri, S. Counting lattice paths by using difference equations with non-constant coefficients. *Bull. Irkutsk. State Univ. Ser. Math.* **2023**, *44*, 55–70. [CrossRef]
7. Abramov, S.A.; Barkatou, M.A.; Petkovšek, M. Linear Difference Operators with Coefficients in the Form of Infinite Sequences. *Comput. Math. Math. Phys.* **2021**, *61*, 1582–1589. [CrossRef]
8. Kruchinin, D.; Kruchinin, V.; Shablya, Y. Method for obtaining coefficients of powers of bivariate generating functions. *Mathematics* **2021**, *9*, 428. [CrossRef]
9. Kruchinin, D.; Kruchinin, V.; Shablya, Y. Method for obtaining coefficients of powers of multivariate generating functions. *Mathematics* **2023**, *11*, 2859. [CrossRef]
10. Krasikov, V.A. A Survey on computational aspects of polynomial amoebas. *Math. Comput. Sci.* **2023**, *17*, 16. [CrossRef]
11. Leinartas, E.K. Multiple Laurent series and difference equations. *Sib. Math. J.* **2004**, *45*, 321–326. [CrossRef]
12. Leinartas, E.K.; Passare, M.; Tsikh, A.K. Multidimensional versions of Poincare's theorem for difference equations. *Sb. Math.* **2008**, *199*, 1505–1521. [CrossRef]
13. Kostić, M.; Koyuncuoğlu, H.C. Multi-dimensional almost automorphic type sequences and applications. *Georgian Math. J.* **2023.** . [CrossRef]
14. Dudgeon, D.E.; Mersereau, R.M. *Multidimensional Digital Signal Processing*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1983.
15. Daubechies, I. *Ten Lectures on Wavelets*; SIAM: Philadelphia, PA, USA, 1992.
16. Apanovich, M.S.; Leinartas, E.K. Correctness of a two-dimensional Cauchy problem for a polynomial difference operator with constant coefficients. *J. Sib. Fed. Univ. Math. Phys.* **2017**, *10*, 199–205. [CrossRef]
17. Apanovich, M.S.; Leinartas, E.K. On correctness cf Cauchy problem for a polynomial difference operator with constant coefficients. *Bull. Irkutsk. State Univ. Ser. Math.* **2018**, *26*, 3–15. [CrossRef]
18. Yakovleva, T.I. Well-posedness of the Cauchy problem for multidimensional difference equations in rational cones. *Sib. Math. J.* **2017**, *58*, 363–372. [CrossRef]
19. Leinartas, E.K.; Lyapin, A.P. On the rationality of multidimensional recursive series. *J. Sib. Fed. Univ. Math. Phys.* **2009**, *2*, 449–455.
20. Lyapin, A.P.; Chandragiri, S. Generating functions for vector partition functions and a basic recurrence relation. *J. Differ. Equ. Appl.* **2019**, *25*, 1052–1061. [CrossRef]
21. Bohner, M.; Cuchta, T. The generalized hypergeometric difference equation. *Demonstr. Math.* **2018**, *51*, 62–75. [CrossRef]
22. Cuchta, T.; Grow, D.; Wintz, N. Discrete matrix hypergeometric functions. *J. Math. Anal. Appl.* **2023**, *518*, 126716. [CrossRef]
23. Lipshitz, L. D-finite power series. *J. Algebra* **1989**, *122*, 353–373. [CrossRef]
24. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales, an Introduction with Applications*; Birkhäuser Boston, Inc.: Boston, MA, USA , 2001.
25. Matthews, T. Probability Theory on Time Scales and Applications to Finance and Inequalities. Ph.D. Thesis, Missouri University of Science, Rolla, MO, USA, 2011.