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Tom Cuchta

Dallas Freeman

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DISCRETE POLYLOGARITHM FUNCTIONS

Tom Cuchta[1,2] — Dallas Freeman[2]

[1]Dept. of Mathematics, Marshall University, Huntington, West Virginia, USA

[2]Dept. of Computer Science and Mathematics, Fairmont State University, Fairmont, West Virginia, USA

ABSTRACT. We investigate a discrete analogue of the polylogarithm function. Difference and summation relations are obtained, as well as its connection to the discrete hypergeometric series.

1. Introduction

Discrete special functions and their applications have been the topic of numerous papers in recent years, see e.g., discrete analogues of Bessel and hypergeometric functions [2, 3, 6], analogues of orthogonal polynomials [4, 5, 8], semidiscrete multivariable models [12–14], and discrete models of physics [1].

On the other hand, the classical polylogarithm function

$$\mathcal{L}\mathrm{i}_s(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^s}.$$

has seen numerous applications in diverse areas such as Fermi-Dirac integrals [16], conformal field theory [11], thermoelectrics [17], and blackbody radiation [15].

We are interested in expanding the theory of discrete special functions to include discrete polylogarithm functions, which we define by

$$\mathrm{Li}_s(t; n, \xi) = \sum_{k=1}^{\infty} \frac{t^{\underline{nk}} \xi^k}{k^s}, \tag{1}$$

where $s, \xi \in \mathbb{C}$ and $n \in \mathbb{R}$, which is the discrete analogue of the polylogarithm function.

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2. Preliminaries and definitions

The forward difference operator Δ is defined by $\Delta f(t) = f(t+1) - f(t)$. The discrete shift for $n, m \in \{0, 1, \ldots\}$ is the relation

$$t^{\underline{n}}(t-n)^{\underline{m}} = t^{\underline{n+m}}. \tag{2}$$

The discrete fundamental theorem of calculus is

$$\sum_{k=a}^{b} \Delta f(k) = f(b+1) - f(a). \tag{3}$$

The classical generalized hypergeometric series is defined by

$${}_p\mathcal{F}_q(\mathbf{a}; \mathbf{b}; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{t^k}{k!},$$

where $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$ for some constants $a_i, b_j \in \mathbb{C}$. The discrete hypergeometric series is

$${}_pF_q(\mathbf{a}; \mathbf{b}; t, n, \xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{t^{\underline{nk}} \xi^k}{k!}.$$

It is known [3, Proposition 2] that the ${}_pF_q$ and ${}_p\mathcal{F}_q$ are related by

$${}_pF_q(\mathbf{a}; \mathbf{b}; t, n, \xi) = {}_{p+n}\mathcal{F}_q\big(\mathbf{a}, \mathbf{t}; \mathbf{b}; \xi(-n)^n\big), \tag{4}$$

where

$$\mathbf{t} = \left(\frac{-t}{n}, \frac{-t+1}{n}, \ldots, \frac{-t+n-1}{n} \right).$$

The falling factorials are defined in terms of the Γ function by

$$a^{\underline{b}} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)}, \tag{5}$$

and ratios of Γ functions obey the following known asymptotic relationship [7]

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \sim x^{\beta} \tag{6}$$

The related Pochhammer symbols are defined for $a \in \mathbb{C}$ by

$$(a)_k = a(a+1)\ldots(a+k-1).$$

The polylogarithm obeys many interesting formulas that can be found in the books [9, 10]. We now express many such properties for the $\mathcal{L}\mathrm{i}$:

$$\mathcal{L}\mathrm{i}_n(t) = t\,{}_{n+1}\mathcal{F}_n(1, 1, \ldots, 1; 2, 2, \ldots, 2; t), \tag{7}$$

$$t\frac{\partial \mathcal{L}\mathrm{i}}{\partial t} = \mathcal{L}\mathrm{i}_{s-1}(t), \tag{8}$$

$$\mathcal{L}\mathrm{i}(t) = \int_0^t \frac{\mathcal{L}\mathrm{i}_s(\tau)}{\tau}\mathrm{d}\tau, \tag{9}$$

and

$$\sum_{j=0}^{p-1} \mathcal{L}\mathrm{i}_s\left(ze^{\frac{2\pi ij}{p}}\right) = p^{1-s}\mathcal{L}\mathrm{i}_s(z^p). \tag{10}$$

3. Discrete polylogarithms

If $t \in \{0, 1, \ldots\}$, then the series (1) converges, since the factor $t^{\underline{nk}}$ will ultimately vanish for sufficiently large k. Now we establish convergence for complex t.

Theorem 3.1. *If $t \notin \{0, 1, \ldots\}$ and $n \in \{0, 1, 2, \ldots\}$, then the series* (1) *converges for $|t| < \sqrt[n]{\frac{1}{|\xi|}}$.*

P r o o f. We assume that $t \notin \{0, 1, 2, \ldots\}$ because such t-values cause the series to terminate due to the factor of $t^{\underline{nk}}$ in the summand. To apply the ratio test, set $a_k = \frac{t^{\underline{nk}}\xi^k}{k^s}$ and consider the limit

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{t^{\underline{nk+n}}\xi^{k+1}k^s}{(k+1)^s t^{\underline{nk}}\xi^k}\right| \\
&= \lim_{k\to\infty}\left|\frac{\Gamma(t-nk+1)\xi k^s}{\Gamma(t-nk-n+1)(k+1)^s}\right| \\
&\approx \lim_{k\to\infty}\left|\xi\left(\frac{k}{k+1}\right)^s t^{-nk+1-(-nk-n+1)}\right| \\
&= \lim_{k\to\infty}\left|\xi\left(\frac{k}{k+1}\right)^s t^n\right| = |\xi t^n|.
\end{aligned}$$

Hence the series converges whenever $|t| < \sqrt[n]{\frac{1}{|\xi|}}$, completing the proof. □

When the subscript is a non-negative integer, the series (1) reduces to a discrete hypergeometric function, analogous to (7).

Theorem 3.2. *If $m, n \in \{0, 1, \ldots\}$, then*

$$\mathrm{Li}_m(t; n, \xi) = \xi t^{\underline{n}}\,{}_{m+1}F_m(1, 1, \ldots, 1; 2, 2, \ldots, 2; t-n, n, \xi). \tag{11}$$

P r o o f. Using (2), compute

$$\xi t^{\underline{n}}{}_{m+1}F_m\left(1,\ldots,1;2,\ldots,2;t-n,n,\xi\right)=\xi t^{\underline{n}}\sum_{k=0}^{\infty}\frac{(1)_k\ldots(1)_k}{(2)_k\ldots(2)_k}\frac{(t-n)^{\underline{nk}}\xi^k}{k!}$$

$$=\sum_{k=0}^{\infty}\frac{(k!)^{m+1}}{\big((k+1)!\big)^m}\frac{t^{\underline{n(k+1)}}\xi^{k+1}}{k!}=\sum_{k=1}^{\infty}\frac{t^{\underline{nk}}\xi^k}{k^m}=\mathrm{Li}_m(t;n,\xi),$$

completing the proof. □

The previous theorem implies a representation as a classical generalized hypergeometric series via (4).

Corollary 3.3. *If* $m,n\in\{0,1,\ldots\}$, *then*

$$\mathrm{Li}_m(t;n,\xi)=\xi t^{\underline{n}}{}_{m+1+n}\mathcal{F}_m\big(1,1,\ldots,1,\mathbf{t};2,\ldots,2;\xi(-n)^n\big),$$

where $\mathbf{t}\in\mathbb{R}^{1\times n}$ *with*

$$\mathbf{t}=\left(\frac{-t+n}{n},\frac{-t+n+1}{n},\ldots,\frac{-t+2n-1}{n}\right).$$

The following theorem is a discrete analogue of (8).

Theorem 3.4. *The functions* (1) *obey the formula*

$$t\Delta\mathrm{Li}_s(t-1;n,\xi)=n\mathrm{Li}_{s-1}(t;n,\xi).\tag{12}$$

P r o o f. Compute

$$t\Delta\mathrm{Li}_s(t-1;n,\xi)=t\Delta\sum_{k=1}^{\infty}\frac{\xi^k(t-1)^{\underline{nk}}}{k^s}=tn\sum_{k=1}^{\infty}\frac{\xi^k(t-1)^{\underline{nk-1}}}{k^{s-1}}$$

$$=n\sum_{k=1}^{\infty}\frac{\xi^k t^{\underline{nk}}}{k^{s-1}}=n\mathrm{Li}_{s-1}(t;n,\xi),$$

completing the proof. □

The following corollary is a discrete analogue of (9).

Corollary 3.5. *The functions* (1) *obey the formula*

$$\mathrm{Li}_s(t-1;n,\xi)=n\sum_{k=1}^{t-1}\frac{\mathrm{Li}_{s-1}(k;n,\xi)}{k}.$$

P r o o f. Divide (12) by t and sum to obtain

$$\sum_{k=1}^{t-1}\Delta\mathrm{Li}_s(k-1;n,\xi)=n\sum_{k=1}^{t-1}\frac{\mathrm{Li}_{s-1}(k;n,\xi)}{k}.$$

Since $\mathrm{Li}_s(0; n, \xi) = 0$, applying (3) on the left-hand side yields

$$\mathrm{Li}_s(t-1; n, \xi) = n \int_1^t \frac{\mathrm{Li}_{s-1}(\tau; n, \xi)}{\tau} \Delta\tau.$$

Recognizing the integral on the right-hand side as a sum completes the proof. □

The summation (10) has the following discrete analogue.

Theorem 3.6. *If* $p \in \{1, 2, 3, \ldots\}$, *then*

$$\sum_{j=0}^{p-1} \mathrm{Li}_n\left(t, n, \xi e^{\frac{2\pi i j}{p}}\right) = p^{1-n}\mathrm{Li}_n(t, np, \xi^p).$$

Proof. Calculate

$$\sum_{j=0}^{p-1} \mathrm{Li}_s\left(t; n, \xi e^{\frac{2\pi i j}{p}}\right) = \sum_{j=0}^{p-1}\sum_{k=1}^{\infty} \frac{t^{\underline{nk}}\xi^k e^{\frac{2\pi k i j}{p}}}{k^s} \sum_{k=1}^{\infty} \frac{t^{\underline{nk}}\xi^k}{k^s} \sum_{j=0}^{p-1} e^{\frac{2\pi k i j}{p}}$$

By the well-known sum of roots of unity

$$\sum_{j=0}^{p-1} e^{\frac{2\pi k i j}{p}} = \begin{cases} p, & k|p, \\ 0, & k \not| p. \end{cases}$$

we obtain

$$\sum_{j=0}^{p-1} \mathrm{Li}_n\left(t, n, \xi e^{\frac{2\pi i j}{p}}\right) = p \sum_{k=1, k|p}^{\infty} \frac{t^{\underline{nk}}\xi^k}{k^s} = p \sum_{\ell=1}^{\infty} \frac{t^{\underline{np\ell}}\xi^{p\ell}}{(p\ell)^s}$$

$$= p^{1-s} \sum_{\ell=1}^{\infty} \frac{t^{\underline{(np)\ell}}\left(\xi^p\right)^\ell}{\ell^s} = p^{1-s}\mathrm{Li}_s(t, np, \xi^p),$$

completing the proof. □

4. Conclusion

We have established discrete analogues of many of the properties of the polylogarithm functions. Future work can expand into looking at applications of these functions, understanding special cases such as representations when n is a negative integer, and other discrete analogues such as inverse tangent integrals and the Legendre χ function.

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Department of Mathematics
Marshall University
1 John Marshall Drive
Huntington, West Virginia 25755
USA
E-mail: cuchta@marshall.edu

Dept. of Computer Science and Mathematics
Fairmont State University
Fairmont, West Virginia 26554
USA
E-mail: dallas.freeman@gmail.com