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New Numerical Code for Black Hole Initial Data

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HyperSolID

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Outline

Problems

Main Problems

- There are no no exact solutions of Einstein Equations that describes a bound system radiating gravitational waves.
- One needs to resort to numerical simulations, or analytical approximation methods.
- Current methods to constrained initial data exhibit *junk radiation* and *ambiguities* about constrained and free data.

Possible Solutions

- It was mathematically proved that given the correct initial data, Einstein equation will yield the expected solution.
- New formulations to calculate the constrains ensuring that the numerical system is well-behaved (hyperbolic equations).

Approaches

Standard Approach

The constraints are formulated as elliptic equations (ex: the conformal thin sandwich method, the gluing technique, etc.)

$$D_j K^j{}_i - D_i K^j{}_j = 0, \quad {}^{(3)}R + \left(K^j{}_j\right)^2 - K_{ij}K^{ij} = 0,$$

New Approach

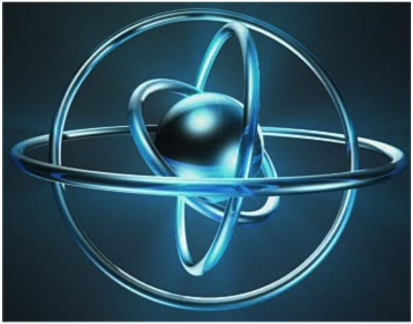
The 3D initial surfaces are further foliated into 2D surfaces, where the constraints form an algebraic-hyperbolic system:

$$\mathcal{L}_{\hat{n}}\mathbf{K} - \hat{D}^l\mathbf{k}_l + F_{\mathbf{K}} = 0, \quad \mathcal{L}_{\hat{n}}\mathbf{k}_i + \mathbf{K}^{-1}(\boldsymbol{\kappa}\hat{D}_i\mathbf{K} - 2\mathbf{k}^l\hat{D}_i\mathbf{k}_l) + F_{\mathbf{k_i}} = 0$$

$$\boldsymbol{\kappa} = (2\mathbf{K})^{-1}\left(2\mathbf{k}^l\mathbf{k}_l - \tfrac{1}{2}\mathbf{K}^2 - \boldsymbol{\kappa}_0\right)$$

HyperSolID

Hyperbolic Solver for Initial Data



Description

1. General 4D metric is given on the initial time slice
2. Free 3D initial data is constructed from the metric
3. Constrained 2D initial data given on the initial sphere

The constrained data is updated by solving the hyperbolic-algebraic system.

Outline

Equations

Evolution Equations

$$
\begin{aligned}
\partial_\rho U &= \tfrac{1}{2}\widetilde{N}\bar{\eth} U + \tfrac{1}{2}\overline{\widetilde{N}}\eth U \\
&+ \tfrac{1}{2}\widehat{N} d^{-1}\left[a(\eth\bar{v} + \bar{\eth} v) - b\bar{\eth}\bar{v} - \bar{b}\eth v\right] - F_U \\
\partial_\rho v &= \tfrac{1}{2}\widetilde{N}\bar{\eth} v + \tfrac{1}{2}\overline{\widetilde{N}}\eth v - \widehat{N} U^{-1} \\
&- \widehat{N} U^{-1}\left\{w\eth U - d^{-1}[(av - b\bar{v})\eth\bar{v} + (a\bar{v} - \bar{b}v)\eth v]\right\} - f_v
\end{aligned}
$$

Algebraic Equation

$$
w = (2U)^{-1}\left[d^{-1}(2av\bar{v} - b\bar{v}^2 - \bar{b}v^2) - \tfrac{1}{2}U^2 - \kappa_0\right]
$$

Variables

Known Variables

The known terms that enter in these formulas are:

$$\widehat{N}, (\widetilde{N}, \overline{\widetilde{N}}), (\eth\widehat{N}, \bar{\eth}\widehat{N}), (\eth\widetilde{N}, \bar{\eth}\widetilde{N}, \eth\overline{\widetilde{N}}, \bar{\eth}\overline{\widetilde{N}}),$$

$$a, (b, \overline{b}), d, (\eth a, \bar{\eth} a), (\eth b, \bar{\eth} b, \eth\overline{b}, \overline{\eth b}), (A, \overline{A}), (B, \overline{B}), (C, \overline{C})$$

$$\boldsymbol{\kappa}_0, \widehat{K}, \overset{\blacklozenge}{K}, (\overset{\diamond}{K}, \overline{\overset{\diamond}{K}}), (\mathring{K}, \overline{\mathring{K}}), \overset{\bullet}{K}, \eth\boldsymbol{\kappa}_0, (\eth\mathring{K}, \bar{\eth}\mathring{K})(\eth\overset{\bullet}{K}, \bar{\eth}\overset{\bullet}{K}).$$

Unknown Variables

The unknown variables are: (U, v, w) and their derivatives:.

$$(\eth U, \bar{\eth} U), (\eth v, \bar{\eth} v, \eth\overline{v}, \bar{\eth}\overline{v})$$

The angular derivatives are the Newman-Penrose $(\eth, \bar{\eth})$ operators

Calculations

Start-up data

We start with a general $4D$ metric, $(g_{ij}, \partial_t g_{ij})$, in Cartesian coordinates (x, y, z). With this data we calculate:

$$g^{ij}, n^i, h_{ij}, h^{ij} = g^{ij} + n^i n^j, (\partial_k h_{ij}, \partial_k h^{ij}, \partial_m \partial_k h_{ij})$$

$${}^3\Gamma^i_{jk}, {}^3R^i_{jlk}, {}^3R_{jk}, {}^3R, K_{ij}, K^i_i$$

Radial Foliation

We choose a simple spherical foliation $\rho = r$, where $r = \sqrt{\delta^{ij} x_i x_j}$. For this foliation we calculate the quantities :

$$\widehat{N} = (h^{ij}\partial_i\rho\partial_j\rho)^{-1/2}, \widehat{N}^i, \widehat{n}^i, \gamma_{ij} = h_{ij} - \widehat{n}_i\widehat{n}_j, \partial_k\gamma_{ij}, \partial_\rho\gamma_{ij}, \widehat{\Gamma}^i_{jk}, \widehat{K}_{ij}, \widehat{K}.$$

Separation

Extrinsic Curvature

The extrinsic curvature is decomposed in:

$$K_{ij} = \boldsymbol{\kappa} \hat{n}_i \hat{n}_j + 2\hat{n}_{(i}\mathbf{k}_{j)} + \mathbf{K}_{ij}, \ \textit{where } \mathbf{K}_{ij} = \gamma_i^k \gamma_j^l K_{kl}$$

Constrained Data

$$\boldsymbol{\kappa} = \widehat{n}^i \widehat{n}^j K_{ij}, \ \mathbf{k}_i = \gamma_i^j \widehat{n}^k K_{jk}, \ \mathbf{K} = \gamma^{ij} \mathbf{K}_{ij}$$

Free Data

$$\mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\gamma_{ij}\mathbf{K}, \ \boldsymbol{\kappa}_0 = {}^3R - \mathring{\mathbf{K}}_{kl}\mathring{\mathbf{K}}^{kl}.$$

Transformation

Change of Coordinates

From Cartesian (x, y, z) to stereographic coordinates (r, q, p)

$$x = \Omega rq, y = \pm\Omega rp, z = \mp\tfrac{1}{2}\Omega r(-1+q^2+p^2),\ \Omega = (1+q^2+p^2)^{-1}.$$

The coordinates of the unit sphere metric $q_{ab} = \Omega^2\delta_{ab}$, are:

$$q^a = \Omega^{-1}(1, i),\ \ q_a = \Omega(1, i),$$

Transformed variables

We transform to the stereographic coordinates the variables:

$$(\gamma_{ij}, \widehat{K}_{ij}, \mathring{\mathbf{K}}_{ij}),\ (\widehat{N}^i, \mathbf{k}_i),\ (\widehat{N}, \widehat{K}, \mathbf{K}, \boldsymbol{\kappa}_0, \boldsymbol{\kappa})$$

Projection

Calculation of the stereographic projections

- Metric terms

$$
\begin{aligned}
&a = \tfrac{1}{2} q^i \overline{q}^j \gamma_{ij}, b = \tfrac{1}{2} q^i q^j \gamma_{ij}, d = a^2 - b\overline{b}, \\
&A = d^{-1} \left[a(2\eth a - \overline{\eth} b) - \overline{b} \eth b \right], B = d^{-1} \left[a \overline{\eth} b - b \eth \overline{b} \right] \\
&C = d^{-1} \left[a \eth b - b(2\eth a - \overline{\eth} b) \right], \widetilde{N} = q_i \widehat{N}^i .
\end{aligned}
$$

- Given terms

$$
\overset{\blacklozenge}{K} = q^i \overline{q}^j \widehat{K}_{ij}, \overset{\lozenge}{K} = q^i q^j \widehat{K}_{ij}, \overset{\bullet}{K} = q^i \overline{q}^j \mathring{\mathbf{K}}_{ij}, \mathring{K} = q^i q^j \mathring{\mathbf{K}}_{ij}.
$$

- Updated terms

$$
U = \mathbf{K}, \; v = q^i \mathbf{k}_i, \; w = \boldsymbol{\kappa}.
$$

Outline

Algorithm

Algebra

1. Module for calculating $(h^{ij}, {}^3R, K_{ij}, K_i^i)$.
2. Module for calculating $(\widehat{N}, \widehat{N}^i, \gamma^{ij}, \widehat{K}_{ij}, \widehat{K}, \mathring{\mathbf{K}}_{ij}, \boldsymbol{\kappa}_0)$ and $\mathbf{k}_i, \mathbf{K}, \boldsymbol{\kappa}$.
3. Module for transformation to stereographic coordinates

Integration

1. Module for the stereographic projections and source terms
2. Module for updating the evolution and algebraic equations

Analysis

1. Module for implementing exact and perturbative test cases
2. Module for calculating errors and convergence rates

Numerics

Numerical Integration: Runge-Kutta 2^{nd} and 4^{th} Order

- Flexible, can start and stop at any given radius
- Allows update of the algebraic equation inside the integrator
- Incorporates internally two types of CFL conditions given by:
  1. Constant (linear) radial grid step $h \propto dqdp$
  2. Variable (logarithmic) radial grid step $h \propto rhodqdp$

Radial dissipation and stereographic interpolation

$$\partial_\rho \to \partial_\rho + \epsilon \eth\bar{\eth} \text{ , and } \partial_\rho \to \partial_\rho - \epsilon^4 \eth^2 \bar{\eth}^2.$$

Spatial Derivatives: Finite-Difference 2^{nd} and 4^{th} Order

- Radial derivatives: $\partial_k \to \Delta_k + \mathcal{O}(\frac{h^n}{\sqrt{3}})$, $\partial_\rho \to \partial_\rho x^k \partial_k$
- Angular derivatives: $\eth$ module previously implemented.

Kerr-Schild Metric

The metric is given on the form: $g_{ab} = \eta_{ab} + 2H\ell_a\ell_b,\ H = \frac{M}{r}$

1. Exact Schwarzschild in centered cartesian coordinates

$$(t, x^i),\ l_a = \left(1, \frac{x_i}{r}, \right),\ r = \rho = \sqrt{\delta_{ij}x^i x^j}$$

2. Exact Schwarzschild in shifted cartesian coordinates

$$(t, \tilde{x} = x - x_0,\ \tilde{y} = y - y_0, z),\ l_a = \left(1, \frac{\tilde{x}_i}{r}, \right),\ r = \sqrt{\delta_{ij}\tilde{x}^i\tilde{x}^j}$$

3. Boosted Schwarzschild in centered cartesian coordinates

$$(\tilde{t} = \gamma(t - vx),\ \tilde{x} = \gamma(x - vt), y,\ z),\ r = \sqrt{\delta_{ij}\tilde{x}^i\tilde{x}^j},\ l_a = \Gamma^b_a\tilde{l}_b,$$

4. Perturbed Schwarzschild in centered cartesian coordinates

$$K^i_i \to K^i_i\left(1 + \frac{Y_{20}}{100}\right)$$

Outline

Timeline

Most of the code is already implemented

The radial foliation, the separation of variables, the stereographic projection, the source terms, the 2^{nd} order integrator.

The code passed two tests

The centered and perturbed Schwarzschild tests prove a clean second order convergence, and long time stability.

Work in progress

- Test the code with the shifted and the boosted Schwarzschild
- Implement and test the 4^{th} order radial integration
- Improve the angular derivative (Fourier or spherical harmonics)
- Implement and test ${}^3R, K_{ij}, K_i^i, \widehat{K}_{ij}, \widehat{K}$