1-1-2012

# On the Logic of Reverse Mathematics

Alaeddine Saadaoui
saadaoui@live.marshall.edu

# On the Logic of Reverse Mathematics

A thesis submitted to

the Graduate College of

Marshall University

In partial fulfillment of

the requirements for the degree of

Master of Arts in Mathematics

by
Alaeddine Saadaoui

Approved by

Dr. Carl Mummert, Committee Chairperson
Dr. John Drost
Dr. Michael Schroeder

Marshall University
May 2012

## ACKNOWLEDGMENTS

I would like to acknowledge and extend my heartfelt gratitude to the following persons who have made the completion of this thesis possible:

The Chair of the mathematics department Dr. Alfred Arkinsete, for his vital encouragement and support.

My advisor Dr. Bonita Lawrence, for her understanding and assistance.

My thesis supervisor Dr. Carl Mummert, who has supported me throughout my thesis with his patience and his knowledge.

My mentor Dr. Anna Mummert, for the help that enriched my teaching experience.

All the math department faculty members and staff, for their assistance and help.

Most especially to my family and friends who have supported and encouraged me with their best wishes.

And to God, who made all things possible.

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## ABSTRACT

On the Logic of Reverse Mathematics

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The goal of reverse mathematics is to study the implication and non-implication relationships between theorems. These relationships have their own internal logic, allowing some implications and non-implications to be derived directly from others. The goal of this thesis is to characterize this logic in order to capture the relationships between specific mathematical works.

The results of our study are a finite set of rules for this logic and the corresponding soundness and completeness theorems. We also compare our logic with modal logic and strict implication logic. In addition, we explain two applications of S-logic in topology and second order arithmetic.

# Chapter 1

# INTRODUCTION

The goal of reverse mathematics is to find the weakest set of axioms that are capable of proving a theorem $T$ and to characterize the exact strength of these axioms. To start, we build a weak axiom system $B$ known as the **base system** which is too weak to prove $T$, but strong enough to state it. The next step consists of finding an axiom system $A$ that is stronger than $B$ and is able to prove $T$. To show that $A$ is *required* to prove $T$, we need to show that $T$ is provable from $A$ and $T$ itself implies $A$, if $B$ is assumed. The second proof shows that $A$ is the weakest axiom system able to prove $T$. In other words, any system that proves $T$ extends $A$.

In reverse mathematics, researchers usually use subsystems of second order arithmetic to formalize the base system and theorems they study. In second order arithmetic, objects are natural numbers or sets of natural numbers. The most frequently used subsystems of second order arithmetic in reverse mathematics are known as the "big five subsystems of second order arithmetic" and are described by Simpson [4]. These subsystems are ordered in terms of logical strength, which means that a system $T$ is stronger than a system $S$ if $B + T$ proves $S$. In our study, we will be interested in these relationships. We will build a deductive system based on a logic that we call S-logic. We will also see that S-logic is different from propositional logic.

To manage and organize relations among reverse mathematical principles, we can use applications created for this purpose. For instance, the Reverse Mathematics Zoo program, created by Damir Dzhafarov,[1] generates a diagram showing known results. The relation could be an implication, a strict implication or a non-implication. A strict implication means that if we have two systems $A$ and $B$, then $A$ *strictly implies* $B$ when $A$ implies $B$ and $B$ does not imply $A$. Figure 1 shows an example of a diagram, which we will not discuss, generated by the Reverse Mathematics Zoo program. The program also regenerated diagrams that were created by hand. The input of the program is a database containing known facts about reverse mathematical principles. By parsing the database in different ways, different diagrams are generated. The benefits of this program are to visualize and capture the relations between different mathematical theorems. However, Dzhafarov indicates that the program has optimization issues that cannot be solved without knowledge of the logic behind the relationships. The goal of our study is to determine the logic behind the relationships between mathematical works.

Our work extends a previous study by Sean Sovine [5], who focused on particular implication and non-implication statements that have the forms of $A \vdash B$ and $A \nvdash B$, and discovered two inference rules. However, the statements studied by Sovine do not contain conjunctions in their hypotheses. We will extend his work by studying implication and non-implication statements which may have conjunctions in their hypotheses.

## 1.1 Propositional logic

In this section, we will introduce propositional logic, which is a standard logic used by logicians. We have two reasons for giving detailed introduction to this logic. The first reason is that the structure of a statement in S-logic is different from the structure of a statement in propositional logic. The second reason is that the semantics of non-implication statements in S-logic are different from the semantics of non-implication statements in propositional logic. We will define the language of S-logic in chapter 2, at which point we will describe the differences between the two logics.

**Propositional logic**, also called sentential logic, is a formal logic used to study specific

[1] The zoo program, the database and the created diagrams are available on the website `http://www.nd.edu/~ddzhafar/The_Zoo.html`.

propositions, also called statements or sentences, to derive more complicated propositions and study the logical properties to derive them. Enderton [1] provides a general reference for propositional logic. The derivation is done by using specific formation rules called **inference rules**, and the result of the derivation is called a **theorem**. The derivation can be considered as a proof of the theorem. The interpretation of a proposition is limited to two possible values: true or false, and its truth or falsity is determined by the truth or falsity of its atomic parts.

## 1.2 Language of propositional logic

### Statements and symbols

In propositional logic, a statement is a sequence of objects called **symbols**, and the set of all possible symbols is called the **alphabet**. We have two types of symbols: **variables** and **sentential connective symbols**. A sentence symbol, also called a variable, is usually represented by a letter used to refer to a proposition that is part of the original one. A sentence symbol could have a value of true or false, but it does not provide a logical description of the relationship between the different parts of a statement. A sentential connective symbol, also called a logical symbol, is used to define the logical relationship within a sentence. The logical symbols are the negation symbol $\neg$, the conjunction symbol $\wedge$, the disjunction symbol $\vee$ (inclusive or), the conditional symbol $\rightarrow$ (if..then..) and the bi-conditional symbol $\leftrightarrow$ (if and only if). To group a subset of symbols, we use the left and right parentheses.

In the following examples, we will take declarative English sentences to translate them to propositions.

**Example 1.1.** *As a first example, the sentence "The weather is sunny" could be translated in propositional logic to one non logical symbol, say, $A$. So the sentence "The weather is not sunny" is a negation of the first sentence and it will be $\neg A$. If we have a compound sentence, we break it down into atomic parts that can be expressed in propositional logic. For example, the sentence "The weather is sunny and the sky is blue" could be divided to* *two parts "The weather is sunny" and "The sky is blue". We already assigned a symbol $A$ to "The weather is sunny" and we will assign a second symbol $B$ to "The sky is blue". The two parts of the sentence are connected by the conjunction "and" which will be translated to the connective symbol $\wedge$ to get the proposition $A \wedge B$ for the whole sentence.*

The sequence of symbols of a sentence has to be written in a defined way by using specific rules. Such sequences are called **well formed formulas** or wffs. For example, the formula "$(A_1 \rightarrow A_2 \vee A_3)$" is a wff and the sequence "$) \rightarrow A_2))$" is not.

A symbol cannot be a finite sequence of two or more symbols. For this reason we call a finite sequence of symbols an **expression**. For example $(\neg A_1)$ is an expression defined by the sequence of symbols $\{(, \neg, A_1, )\}$.

### Truth assignment

A **truth assignment** for a set $S$ of sentence symbols is a function $v$ defined by $v : S \rightarrow \{F, T\}$. $F$ is called **falsity** and $T$ is called **truth**. We also use 0 and 1 instead of $F$ and $T$, respectively. The function $v$ assigns to each symbol of $S$ either $T$ or $F$. When $v(A) = T$, we say that $v$ **satisfies** $A$.

**Example 1.2.** *Let $S$ be a set of symbols such that $S = \{A_1, A_2, A_3\}$. One truth assignment for $S$ is the $v$ such that $v(A_1) = T$, $v(A_2) = F$ and $v(A_3) = F$.*

### Truth assignment for well formed formulas

The well formed formulas are built by using the sentence and connective symbols. So we can assign the values $F$ or $T$ to a well formed formula. Suppose $\bar{S}$ is the set of wffs built by using the symbols from $S$. We will define the truth assignment $\bar{v} : \bar{S} \rightarrow \{F, T\}$ such that:

1. $\forall A \in S, \bar{v}(A) = v(A)$.
2. $\bar{v}(\neg A) = \begin{cases} T & \text{if } \bar{v}(A) = F, \\ F & \text{if } \bar{v}(A) = T. \end{cases}$

3. $\bar{v}(A \wedge B) = \begin{cases} T & \text{if } \bar{v}(A) = T \text{ and } \bar{v}(B) = T, \\ F & \text{if } \bar{v}(A) = F \text{ or } \bar{v}(B) = F \text{ (or both).} \end{cases}$

4. $\bar{v}(A \vee B) = \begin{cases} T & \text{if } \bar{v}(A) = T \text{ or } \bar{v}(B) = T \text{ (or both),} \\ F & \text{if } \bar{v}(A) = F \text{ and } \bar{v}(B) = F. \end{cases}$

5. $\bar{v}(A \to B) = \begin{cases} F & \text{if } \bar{v}(A) = T \text{ and } \bar{v}(B) = F, \\ T & \text{if } \bar{v}(A) = F \text{ or if } \bar{v}(A) = T \text{ and } \bar{v}(B) = T. \end{cases}$

6. $\bar{v}(A \leftrightarrow B) = \begin{cases} T & \text{if } \bar{v}(A) = \bar{v}(B), \\ F & \text{if } \bar{v}(A) \neq \bar{v}(B). \end{cases}$

**Example 1.3.** *Let $S = \{A_1, A_2, A_3\}$, $\bar{S} = \{A_1 \vee A_2, \neg A_3 \wedge A_2, A_2 \to \neg A_1\}$. Let $v$ be a truth assignment for $S$ such that $v(A_1) = T$, $v(A_2) = F$ and $v(A_3) = T$. Let $\bar{v}$ be the truth assignment for $\bar{S}$. We have:*

1. *$v(A_1) = T$ implies $\bar{v}(A_1) = T$ and $v(A_2) = F$ implies $\bar{v}(A_2) = F$, so $\bar{v}(A_1 \vee A_2) = T$.*

2. *$v(A_3) = T$ implies $\bar{v}(A_3) = T$, so $\bar{v}(\neg A_3) = F$ and $\bar{v}(A_2) = F$ implies $\bar{v}(\neg A_3 \wedge A_2) = F$.*

3. *$\bar{v}(A_1) = T$, so $\bar{v}(\neg A_1) = F$ and $\bar{v}(A_2) = F$ implies $\bar{v}(A_2 \to \neg A_1) = T$.*

**Tautologies**

Let $S$ be a set of formulas and $A$ be a formula. We say that $S$ **tautologically implies** $A$ and we write $S \models A$ if every truth assignment that satisfies every formula of $S$ also satisfies $A$.

**Example 1.4.** *If we have $A \in S$ then $S \models A$. Because if we have a truth assignment that satisfies every symbol of $S$, it will also satisfy $A$ as $A \in S$.*

*Suppose $S = \{A, A \to B\}$. If there exist a truth assignment $\bar{v}$ such that $\bar{v}(A) = T$ and $\bar{v}(A \to B) = T$, we get $\bar{v}(B) = T$. Thus, we can conclude $S \models B$.*

## 1.3 Summary of results

In the remainder of this thesis, by studying the nature of implication and non-implication relations, we define a logic that is capable of stating these relations. As we have two types of relationships, we define the rules of each type, prove their soundness and their completeness.

The second chapter presents S-logic and contains two parts. The first part focuses on positive statements and defines their rules. The second part of this chapter focuses on negative statements and also presents its rules. For each rule, we prove its soundness and at the end of each part we prove the completeness of the rules.

The third chapter shows two applications of S-logic. The first example is an application in second order arithmetic and the second example is an application in topology. For each example, we define the field, its semantics, and we show all steps needed to interpret each field into S-logic. Also, we prove the properties of S-logic that we used for each example.

The fourth chapter presents a comparison between S-logic, modal logic and strict implication logic. First, we define each logic and describe its semantics. Second, we define the interpretation between the three logics. Finally, we discuss the difference between S-logic and the two other logics.

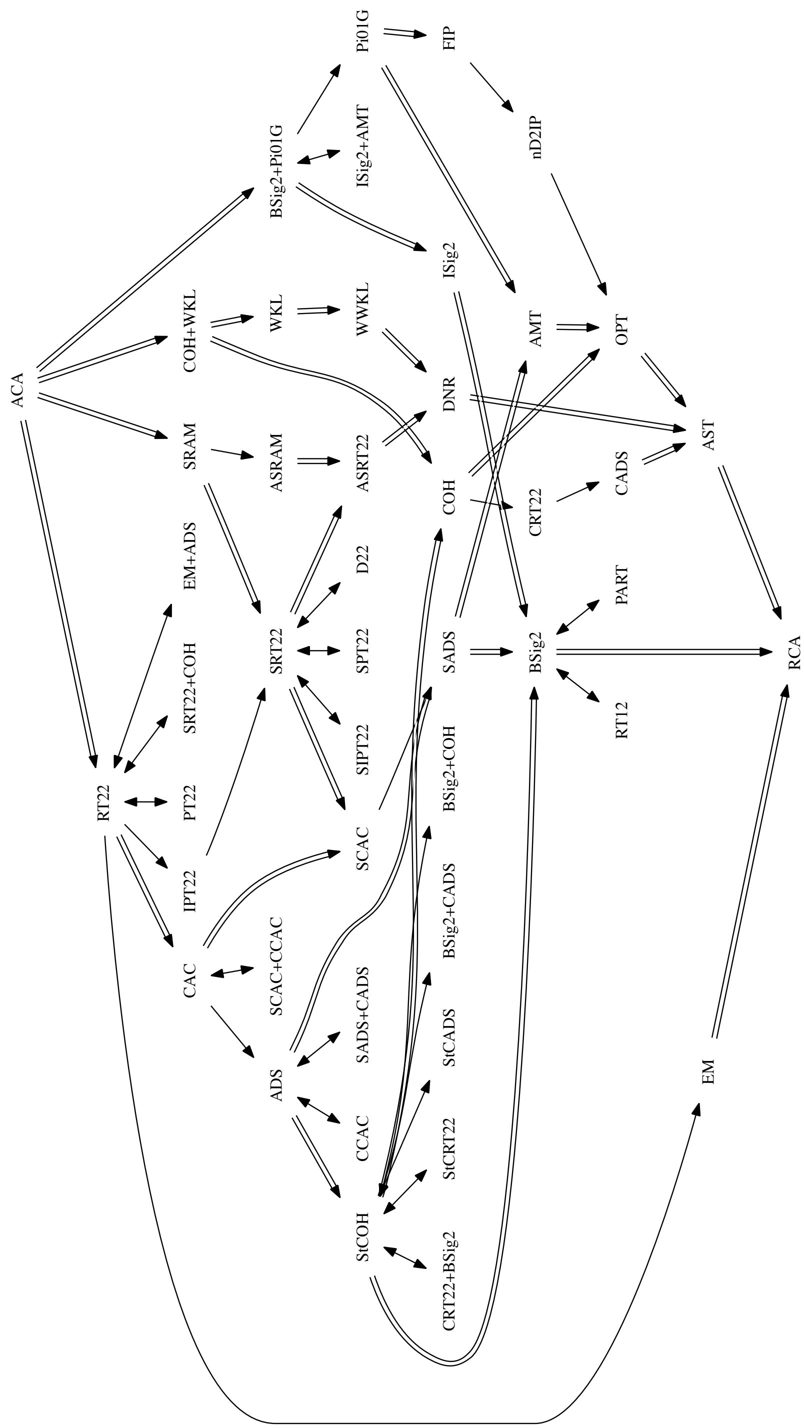


Figure 1: Diagram of implications in reverse mathematics generated by the Reverse Mathematics Zoo program. The diagram shows reverse mathematics of some mathematical principles. Arrows indicate implications and double arrows indicate strict implications.

# Chapter 2

# S-LOGIC

In this chapter we will introduce S-logic, a new logic we create to study the relationships between theorems. We first define the language of S-logic, its semantics, and several inference rules. We prove the soundness of each rule as it is defined, and at the end we prove the completeness of all the rules together.

This chapter is divided into three parts. The first part contains the definitions of the different terms of S-logic. The second part contains the rules for positive statements, and the last part contains the rules for negative statements.

**Definition 2.1.** *An* ***alphabet****, also called a set of symbols, represents a set of variables. Each variable is represented by a letter. A* ***sequent*** *is defined by a finite set of variables* $\Gamma$ *and a variable* $x$*. There are two types of sequents: A* ***positive sequent*** *has the form of* $\Gamma \vdash x$ *and a* ***negative sequent*** *has the form of* $\Gamma \nvdash x$*.*

An initial difference between S-logic and propositional logic is that any sequent in S-logic has the form of $\Gamma \vdash x$ or $\Gamma \nvdash x$, where $x$ is a letter. Although $\Gamma$ is a set of variables, we think of it as a conjunction of variables. However, in propositional logic, we can have more complicated statements like $\neg(A \to B) \to (C \to \neg D)$.

The intended meaning of $\Gamma \vdash x$ is that if all variables of $\Gamma$ are true, then $x$ is true. So every situation that makes $\Gamma$ true also makes $x$ true. $\Gamma \nvdash x$ is intended to mean that there is a situation where all variables of $\Gamma$ are true and $x$ is false. In standard logic models, $\Gamma$ implies $x$ is defined by the formula $(\bigwedge \Gamma) \to x$, where $\bigwedge \Gamma$ is the conjunction of the formulas in $\Gamma$. Note that the formula for "$\Gamma$ does not imply $x$" is $\neg((\bigwedge \Gamma) \to x)$ and means $\Gamma$ is true and $x$ is false, in propositional logic, which is different from the meaning of a negative sequent in S-logic. The next definition is about the function that assigns the value of true or false to each variable.

**Definition 2.2.** *An* ***S-theory*** *is a fixed choice of an alphabet and a particular set of sequents with letters from this alphabet.*

The focus of this chapter is to determine which other sequents can be deduced from a given S-theory.

**Example 2.3.** *We can define an S-theory by an alphabet $\mathcal{A}$ and a set of sequents $S_0$ where $\mathcal{A} = \{A, B, C, D, E, F\}$ and $S_0 = \{A \vdash B, B \vdash C, \{C, D\} \vdash F, D \vdash E, \{A, E\} \vdash F, A \nvdash F\}$.*

**Definition 2.4.** *A* ***world*** *$w : A \to \{T, F\}$ is a function that makes each letter true or false. If $S$ is an S-theory, an* ***S-world*** *is a world that makes every positive sequent of $S$ true.*

Suppose $\Gamma$ is a set of letters and $A$ is a letter. Given a world $w$, we say $w$ makes $\Gamma \vdash A$ true if either $w$ makes some letters of $\Gamma$ false or $w$ makes $A$ true. $w$ makes $\Gamma \nvdash A$ true if $w$ makes all letters of $\Gamma$ true and $A$ false.

Suppose we have a set of variables $\Gamma$ and a world $w$, we say that $w$ makes $\Gamma$ true if $w$ makes every letter of $\Gamma$ true.

A world in S-logic is like a truth assignment in standard propositional logic. The only difference between a world and a truth assignment is that a world assigns to both a letter and sequent the value of true or false. However, in standard propositional logic we have a function that assigns the value of true or false to variables and another function that does the same for formulas.

The concept of an S-world is important because we need it to define the semantics of S-logic.

**Definition 2.5.** *Suppose $S$ is an S-theory. $S$ is called* ***consistent*** *if there exists a set of worlds $W$ such that:*

- *For every world $w \in W$ and every positive sequent $\Gamma \vdash x \in S$, $w$ makes $\Gamma \vdash x$ true.*

- *For every negative sequent $\Gamma \nvdash x \in S$, there exists a world $w \in W$ such that $w$ makes $\Gamma \nvdash x$ true.*

*In other words, each negative sequent must be true at least one S-world. If $W$ is a set of worlds that verifies $S$ is consistent we say that $W$ is **sufficient** for $S$.*

A set of worlds $W$ sufficient for $S$ guarantees the consistency of $S$. If $S$ contains only positive sequents then $W$ could be empty. If $S$ contains both positive and negative sequents, then $W$ must at least contain enough set of S-worlds to make each negative sequent true in at least one S-world. A single S-world could make more than one negative sequent true and a negative sequent could be true in more than one S-world.

We are only interested in consistent sets of sequents. We will first define the rules of S-logic for positive sequents and second for negative sequents. For each rule we will prove its soundness. **Soundness** of a rule means that a set of worlds sufficient for a set of sequents is also sufficient for the new set deduced by applying the rule. In other words, the rule preserves consistency in a strong way.

**Example 2.6.** *Let $S_0$ be the theory $\{A \vdash B, B \vdash C, \{C, D\} \vdash F, D \vdash E, \{A, E\} \vdash F, A \nvdash F\}$ from Example 2.3. We will suppose $S_0$ is consistent and a set of worlds $W$ is sufficient for $S_0$. As $S_0$ contains one negative sequent $A \nvdash F$, $W$ has to contain at least one S-world that makes $A \nvdash F$ true.*

We now begin to define our deduction rules.

**The I Rule** (Identity). The identity rule states: for each letter $A$,

$$\frac{}{A \vdash A}\ (\mathrm{I})$$

In plain language, we can deduce $A \vdash A$ without any hypothesis.

**Theorem 2.7.** *(Soundness of I) If $S$ is a consistent S-theory and $A$ is a letter from the alphabet of $S$, then the set $S' = S \cup \{A \vdash A\}$ is also consistent.*

*Proof.* It is clear that every S-world makes $A \vdash A$ true, so if $S$ is a consistent S-theory, then $S' = S \cup \{A \vdash A\}$ is also consistent. $\square$

**Example 2.8.** *Because $S_0$ is consistent, $S' = S_0 \cup \{A \vdash A\}$ is also consistent.*

**The MP Rule** (Modus ponens). The Modus ponens rule is defined by:

$$\frac{\{A_1, \ldots, A_n\} \vdash X \qquad \{B_1, \ldots, B_k\} \vdash A_1}{\{B_1, \ldots, B_k, A_2, \ldots, A_n\} \vdash X} \text{(MP)}$$

In plain language, we suppose that we have an S-theory $S$. If $\{A_1, \ldots, A_n\} \vdash X$ is in $S$ and $\{B_1, \ldots, B_k\} \vdash A_1$ is in $S$ then we can deduce $\{B_1, \ldots, B_k, A_2, \ldots, A_n\} \vdash X$ from $S$.

The MP rule is important to deduce positive sequents from a set of positive sequents.

**Example 2.9.** *By using the set of sequents $S_0$, $A \vdash B$ and $B \vdash C$ imply $A \vdash C$. Also, $B \vdash C$ and $\{C, D\} \vdash F$ imply $\{B, D\} \vdash F$.*

The next theorem states the soundness of the MP rule.

**Theorem 2.10.** *(Soundness of MP) Suppose we have a consistent theory $S$, a set of variables $\Gamma = \{a_1, ..., a_n\}$ such that $\Gamma \vdash b \in S$, and a set of variables $\Gamma' = \{c_1, ..., c_m\}$ such that $\Gamma' \vdash a_1 \in S$. Let $S' = S \cup \{\Gamma \cup \Gamma' - \{a_1\} \vdash b\}$. If a set of worlds $W$ is sufficient for $S$ then $W$ is also sufficient for $S'$.*

*Proof.* Let $W$ be a set of worlds sufficient for $S$ and fix a world $w$ in $W$. We know that every sequent $X \vdash y \in S$ is also a sequent in $S'$. For $\Gamma \cup \Gamma' - \{a_1\} \vdash b$, take $w$ that makes $\Gamma \cup \Gamma' - \{a_1\}$ true, so it will make all variables of $\Gamma'$ true. Therefore we have $w(c_j) = T$ for all $j$, $1 \leqslant j \leqslant m$. As $\{c_1, ..., c_m\} \vdash a_1$, this implies $w(a_1) = T$ because $S$ is consistent and we know that $w(a_j) = T$ for $1 < j \leqslant n$. So $w(a_j) = T$ for all $j$, $1 \leqslant j \leqslant n$, which implies that $\Gamma$ is true. And since $W$ is sufficient for $S$, we get $w(b) = T$.

For any sequent $X \nvdash y$ of $S'$, $X \nvdash y$ is also an element of $S$ and as $S$ is consistent, there exist at least one world that makes $X$ true and $y$ false. □

**Example 2.11.** *Let $S_0$ be the theory from Example 2.3. Assume a set of worlds $W$ is sufficient for $S_0$. Let $\Gamma = \{C, D\}$ so we have $\Gamma \vdash F \in S_0$. Let $\Gamma' = \{B\}$ and we get $\Gamma' \vdash C \in S_0$. By applying MP rule, we obtain $W$ is also sufficient for $S_0 \cup \{\Gamma \cup \Gamma' - \{C\} \vdash F\}$, which means that $W$ is also sufficient for $S_0 \cup \{\{B, D\} \vdash F\}$.*

**The W Rule** (Weakening). The weakening rule is defined by:

$$\frac{\Gamma \vdash X \qquad \Gamma \subseteq \Delta}{\Delta \vdash X} \text{ (W)}$$

In plain language, we suppose we have an S-theory $S$. If $\Gamma \vdash X$ is in $S$ and $\Gamma \subseteq \Delta$ then we can deduce $\Delta \vdash X$ from $S$.

The W rule reflects that $\Gamma \vdash X$ is stronger than $\Delta \vdash X$ for any set of variables $\Delta$ such that $\Gamma \subseteq \Delta$, and $\Delta$ does not change the consistency of $S'$ formed by $S$ and $\Delta \vdash X$. The following theorem states the soundness of W rule.

**Theorem 2.12.** *Suppose we have an S-theory $S$, a sequent $\Gamma \vdash b$ of $S$, a set of variables $\Gamma'$ and a set $S' = S \cup \{\Gamma \cup \Gamma' \vdash b\}$. If a set of S-worlds $W$ is sufficient for $S$ then $W$ is sufficient for $S'$.*

*Proof.* (Soundness of W rule) Take an S-world $w$. If $w(b) = T$, then $w$ satisfies $\Gamma' \vdash b$. Otherwise, because $w$ satisfies $\Gamma \vdash b$, $w(\Gamma) = F$ so $w(\Gamma \cup \Gamma') = F$ and therefore $w$ satisfies $\Gamma \cup \Gamma' \vdash b$. □

**Example 2.13.** *Let $\Gamma = \{A\}$ and $\Gamma' = \{E\}$. Thus $\Gamma \vdash B \in S_0$. As $W$ is sufficient for $S_0$, by applying the W rule, we see that $W$ is also sufficient for $\Gamma \cup \Gamma' \vdash B$, which means $W$ is also sufficient for $\{A, E\} \vdash B$.*

**Definition 2.14.** *A sequent $\sigma$ can be* ***derived*** *from $S$ if there is a sequence of sequents such that each is either in $S$ or is derived from previous sequents by a single rule, and the sequence ends with $\sigma$.*

The next theorem states the completeness of the I, MP, and W rules for positive sequents. **Completeness** means that if every set of worlds sufficient for a set of sequents $S$ is also sufficient for a positive sequent $\sigma$, then $\sigma$ could be deduced from $S$ by using the rules of S-logic.

**Theorem 2.15.** *(Completeness of I, MP and W rules for positive sequents) Let $S$ be a set of positive sequents. If every S-world makes a positive sequent $\sigma$ true then $\sigma$ can be derived from $S$ by using some sequence of the I, MP and W rules.*

*Proof.* The idea of the proof is to take a sequent $\sigma$ such that every S-world makes $\sigma$ true and prove that we can derive $\sigma$ from $S$. Suppose $\Gamma \vdash b$ is an arbitrary sequent. We have two cases:

**Case 1:** If $b \in \Gamma$, then by using the I rule, $b \vdash b$ can be derived from $S$. By applying the W rule on $b \vdash b$, $\Gamma \vdash b$ can be derived from $S$.

**Case 2:** If $b \notin \Gamma$, we prove the contrapositive of the theorem, which says that if we cannot derive $\Gamma \vdash b$ from $S$ by using the MP, I and W rules then there is a world satisfying $S$ and $\Gamma$ in which $b$ is false.

We define a world $w$ as follows. The world $w$ makes every variable in $\Gamma$ true and $b$ false. because $b$ is not in $\Gamma$. For any variable $x \notin \Gamma \cup \{b\}$, if $\Gamma \vdash x \in S$ we make $w(x) = T$, otherwise we make $w(x) = F$.

We need to check that $w$ satisfies $S$. In other words, if $\Sigma \vdash c \in S$ and every variable in $\Sigma$ is true in $w$, then $c$ is true in $w$. If $\Sigma \subseteq \Gamma$, apply the W rule to deduce $\Sigma \cup \Gamma \vdash c$, which means $w(c) = T$.

Now, suppose $\Sigma$ is not a subset of $\Gamma$. Let $\Sigma = \{v_1, ..., v_n\}$ and $\Gamma = \{w_1, ..., w_k\}$. For each variable $v_i \in \Sigma$, $1 \leqslant i \leqslant n$, $w(v_i) = T$, so $v_i \in \Gamma$ or $\Gamma \vdash v_i \in S$. So by rearranging the elements of $\Sigma$, $\Sigma = \Gamma' \cup \{v'_1, ..., v'_p\}$ such that $\Gamma' \subseteq \Gamma$, $\Gamma \vdash v'_i \in S, 1 \leqslant i \leqslant p$ and then $\Gamma' \cup \{v'_1, ..., v'_p\} \vdash c$. By applying repeatedly the MP rule on each variable $v'_i$, $1 \leqslant i \leqslant p$ we deduce $\Gamma' \cup \Gamma \vdash c$ and therefore $\Gamma \vdash c$. As the $w(\Gamma) = T$, $w(c) = T$. □

By analyzing the proof of Theorem 2.15, we can obtain a stronger result stated in the next lemma.

**Lemma 2.16.** *If $S$ is consistent, $\Delta \nvdash c$ is in $S$ and $x$ is such that $S$ does not derive $\Delta \vdash x$, then there is an S-world in which $\Delta$ is true, $x$ is false and $c$ is false.*

*Proof.* We cannot derive $\Delta \vdash x$ from $S$ so by Theorem 2.15, there exists a world $w$ such that $w(\Delta) = T$ and $w(x) = F$. $\Delta \nvdash c$ is in S, so the same world makes $c$ false. □

Next, we define a rule for negative sequents, then prove its soundness and completeness with the previous rules.

**The N rule**(Negativity). The negativity rule is defined by:

$$\frac{\Delta \nvdash c \qquad \Delta \cup \{b\} \vdash c \qquad \Delta \vdash \Gamma}{\Gamma \nvdash b} \text{(N)}$$

In plain language, the rule means that if we have an S-theory $S$, $\Delta \nvdash c$ is in $S$, $\Delta, b \vdash c$ is in $S$ and $\Delta \vdash \Gamma$ is in $S$ then we can deduce $\Gamma \nvdash b$ from $S$. To prove the soundness of the N rule, we need to find a world $w$ such that $w(\Gamma) = T$ and $w(b) = F$.

**Theorem 2.17.** *(Soundness of the N rule) If $W$ is a set of S-worlds sufficient for $S$ and $\Gamma \nvdash b$ is derived from $S$ by the N rule. Then $W$ is sufficient for $S \cup \{\Gamma \nvdash b\}$.*

*Proof.* The sequent $\Delta \nvdash c \in S$, so there exists an S-world $w$ in $W$ such that $w(\Delta) = T$ and $w(c) = F$. The sequent $\Delta \vdash \Gamma \in S$ so $w(\Gamma) = \mathit{true}$. We need to prove that $w(b) = \mathit{false}$. We proceed by contradiction. Assume $w(b) = T$, thus $\Gamma \vdash b$ can be deduced from $S$. By deduction, we obtain $\Delta \vdash b$. As we have $\Delta, b \vdash c \in S$, by applying the MP rule we get $\Delta \vdash c$, a contradiction. So we have $w(b) = \mathit{false}$ and therefore w satisfies $\Gamma \nvdash b$. □

**Example 2.18.** *If we take the set of sequents $S_0$ from Example 2.3 and consider $\Delta = \{A\}, \Gamma = \{A\}, c = F, b = E$. From $S_0$ we have:*

$$\frac{A \nvdash F \qquad \{A, E\} \vdash F \qquad \dfrac{}{A \vdash A}\,(I)}{A \nvdash E}\,(N)$$

*First, we derive $A \vdash A$ by using the I rule. Then, we derive $A \nvdash E$ by using $A \vdash A$ and two sequents in $S_0$.*

*The next theorem states the completeness of I, MP, W and N rules.*

**Theorem 2.19.** *Suppose $S$ is a consistent S-theory. If every set of worlds that is sufficient for $S$ is also sufficient for $\Gamma \nvdash b$, then we can derive $\Gamma \nvdash b$ from $S$.*

*Proof.* We will prove the theorem by contraposition. We assume we cannot derive $\Gamma \nvdash b$ from $S$ and we prove that there is a set of S-worlds sufficient for $S$, but not for $S \cup \{\Gamma \nvdash b\}$. Let $S^N = \{\sigma \in S : \sigma \text{ is a negative sequent}\}$.

Let $S_1 = \{\sigma \in S^N : \text{there is an S-world } w \text{ such that } w(\sigma) = T \text{ and } w(\Gamma) = F\}$. If $S^N = S_1$ then we get a set of worlds sufficient for $S$ but not for $\Gamma \nvdash b$. If $\sigma = \Delta_\sigma \nvdash c_\sigma$ is in $S^N - S_1$, every $w$ that makes $\sigma$ true makes $\Gamma$ true. By using Lemma 2.16, we can derive $\Delta_\sigma \vdash \Gamma$.

Let $S_2$ be the set of all negative sequents $\sigma \in S^N - S_1$ such that if $\sigma = \Delta_\sigma \nvdash c_\sigma$ there is an S-world satisfying $\Delta_\sigma$ and $b$ in which $c_\sigma$ is false. If $S^N = S_1 \cup S_2$, we get a set of worlds sufficient for $S^N$ but not for $\Gamma \nvdash b$. Otherwise if $\sigma = \Delta_\sigma \nvdash c_\sigma$ is in $S^N - (S_1 \cup S_2)$, every S-world satisfying $\Delta_\sigma$ and $b$ makes $c_\sigma$ true, so we can derive $\Delta_\sigma, b \vdash c_\sigma$. By using the N rule, we can derive $\Gamma \nvdash b$ which is impossible by assumption. $\square$

Figure 2 shows all the rules of S-logic. The next theorem summarizes our results.

**Theorem 2.20.** *The rules I, MP, W and N are sound and complete in the following sense:*

1. *A positive sequent $\sigma$ can be derived from an S-theory with these rules if and only if $\sigma$ holds in every S-world.*

2. *A negative sequent $\sigma$ can be derived from a consistent S-theory if and only if every set of worlds sufficient for $S$ is also sufficient for $\sigma$.*

*Identity:*

$$\frac{}{A \vdash A}\ (\mathrm{I})$$

*Weakening:*

$$\frac{\Gamma \vdash X \qquad \Gamma \subseteq \Delta}{\Delta \vdash X}\ (\mathrm{W})$$

*Modus ponens:*

$$\frac{\{A_1, \ldots, A_n\} \vdash X \qquad \{B_1, \ldots, B_k\} \vdash A_1}{\{B_1, \ldots, B_k, A_2, \ldots, A_n\} \vdash X}\ (\mathrm{MP})$$

*Negativity:*

$$\frac{\Delta \nvdash c \qquad \Delta \cup \{b\} \vdash c \qquad \Delta \vdash \Gamma}{\Gamma \nvdash b}\ (\mathrm{N})$$

Figure 2: Inference rules for S-logic.

# Chapter 3

# APPLICATIONS OF S-LOGIC

The goal of S-logic is to study the relations between particular mathematical theorems. S-logic can be used in two different ways. The first way is to deduce new theorems from existing ones, and the second way is to check whether a theorem is immediately provable from other ones. To be able to apply S-logic, we need to construct from the existing theorems an alphabet, a set of sequents and a set of models. A model makes each statement true or false, so it gives a world in S-logic. The set of statements that we want to study will be the S-theory.

Because S-logic is an effective deductive system, it can be implemented by creating a computer program that uses known facts and the rules of S-logic to deduce new statements or check whether a statement could be deduced. In the following examples, we will show two applications of S-logic. The first example demonstrates an application of S-logic in reverse mathematics; the second demonstrates an application of S-logic in topology.

## 3.1 Second order arithmetic

**Definition 3.1.** *Second order arithmetic is an axiomatic system designed for two kinds of variables: the natural numbers and the subsets of natural numbers. The formal system of second order arithmetic is denoted by* $Z_2$.

A **model** of second order arithmetic, also called **structure**, is a set of objects that are natural numbers or sets of natural numbers.

The two following theorems are well known and proved by Simpson [4].

**Theorem 3.2.** *(Completeness theorem) Let $B$ be the base system and $A$ be a set of axioms. A formula $X$ is provable from $B + A$ if and only if $X$ holds in every model of $B + A$.*

**Theorem 3.3.** *(Soundness theorem) Let $B$ be the base system and $A$ be a set of axioms. $B + A$ is sound if whenever a formula $X$ is deduced from $A + B$, every model satisfying $A + B$ satisfies $X$.*

### 3.2 Interpretation of second order arithmetic into S-logic

First, we need to pick a collection of formulas $A_1, A_2, ..., A_n$ corresponding to subsystems of $Z_2$. The alphabet $A$ will be the set of these formulas and a model $M$ will be a world in S-logic. Let $\Gamma$ be a conjunction of formulas and $X$ be a formula. An S-theory $T_1$ will be defined as follows:

1. Put $\Gamma \vdash X$ into $T_1$ if there exists a proof in literature such that B+$\Gamma$ proves $X$.
2. Put $\Gamma \nvdash X$ into $T_1$ if there is a model $M$, constructed in the literature, where $M$ satisfies $\Gamma$ and not $X$.

The set of sequents of $T_1$ has to be fixed, so we take a fixed set of results from the literature to construct $T_1$.

**Lemma 3.4.** *Each model $M$ of the base system of $Z_2$ gives a world in S-logic that satisfies $T_1$.*

*Proof.* For any $\Gamma \nvdash X \in T_1$, there exists a model $M$ such that $M$ satisfies $B$ and $\Gamma$ but not $X$, so $M$ is a world satisfying $\Gamma$ but not $X$. □

Let $W_1$ be the set of all models in literature that satisfy the base system $B$.

**Lemma 3.5.** *$W_1$ is sufficient for $T_1$.*

*Proof.* By construction, for every sequent $\Gamma \nvdash X \in T_1$, there exists a model $M \in W_1$ such that $M$ satisfies $\Gamma$ and $B$ but not $X$. □

Let cl($T$) be the closure of an S-theory $T$.

**Theorem 3.6.** *$W_1$ is sufficient for* $\mathrm{cl}(T_1)$ *under the rules of S-logic.*

*Proof.* $W_1$ is sufficient for $T_1$ and the rules of S-logic are sound. Therefore by Theorem 2.19, $W_1$ is sufficient for $\mathrm{cl}(T_1)$ under the rules of S-logic. □

$T_1$ is limited to the literature and the proofs that we have discovered so far. By contrast, we can construct a new S-theory $T_2$ such that:

1. $A_2$ is the alphabet of $T_2$ such that $A_2$ is the set of all formulas of $Z_2$.

2. $\Gamma \vdash X \in T_2$ if there is a proof of $X$ in principle assuming $\Gamma$ and $B$.

3. $\Gamma \nvdash X \in T_2$ if there exists a model $M$ that makes $B$ and $\Gamma$ true but $X$ false.

The S-theory $T_2$ contains all information and $T_1$ just contains partial information that we are sure of, so $T_1$ is a subset of $T_2$. Similarly, $W_1$ is the set of models discovered so far, so we will define the set of models $W_2$ for $T_2$ such that $W_2$ is the set of every possible model.

**Lemma 3.7.** *$W_2$ is sufficient for $T_2$.*

**Theorem 3.8.** *$W_2$ is sufficient for* $\mathrm{cl}(T_2)$*under the rules of S-logic.*

The proof of this theorem is similar to the proof of Theorem 3.6.

## 3.3 Topology

In this section, we apply S-logic to capture the relationship between topological properties of topological spaces. We will define an alphabet and show the steps to construct the S-theory.

## 3.4 Interpretation of topology into S-logic

We take a set of properties $P_1, ..., P_n$ of topological spaces. The alphabet $A_3$ is the set of these properties, and a topological space $M$, for example, gives a world in S-logic. Let $\Gamma$ be a conjunction of topological properties from $A_3$, $X$ be a topological property in $A_3$. We will construct the S-theory $T_3$ as follows:

1. We put $\Gamma \vdash X$ into $T_3$ if there is a proof in literature for $\Gamma \vdash X$.

2. We put $\Gamma \nvdash X$ into $T_3$ if there is a topological space $M$ in literature that satisfies $\Gamma$ and not $X$.

Figure 3 shows counterexamples for a fixed set of topological properties that can be used to construct negative sequents.

**Theorem 3.9.** *Let $W_3$ be the set of topological spaces sufficient for $T_3$. $W_3$ is sufficient for* $\mathrm{cl}(T_3)$ *under the rules of S-logic.*

The proof of this theorem is similar to the proof of Theorem 3.6.

The S-theory $T_3$ just contains the statements for which we have proofs or counterexamples. We can construct another S-theory $T_4$ such that:

1. $A_4$ is the alphabet of $T_4$ and it is the set of all topological properties. $W_4$ is the set of all topological spaces.

2. We put $\Gamma \vdash X$ into $T_4$ if there is a proof for $X$ in principle assuming $\Gamma$.

3. We put $\Gamma \nvdash X$ into $T_4$ if there is a topological space $M$ in $W_4$ that satisfies $\Gamma$ and not $X$.

**Lemma 3.10.** *$W_4$ is sufficient for $T_4$.*

**Theorem 3.11.** *$W_4$ is sufficient for* $\mathrm{cl}(T_4)$ *under the rules of S-logic.*

The proof of this theorem is similar to the proof of Theorem 3.6.

## 3.5 Other applications

S-logic can be applied in other situations where we have a set of objects and a set of properties such that we can construct the set of worlds from the set of objects and the S-theory from the relationships between the different properties. The set of worlds has to be sufficient to verify non-implication relationships.

| 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 0 | | 0 | 1 | 1 | 1 | $T_0$ |
| 1 | 0 | 1 | 1 | 1 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | $T_1$ |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | $T_2$ |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | $T_{2\frac{1}{2}}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 1 | | 1 | 1 | 1 | 1 | $T_3$ |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 1 | | 1 | 1 | 1 | 1 | $T_{3\frac{1}{2}}$ |
| 0 | 0 | 0 | 0 | 0 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 0 | | 1 | | 1 | 1 | 1 | 1 | $T_4$ |
| 0 | 0 | 0 | 0 | 0 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 0 | | 1 | | 1 | 1 | 1 | 1 | $T_5$ |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Urysohn |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Semiregular |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Regular |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Completely Regular |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Normal |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Completely Normal |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Perfectly Normal |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 0 | | 1 | 0 | 0 | 1 | Compact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 1 | 1 | | 1 | | 1 | 0 | 1 | 1 | $\sigma$-Compact |
| 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 1 | 1 | | 1 | | 1 | 0 | 1 | 1 | Lindelöf |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 0 | | 1 | 0 | 0 | 1 | Countably Compact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 0 | | 1 | 0 | 0 | 1 | Sequentially Compact |
| 1 | 1 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 1 | | 1 | 0 | 0 | 1 | Weak. Count. Compact |
| 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 0 | | 1 | 0 | 0 | 1 | Pseudocompact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 1 | | 1 | 1 | 1 | 1 | Locally Compact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 1 | | 1 | 1 | 1 | 1 | Strong Loc. Compact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 1 | 1 | | 1 | | 1 | 0 | 1 | 1 | $\sigma$-Locally Compact |
| 1 | 0 | 0 | 1 | 1 | 0 | | 0 | 1 | 1 | | 1 | 1 | 1 | 1 | | 1 | | 1 | 0 | 1 | 1 | Separable |
| 1 | 0 | 0 | 0 | 1 | 0 | | 0 | 1 | 1 | | 1 | 0 | 1 | 1 | | 1 | | 1 | 0 | 1 | 1 | Second Countable |
| 1 | 0 | 0 | 0 | 1 | 1 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 1 | | 1 | 1 | 1 | 1 | First Countable |
| 1 | 1 | 1 | 1 | 1 | 0 | | 0 | 1 | 1 | | 1 | 1 | 1 | 1 | | 1 | | 1 | 0 | 1 | 1 | Count. Chain Cond. |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 1 | | 1 | 1 | 1 | 1 | Paracompact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 1 | | 1 | 1 | 1 | 1 | Metacompact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 1 | | 1 | 1 | 1 | 1 | Count. Paracompact |
| 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 0 | 0 | 1 | | 1 | | 1 | 1 | 1 | 1 | Count. Metacompact |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Fully Normal |
| 0 | 0 | 0 | 0 | 0 | 1 | | 1 | 1 | 1 | | 0 | 0 | 0 | 0 | | 1 | | 1 | 1 | 1 | 1 | Fully $T_4$ |
| 1 | 1 | 1 | 1 | 1 | 0 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 0 | | 1 | 0 | 0 | 0 | Connected |
| 1 | 0 | 0 | 1 | 0 | 0 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 0 | | 1 | 0 | 0 | 0 | Path Connected |
| 1 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 1 | 0 | 0 | 0 | Arc Connected |
| 1 | 1 | 1 | 1 | 1 | 0 | | 0 | 0 | 0 | | 1 | 1 | 1 | 1 | | 0 | | 1 | 0 | 0 | 0 | Hyperconnected |
| 0 | 0 | 0 | 0 | 0 | 0 | | 1 | 1 | 1 | | 1 | 0 | 0 | 0 | | 0 | | 1 | 0 | 0 | 0 | Ultraconnected |
| 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 1 | | 1 | 1 | 1 | 1 | Locally Connected |
| 1 | 0 | 0 | 1 | 0 | 1 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 1 | | 1 | 1 | 1 | 1 | Local Path Connected |
| 1 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 1 | 1 | 1 | 1 | Local Arc Connected |
| 0 | 0 | 0 | 0 | 0 | 0 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 0 | | 1 | 0 | 0 | 0 | Biconnected |
| 0 | 0 | 0 | 0 | 0 | 0 | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | 0 | | 1 | 0 | 0 | 0 | Has Dispersion Point |
| 0 | 0 | 1 | 0 | 1 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Totally Path Disc. |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Totally Disconnected |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Totally Separated |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | | 0 | 1 | 1 | 1 | Extremally Disc. |
| 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 1 | | 1 | 1 | 1 | 1 | Zero Dimensional |

Figure 3: Excerpt from *Counterexamples in Topology* [6]. Each row is a particular property of a space and each column is a specific topological space.

# Chapter 4

# MODAL LOGIC AND STRICT IMPLICATION LOGIC

**Modal logic** is an extension of propositional logic that includes modality operators [2]. **Modality** consists of qualifying a proposition by adding expressions like "usually," "necessarily," "possibly,"... The words used to express modality are called **modals**. For example, the sentence "The weather is sunny" could be qualified by saying "The weather is possibly sunny," or "The weather is usually sunny." Modals give an idea about the truthfulness of a statement by stating the circumstances under which the statement is true. Modal logic studies focus on the two modals "necessarily" and "possibly." However, the term modal logic, in general, refers to a family of logics that use different modals.

## 4.1 Language of modal logic

Modal logic is an extension of propositional logic that adds two unary modal operators $\Box$ and $\Diamond$ for necessarily and possibly, respectively [7]. The operator $\Box$ could be expressed in terms of the operator $\Diamond$, and vice versa, through the following properties:

1. $\Box A \leftrightarrow \neg\Diamond\neg A$

2. $\Diamond A \leftrightarrow \neg\Box\neg A$

**Example 4.1.** *In the following examples, we will use English sentences.*

1. *It is necessary that it will rain if and only if it is not possible that it will not rain today. In this example, the variable $A$ is "It will rain" and the sentence shows $\Box A \leftrightarrow \neg\Diamond\neg A$,*

2. *It is possible that Marc will come today if and only if it is not necessary that Marc will not come today. In this example, the variable A is "Marc will come today" and the sentence shows* $\Diamond A \leftrightarrow \neg\Box\neg A$.

## 4.2 Modal logic rules

The base system of modal logic is called $K$. $K$ is obtained by adding two rules to the propositional logic rules:

1. Necessitation rule: If $A$ is a theorem of $K$, then so is $\Box A$. In other words, any theorem of $K$ is necessary.

2. Distribution axiom: $\Box(A \rightarrow B) \vdash (\Box A \rightarrow \Box B)$. In other words, if it is necessary that $A$ implies $B$, then necessary $A$ implies necessary $B$.

The operator $\Diamond$ is not used in the two rules because we can express it by using $\Box$ as $\Diamond A = \neg\Box\neg A$.

## 4.3 The semantics of modal logic

The semantics of modal logic are defined using frames. A **frame** is a pair $\langle W, R\rangle$ where $W$ is a set of worlds and $R$ is an **accessibility relation** [7]. $xRy$ means that the world $y$ is "accessible" from the world $x$. When every world is accessible from every other, we call the frame a **Euclidean frame**. The system $S_5$ is a complete modal logic system that has a Euclidean frame.

The accessibility relation is used to define truth values of formulas with modal operators. For example, $\Box A$ is true in a world $w$ if $A$ is true in every world $v$ such that $wRv$. In other words, $\Box A$ is true in a world $w$ if $\Box A$ is true in every world accessible from $w$. $\Box A$ implies $A$ if the relation $R$ is reflexive. In other words, $\Box A$ implies $A$ if every world is accessible from itself.

In case we have a Euclidean frame, $A$ **necessarily implies** $B$ means that every model that satisfies $A$ also satisfies $B$. In other words, there is no model that satisfies $A$ and does not satisfy $B$. However, $A$ **possibly implies** $B$ means that there is a least one model that satisfies $A$ and $B$. So we can deduce that "A necessarily implies B" necessarily implies "A possibly implies B" and we get then $\Box(\Box(A \to B) \to \Diamond(A \to B))$.

## 4.4 Relationship between S-logic and modal logic

Suppose we have an alphabet $A$, $\Gamma$ is a conjunction of letters $A_1, ..., A_n$ from $A$, a letter X from $A$ and an S-theory $T$. $\Gamma \vdash X$ is in $T$ means that every world that makes $\Gamma$ true also makes X true. So we can deduce that the conjunction of $A_1, ..., A_n$ *necessarily implies* X and then the translation of $\Gamma \vdash X$ in modal logic will be $\Box((A_1 \wedge ... \wedge A_n) \to X)$.

The sequent $\Gamma \nvdash X \in T$ means that there is a world that satisfies all the variables of $\Gamma$ and not X. So the translation of $\Gamma \nvdash X$ in modal logic will be $\Diamond((A_1 \wedge ... \wedge A_n) \wedge \neg X)$ or $\neg\Box((A_1 \wedge ... \wedge A_n) \to X)$.

**Definition 4.2.** *Given $\Gamma = \{A_1, ..., A_n\}$, we define a function $\tau$ that translates sentences of S-logic into sentences of modal logic such that:*

1. $\tau(\Gamma \vdash x) = \Box((A_1 \wedge ... \wedge A_n) \to x)$
2. $\tau(\Gamma \nvdash x) = \Diamond((A_1 \wedge ... \wedge A_n) \wedge \neg X)$
3. *If $T$ is an S-theory, $\tau(T) = \{\tau(\sigma) : \sigma \in T\}$*

**Theorem 4.3.** *Given any S-theory $T$ and a sentence $\sigma$, $\sigma$ is deducible from $T$ in S-logic if and only if $\tau(\sigma)$ is deducible from $\tau(T)$ in the modal system $S_5$.*

*Proof.* By using Theorem 2.20, $T$ is complete in S-logic so $\sigma$ is deducible from $T$ if and only if every set of worlds $W$ sufficient for $T$ is sufficient for $T \cup \{\sigma\}$. The system $S_5$ is complete in modal logic, so $\tau(\sigma)$ is deducible from $\tau(T)$ if and only if every Euclidean frame $F$ sufficient for $\tau(T)$ is sufficient for $\tau(T \cup \{\sigma\})$.

The relationship between the set of worlds $W$ and the Euclidean frame $F$ can be defined in this way. Given $W$, we can define $F = \langle W, R\rangle$, where $R$ is an accessibility relation such that every world in $W$ is accessible from every other world in $W$. Conversely, if we have a Euclidean frame $F = \langle W, R \rangle$, we take the set of worlds $W$ for the S-theory $T$ in S-logic. Then a set of worlds $W$ is sufficient for an S-theory $T$ if and only if $\langle W, R \rangle$ satisfies $\tau(T)$ $\square$

### 4.5 Difference between S-logic and modal logic

The difference between S-logic and modal logic is the statement structure of the statements. In modal logic, we can have nested connectives like $\square(\neg\Diamond A \to B)$, which is not possible in S-logic. Also the hypothesis of a statement in S-logic can just be a conjunction of letters but in modal logic we can have more complicated hypothesis like $\square(\Diamond(A \to B) \to D)$.

### 4.6 Strict implication logic

The implication $A \to B$ in standard propositional logic have sometimes interpretation problems in natural language. For example, if we take the sentence "If the sky is green, then 1+1=2". This sentence is true in standard propositional logic because "The sky is green" is false and "1+1=2" is true, so $F \to T$ is true. However the implication in the original sentence is not true. So the strict implication logic defines a new implication by using the modal operator $\square$ used in modal logic and the propositional logic implication $A \vdash B$ [3]. We say that $A$ **strictly implies** $B$ by using the formula $\square(A \vdash B)$, which means that $A$ implies $B$ in every possible world. In strict implication logic, we use the symbol $\supset$ and $A \supset B$ means $A$ strictly implies $B$.

### 4.7 Relationship between S-logic and strict implication logic

$A \supset B$ in modal logic is $\square(A \to B)$ so the translation of $\Gamma \vdash X$ in strict implication logic will be $A_1 \wedge A_2 .... \wedge A_n \supset X$. Similarly, $A \not\supset B$ in modal logic is $\neg\square(A \to B)$ so the translation of $\Gamma \nvdash X$,in strict implication logic, will be $A_1 \wedge A_2 .... \wedge A_n \not\supset X$.

Thus, if we have a complete system in strict implication logic, we can define a function that translates sentences of S-logic into sentences of strict implication logic like the function $\tau$ defined in modal logic, and we obtain the same results of Theorem 4.3.

## REFERENCES


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## Alaeddine Saadaoui

**Born** September 14, 1982 in Feriana, Gasserine, Tunisia

**Education**

- Master of Science. National School of Computer Sciences,Tunisia, May 2008. Thesis Advisor: Dr. Yassine Jamoussi.
- Bachelor of Science. National School of Computer Sciences, Tunisia, May 2006.

**Publications**

1. Web Services Composition Guided by the User's Intentions, Master's Thesis, National School of Computer Sciences, Tunisia, May 2008.
2. On the Logic of Reverse Mathematics, Master's Thesis, Marshall University, USA, May 2012.