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Population Modeling by Differential Equations

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Population Modeling by Differential Equations

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Abstract

A general model for the population of Tibetan antelope is constructed. The present model shows that the given data is reasonably logistic. From this model the extinction of antelopes in China is predicted if we don't consider the effects of humans on the population. Moreover, this model shows that the population is limited. A projected limiting number is given by this model. Some typical mathematical models are introduced such as exponential model and logistic model. The solutions of those models are analyzed.

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TABLE OF CONTENTS

Acknowledgments iii

Table of Contents iv

Chapter One: Introduction 1

Chapter Two: Basic Mathematical Techniques……………………………………..3
2.1 A Standard Equation for General Population Growth…………………...…….3
2.2 Model 1 (Both B and D are constants)……………………………………....3
2.3 Model 2 (Birth rate is a linearly decreasing function)………………………….5
2.4 Model 3 (Death rate is a linearly increasing function)…………………………....9
2.5 Using the Discrete System to Describe the Logistic Model……………………..10

Chapter Three: Problem Description…………………………...…..............…...12
3.1 Introduction of Tibetan Antelope in China………………………………......12
3.2 Data for Tibetan Antelope……………………………………………………....14

Chapter Four: Modeling the Logistic Model…………………….……………...…..15
4.1 Graph the Data……………………………….…………………………15
4.2 The Mathematical Model……………...…...………………………………....16
4.3 Logistic Model for the Given Data...……………………….…………………...16
4.4 Determining the r and K …………...………………………………...……..19
4.5 The Solution for the Logistic Model……………...……. ………………………20
4.6 Comparison of Logistic Model with Actual Data……...……………………......21

Chapter Five: Conclusion ……………………………………...…..……………...25

References…………………………………...……………………………………..26

Chapter 1
Introduction

Extinction has played a major role in the wildlife world. Today, species are currently becoming extinct at a faster rate than at any time in the past. The number of threatened animals and plants species has exceeded 16,000, a new environmental report said on May 2, 2006, [9]. Extinction of most species is due to changes in their natural habitat, climate changes, pollution, and other types of environmental situations which are difficult to determine. Tibetan Antelopes, on the verge of extinction, are still threatened by poaching and habitat damage despite the progress in anti-poaching campaigns in China, according to Chinese wild animal protection authorities, [8].

Over the past few years, a group of Chinese and American scientists researching the antelope's birthing and breeding behaviors have discovered that the stability of the Tibetan Antelope population has been seriously undermined as many adult female antelopes have been killed. Pregnant antelopes run a higher risk of being hunted because they travel in large groups along routine paths to give birth in certain areas each year, thus becoming easy targets for poachers.

The researchers found in the 1998-1999 time periods, when the poaching was rampant, that the percentage of pregnant females dropped dramatically. At the same time an average of 20,000 antelopes, [6], or more than 30 percent of the estimated total, were slaughtered annually. Tibetan Antelopes are long-term residents on the Tibet-Qinghai Plateau in the west of China. The population has dropped from several million to below

70,000 in the past two decades due to extensive poaching and the damage of the animals' habitat in the wake of a gold rush, [9].

Population dynamics, especially the equilibrium states and their stability, have traditionally been analyzed using mathematical models, [1]. Some models are difference equation models and some are differential equation models. Of interest in both the continuous and discrete models are the equilibrium states and convergence toward these states. In such cases, an interesting question to ask is how fast the population will approach the equilibrium state. To make a mathematical model useful in practice we need to use quantitative methods that allow us to forecast a population's future and express the numerical results, [3]. The need to make forecasts leads to the development of models. A model is a mathematical description of changes in population magnitude. The model may be as simple as an equation with only one variable or as complex as a computer program with thousands of lines. One of the difficulties of building a mathematical model is that we need to consider the particular situation. For example, we need to incorporate some details about the concerned species into the model.

In this paper I will review some simple mathematical models. In particular, I will focus on the logistic growth model. From that we can see the limitations for the population of Tibetan Antelope in China. Based on it, a general model is constructed. It may either show the time until extinction or until the population has either decreased or increase to reach an equilibrium level.

Chapter 2
Basic Mathematical Techniques

2.1 A Standard Equation for General Population Growth

General population models can always be written in the form of a standard equation. It looks like the following:

(Rate of change in quantity) = (Number of births) – (Number of deaths) (2.1.1)

where the quantity is related to the number of members of given population. Let $P(t)$ represent the amount of the species of interest present at that time t. So (2.1.1) can be written as the following:

$P'(t)$ = Changes due to birth - Changes due to Death (2.1.2)

Suppose that B represents the birth rate and D represents the death rate. Then, equation (2.1.2) is equivalent to the following:

$$P'(t) = BP(t) - DP(t) \tag{2.1.3}$$

The above equation can be written as the following:

$$P'(t) = (B - D)P(t) \tag{2.1.4}$$

Note that B and D could be functions of time t or they could be related to the population. It depends on particular species and environment conditions. Base on above equation, let's make some additional assumptions.

2.2 Model 1. Both B and D are constants

Suppose that $r = B - D$, then r is a constant. Equation (2.1.4) becomes

$$P'(t) = rP(t) \tag{2.2.1}$$

In this simplest model, r tells us how fast the population is changing at any given population level. It could be positive or negative. If r is positive, it means the population is increasing. If r is negative, it means the population is decreasing. So we can call r the rate of growth of the population or the rate of decrease of the population. And this model is called the exponential model.

For the differential equation (2.2.1), we can find the solution easily with the known initial data. Note, r can be positive or negative. It depends on which rate term is dominant. To solve this differential equation, we want to review the definition of the solution of such an equation.

Definition 2.1 [4] (Solution of a first order initial value problem): Let $(t_0, x_0) \in (a,b)\times(c,d)$ and assume f is continuous on $(a,b)\times(c,d)$. We say that the function x is a solution of the initial value problem (IVP)

$$x' = f(t,x), \quad x(t_0) = x_0$$

on an interval $I \subset (a,b)$ provided $t_0 \in I$, $x(t)$ is a solution of the IVP on I, and

$$x(t_0) = x_0 .$$

Note, for example, that if $(a,b)\times(c,d) \subset (-\infty,\infty)$, then the function m defined by

$$m(t) = 500e^{-kt},$$

$t \in (-\infty,\infty)$ is a solution of the IVP

$$m' = -km, \quad m(0) = 500$$

on the interval $I \subset (-\infty,\infty)$.

From the above definition, the differential equation (2.2.1) is easily solved as a first order differential equation, leading to a general solution of the following term:

$$P(t) = P_0 e^{rt} \tag{2.2.2}$$

where P_0 represents the initial population size.

Mathematically, differential equation (2.2.1) can be described as the change in P over time is proportional to the size of the population present. This model presents exponential growth without limit. However, in our real world, this case does not happen, because we need to consider the environmental factors, including weather, food, disease, illegal hunting etc. So why do I choose to present this model? I would like to start from the simplest model and then based on it try to involve some factors step by step coming closer to describing the actual situation and approaching the goal.

2.3 Model 2. Either birth rate or death rate is a constant.

Assume that death rate is a constant. Most populations are limited by some factors. For example, it may be restricted by physical space, food supplies, and competition with other species and so on. The instability of the environment is one of the main factors that make the birthrate decrease. For example, if there is no water because of a drought and the grass is gone. If the population needs the grass for food, they have to move to a new place where there is grass. Otherwise, the population will face extinction. Another reason that may cause a decreasing birthrate year by year can be the decrease in the reproductive ability of the species. Perhaps the decrease was due to their biologically changes or other genetic problems as well.

In this case, we can construct it a simple mathematical model where birth rate is a linearly decreasing function of the population size. In other words, the birth rate is of the form,

$$B = B_0 - B_1 P(t)$$

where B_0 and B_1 are constants.

How can we control the birth rate?

Since B is the birth rate, it should be between 0 to 1. That is,

$$0 \le B_0 - B_1 P \le 1 \qquad (2.3.1)$$

where B_0 is the initial rate, and also P is the population at that time t. If we want to control the value of B, we need to find the range of B_1. Let's rewrite (2.3.1) as the following equations:

It is easily seen that

$$0 \le B_0 - B_1 P \le 1$$

implies

$$\frac{B_0 - 1}{P} \le B_1 \le \frac{B_0}{P} \quad . \qquad (2.3.2)$$

since $P > 0$.

That is to say, the value B_1 is between $\frac{B_0 - 1}{P}$ and $\frac{B_0}{P}$.

Let D_0 represent the initial death rate at that time, then the population equation (2.1.1) becomes:

$$P'(t) = (B_0 - B_1 P - D_0)P = B_0 P - B_1 P^2 - D_0 P = (B_0 - D_0)P - B_1 P^2 \qquad (2.3.3)$$

To simplify this expression a little, we define two new terms,

$$k = B_1 \text{ and } M = \frac{(B_0 - D_0)}{B_1}$$

With above definitions, we can rewrite the differential equation (2.3.3) as:

$$P'(t) = kMP - kP^2$$

$$= kMP(1 - \frac{kP}{kM})$$

$$= kMP(1 - \frac{P}{M}) \quad (k \neq 0) \qquad (2.3.4)$$

Equation (2.3.4) describes logistic growth. In order to analyze the model and find out the solution, we need to review the Verhulst Equation, [1]. The logistic law of population growth is described by the first order differential equation,

$$N' = rN(1 - \frac{N}{K}),$$

where N is the number of individuals in the population, and r is the intrinsic rate of change in population, and K is carrying capacity of the environment.

First notice that the derivative will be zero at N = 0 and N = K. Also notice that these are in fact solutions to the differential equation. These two values are called equilibrium solutions.

If we start with a population of zero, there is no growth and the population stays at zero. If we start with a population in the range 0< N < K, then from our differential equation we know that $N' > 0$ and hence N is increasing. If we start at N=K, the population stays at this level. Similarly, if start with N > K, then we have $N' < 0$ and hence N is decreasing. Using our analysis we construct the following phase line diagram shown in Figure 1.

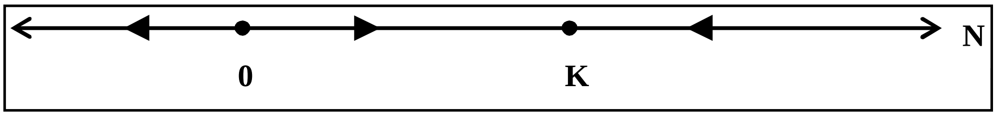


Figure 1

From that phase line diagram, we see that solutions tend toward the equilibrium at K and hence the solution N=K is stable while the equilibrium at 0 is unstable. According to this model, if the population is above 0, it will go to the carrying capacity K eventually.

Let's return to our Model 2, differential equation (2.3.5).

We rewrite it as,

$$\frac{dP}{dt} = kMP(1 - \frac{P}{M})$$

Separating variables in this equation we obtain

$$\frac{M}{P(M-P)}dP = kMdt$$

Using the separation of variable methods we can get the general solution for the differential equation (2.3.5), [10], that is:

$$P(t) = \frac{e^{kMt}M}{e^{kMt} - e^{c}}$$

where c can be calculated from given initial conditions.

Dividing numerators and denominators by e^{kMt}, we get:

$$P(t) = \frac{M}{1 - \frac{e^{c}}{e^{kMt}}}.$$

Simplifying the solution, we can get the following result:

$$P(t) = \frac{M}{1 - \frac{1}{e^{kMt-c}}}.$$

where $M = \frac{(B_0 - D_0)}{B_1}$ and $k = B_1$, and B_0 and B_1 are positive constants, and c is a constant.

Let's analyze this solution. We see that as time increases, the size of population reaches a finite limit. Mathematically,

$$P(t) \to M \text{, as } t \to \infty$$

The value M is defined in terms of the birth rate and death rate expressions and is referred to as the limitation of the population. This result was expected in the real world.

2.4 Model 3. Either birth rate or death rate is a constant.

If B is a constant, let's suppose that the death rate is a linearly increasing function of the population size.

It follows that

$$D = D_0 + D_1 P$$

where D_0 and D_1 are positive constants.

The differential equation becomes,

$$P'(t) = (B_0 - D_0 - D_1 P)P = (B_0 - D_0)P - D_1 P^2$$

Similarly, we define two terms,

$$k = D_1 \text{ and } M = \frac{(B_0 - D_0)}{D_1}$$

Then the differential equation becomes,

$$P'(t) = kMP(1 - \frac{P}{M})$$

We can see that it has the same form as the Model 2, which is differential equation (2.3.5). So we can assume that the solution of the above differential equation has the same form as the solution of differential equation (2.3.5). That is to say, the size of population goes to a finite number as time goes on. Also the finite value is related to the initial data, including the birth rate, death rate and so on.

Therefore, for this model the result is expected in the real world.

2.5 Using the discrete system to describe the logistic model

Sometimes it is difficult to find a value of the population, P, at any given time; It's hard to find continuous data for various seasons. For example, the female is not fertile in winter. Thus our data is discrete.

The simple logistic equation is a formula for describing the evolution of an animal population over time. Many animal species are fertile only during the specific time of the year. And also some young are born in a particular season. Since not every existing animal will reproduce, not every female will be fertile. For that reason, the system might be better described by a discrete equation than a continuous differential equation.

A difference equation is an equation involving differences. If N_n is the number of animals this year and N_{n+1} is the number next year, then the simplest first order model is

$$N_{n+1} = rN_n$$

where r is the growth rate. This model will approximate the evolution of the population. And this exponential population grows without limit. Since every population is bound by the physical limitations of it's surrounding. Then the logistic equation becomes

$$N_{n+1} = rN_n(1 - N_n)$$

A solution of a difference equation is an expression (or formula) that makes the difference equation true for all values of the integer variable n. The nature of a difference equation allows the solution to be calculated recursively. It is easier to see the solution of the difference equation through algebraic equations.

Chapter 3
Problem Description

3.1 Introduction of Tibetan Antelope

The Tibetan Antelope is a remarkable runner. Despite the thin atmosphere on the high level, it can run as fast as 50 mile per hour. This is not only because it is so light and nimble, but also because its muzzle is particularly swollen and it has many air sacs in its nostrils, aiding its breathing.

Since the thin atmosphere on the high level, there is few animal or human living in that area. That's why it's hard to obtain research for the Tibetan Antelope. Due to the geographic distribution of Tibetan Antelope and the circumstance of Chinese government, the outcome of statistical analysis of Tibetan Antelope still remains unsolved before the 1950. There are some reasons for that.

First, Tibetan Antelopes are distributed extensively in China. Tibet currently has approximately **149,930** Tibetan Antelopes in a 698,000-sq-km area across 103 villages and 18 towns, [6]. And also there are many Tibetan Antelopes distributed in area of very high elevations where the air is thin and no humans inhabit the regions, so performing the censuses are very difficult, and nothing can be done for the statistic analysis of the wildlife populations. Secondly, the Chinese government was established in 1949. At that time the Chinese government had limited funds. Tibet theoretically belongs to China, but objectively, Tibet still intends to be independent. Therefore, the people who live in Tibet have a lot of conflicts with the Chinese government, constantly armed clashes and other acts of violence takes place. In this kind of situation, almost no one cares about the statistical work for the Tibetan Antelope. There were only few research papers regarding

Tibetan Antelope that were written by American scientists, but there was no official statistical data that can be investigated or verified.

The official data being used in this paper was initially collected after 1950. Because of the special habitat environment of Tibetan Antelopes, we can surmise that from 1900 to 1970's the change in quantities of Tibetan Antelopes depended on the habitat. In other words, the influence that humans might have on to the habitat environment of Tibetan Antelopes has been minimal and almost can be neglected. We can assume in that period the changes in quantities of the Tibetan Antelopes was completely based on in the system of Tibetan Antelopes the Tibetan Antelopes and their environment . Therefore, this model is created based on the assumption according to this analysis.

According to statistics, the quantity of Tibetan Antelopes is 650,000 in 1974, [7]. After that, in very short 15 years, the quantity of Tibetan Antelopes was enormously reduced to 50,000 in 1989. There are many reasons for that outcome, for example, in 1975, China was a very poor country. The people living in the villages had nothing to eat, hence they hunted for the wildlife for their hunger, and as it had always happened in the past. Moreover, at that time there was no law to prevent and disallow people to hunt or to kill the Tibetan Antelopes. Moreover, the very important reason is, at the beginning of 80's, the luxurious custom very popular in Europe and America of wearing, to wear the animal furs was the symbol of prestige and high classes. Hence, the people did not hesitate to purchase the expensive furs. The furs of the Tibetan Antelopes are very soft and some of the warmest wools in the world, and the production process of the furs was very simple and easy. Therefore, the Tibetan Antelopes became the sacrifice animal for the luxuriously dressed. The furs of an adult Tibetan Antelope sold for $8000. Since 1979, the animal has been recognized as an endangered species and protected under the Convention on the International Trade in Endangered Species, [9]. Although there is a policy for protection, the population of the antelopes still was shrinking sharply because of the hunting. I think that is the major reason for the decreasing sharply population for Tibetan Antelope.

3.2 Data for Tibetan Antelope

According to the IUCN (International Union for Conservation of Nature and Natural resources), [5], population estimate between 1950 and 1960 ranged from 500,000 to 1,000,000. The following Table 1 shows the available data about Tibetan Antelope between 1950 and 1960.

Date (Years)	**Population (Million)**
1950	0.500
1951	0.550
1952	0.601
1953	0.645
1954	0.695
1955	0.75
1956	0.816
1957	0.890
1958	0.958
1959	1.041
1960	1.130

Table 1

Chapter 4
Modeling The Logistic Model

4.1 Graph the Data

Considering the population sizes for Tibetan Antelope for the years between 1950 and 1960, we will derive a mathematical model for the Tibetan Antelope.

Using these data, we can plot the graph. It is the following Figure 2.

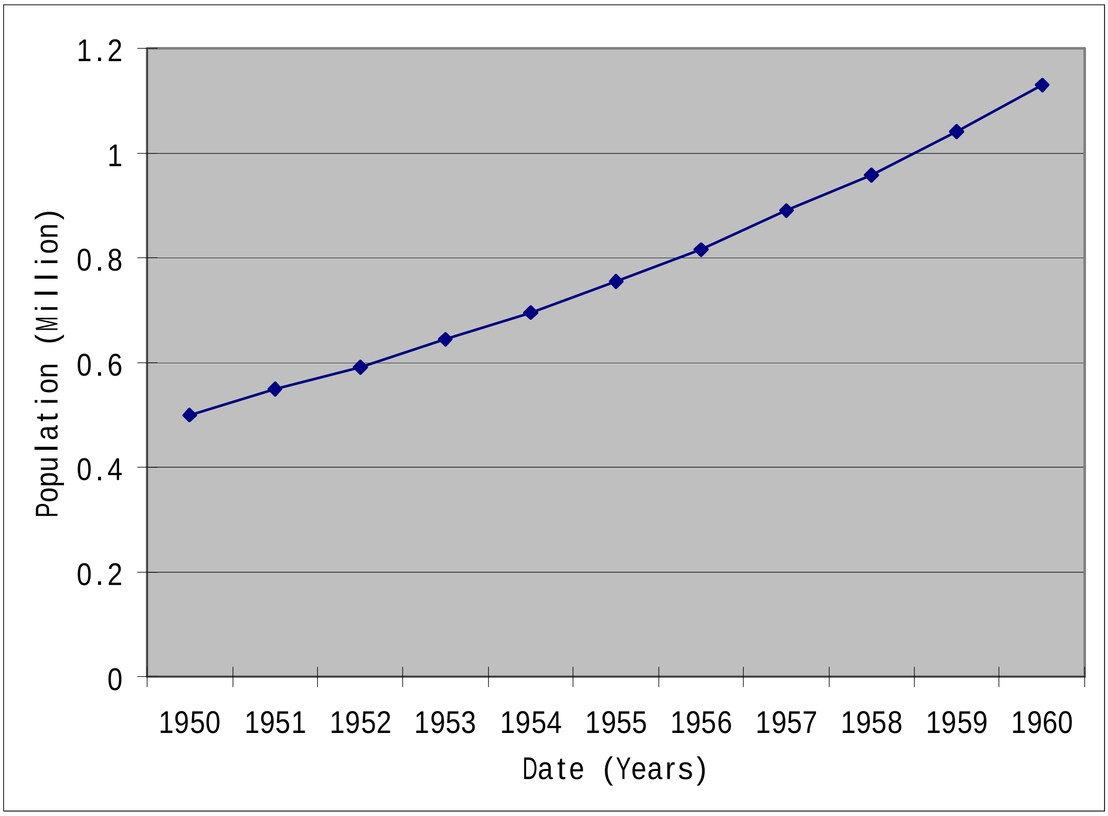


Figure 2

4.2 The Mathematical Model

Let's consider that the logistic growth model with form:

$$P' = rP(1 - \frac{P}{K}) \tag{4.2.1}$$

In order to show model (4.2.1) is logistic, we need to focus on the following questions:

- How to tell whether a given set of data is reasonably logistic?
- What parameters r and K will be good fit?

4.3 Logistic model for the given data

Since we have discrete data, then we describe the model using a difference equation. We use previous values from the systems to calculate the new ones. The equation (4.2.1) can be expressed by the difference equation version as the following equation:

$$P(t+1) - P(t) = rP(1 - \frac{P}{K})$$

It can be rewritten as:

$$\frac{\Delta P}{P} = r(1 - \frac{P}{K}) \tag{4.3.1}$$

The equation says that the ratio of ΔP to P is a linear function of P.

Now we have testing of logistic behavior for our model:

First of all, let's consider the left hand side (LHS) of equation (4.3.1). We calculate the difference of the populations for two consecutive years, and then use those differences against the corresponding function values.

Next, we plot the ratios and the corresponding function values.

At last, if we can show that the plots are approximately linear, then the model equation (4.3.1) is reasonable. That is to say, the model has the form (4.2.1) and it is logistic.

Calculating the ratios on the left hand side of (4.3.1) yields:

1. $a_1 = \dfrac{P(1951)-P(1950)}{P(1950)} = \dfrac{0.55-0.5}{0.5} = 0.1$;

2. $a_2 = \dfrac{P(1952)-P(1951)}{P(1951)} = \dfrac{0.601-0.55}{0.55} \approx 0.0927$;

3. $a_3 = \dfrac{P(1953)-P(1952)}{P(1952)} = \dfrac{0.645-0.601}{0.601} \approx 0.0732$ **;**

4. $a_4 = \dfrac{P(1954)-P(1953)}{P(1953)} = \dfrac{0.695-0.645}{0.645} \approx 0.0775$;

5. $a_5 = \dfrac{P(1955)-P(1954)}{P(1954)} = \dfrac{0.755-0.695}{0.695} \approx 0.0863$;

6. $a_6 = \dfrac{P(1956)-P(1955)}{P(1955)} = \dfrac{0.816-0.755}{0.755} \approx 0.0807$;

7. $a_7 = \dfrac{P(1957)-P(1956)}{P(1956)} = \dfrac{0.890-0.816}{0.816} \approx 0.0906$;

8. $a_8 = \dfrac{P(1958)-P(1957)}{P(1957)} = \dfrac{0.958-0.890}{0.890} \approx 0.0764$;

9. $a_9 = \dfrac{P(1959)-P(1958)}{P(1958)} = \dfrac{1.041-0.958}{0.958} \approx 0.0866$;

10. $a_{10} = \dfrac{P(1960)-P(1959)}{P(1959)} = \dfrac{1.13-1.041}{1.041} \approx 0.0854$.

Thus, we have the following list of data:

a	$P(t)$
0.1	0.500
0.0927	0.550
0.0732	0.601
0.0863	0.695
0.0807	0.755
0.0906	0.816
0.0764	0.890
0.0866	0.958
0.0854	1.041

Table 2

Plotting the Least Square Approximation graph by using the data from Table 2, we obtain following graph:

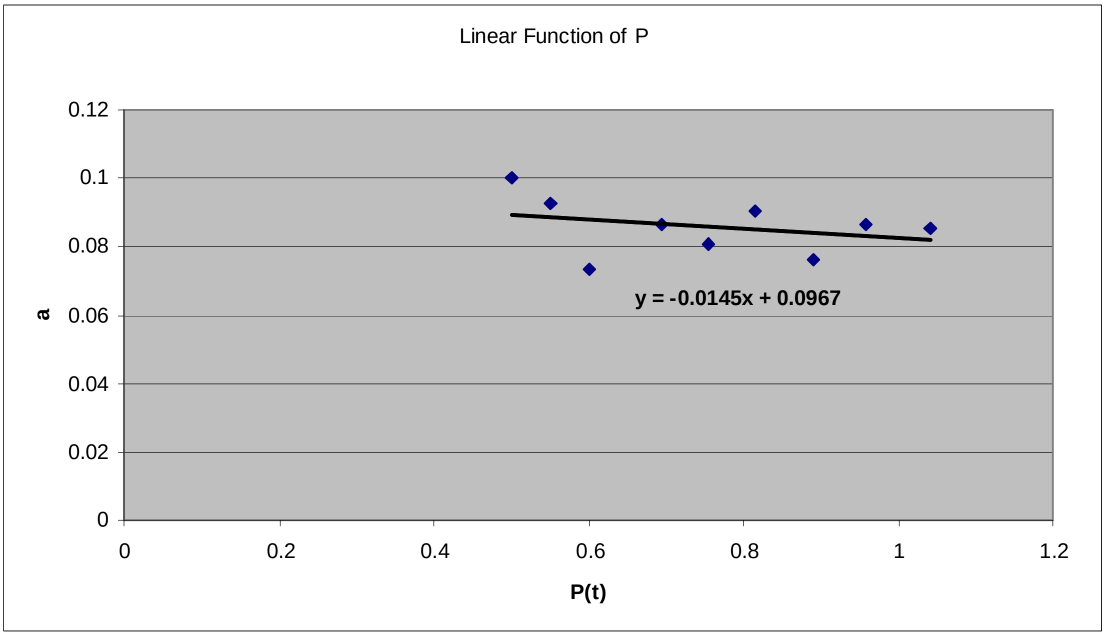


Figure 3

As you can see in Figure 3, at various population plotted levels $P(t)$ at time t, we can calculate corresponding ratios a. Based on these points, we plot Least Square Approximation graph.

Looking at the graph, we can see that most of our data points are close to this line. The overall resulting plot is approximately linear. Therefore, our assumption for the equation (4.2.1) is reasonable. That is the present model (4.2.1) shows that the given data is logistic.

4.4 Determining the values of r and K

In the Least Square Approximation graph (Figure 3), we know the equation for the line, which is,

$$y = -0.0145x + 0.0967$$

Substituting the point P(1950) into this equation, we obtain,

$$y_1 = -0.0145*(0.5) + 0.0967 = 0.08945$$

Similarly, substituting P(1951) into it, we obtain,

$$y_2 = -0.0145*(0.55) + 0.0967 = 0.0887$$

That is to say, we can get values of the ratio, a, where y=a. Then we have,

$$y_1 = 0.08945 \text{ and } y_2 = 0.0887$$

Substituting the data of 1950, 1951 and 1952 to the equation (4.3.1). We have the following two equations:

$$r(1 - \frac{0.5}{K}) = 0.08945 \qquad (4.4.1)$$

$$r(1 - \frac{0.55}{K}) = 0.0887 \qquad (4.4.2)$$

Suppose that $r, K \neq 0$, and divide (4.4.1) by (4.4.2), we can get that:

$$\frac{1-\frac{0.5}{K}}{1-\frac{0.55}{K}} = \frac{0.08945}{0.0887}.$$

Simplifying it, we have:

$$K = 6.463$$

From (4.4.1), then we obtain the value for r,

$$r = 0.097$$

Therefore, the model is:

$$P' = 0.097P(1-\frac{P}{6.463}) \qquad (4.4.3)$$

As we know, the size of population for the logistic model tends to the carrying capacity K. In this case, the size is bound by 6.463 million. In another words, the limiting number for this population model is 6.463 million.

4.5 The Solution for the Logistic Model.

Rewriting the (4.4.3), we have,

$$\frac{dP}{dt} = 0.097P - 0.015P^2 \qquad (4.5.1)$$

Since equation (4.5.1) is one in which variables are separable, we have

$$\int \frac{dP}{P(0.097-0.015P)} = t + c$$

Also because

$$\frac{1}{P(0.097-0.015P)} = \frac{1}{0.097}(\frac{1}{P} + \frac{0.015}{0.097-0.015P})$$

the equation can be written as,

$$\frac{1}{0.097}\int\left(\frac{1}{P}+\frac{0.015}{0.097-0.015P}\right)dP = t + c \qquad (4.5.2)$$

Let t=0 corresponds to the size of population in 1950, 0.5. Then we have

$$P_0 = 0.5$$

Using the condition, $P_0 = 0.5$ at t=0, we get

$$c = \frac{1}{0.097}\left[\ln(0.5) - \ln(0.097 - 0.015\times 0.5)\right] = 17.736$$

Thus, equation (4.5.2) becomes

$$\frac{1}{0.097}(\ln P - \ln(0.097 - 0.015P)) = t + 17.736$$

Solving for P,

$$P(t) = \frac{6.463}{1 + 11.926e^{-0.097t}} \qquad (4.5.3)$$

If we take the limit of solution (4.5.3) as $t \to \infty$, we can see that,

$$P(t) \to 6.463$$

This shows that there is a limit to the growth of P. The limiting number is 6.463 million. This conclusion is the same as the one that we discussed in the previous chapter.

4.6 Comparison of the Logistic Model with Actual Data.

To illustrate the solution (4.5.3) for the differential equation (4.4.3) related to the Tibetan Antelope's data that we have from the time period 1950 to 1960, we want to plot the solution equation (4.5.3) and see how it matches the curve in Figure 2.

Since we have equation (4.5.3), we can get the population for each year. Then we have the following table.

t	P(t)
0	0.5
1	0.546623
2	0.597157
3	0.651849
4	0.710943
5	0.77468
6	0.843293
7	0.917003
8	0.996014
9	1.08051
10	1.17063

Table 3

According to the Table 1 and Table 3, in order to check if the plotting of differential equation matches the data of Tibetan Antelope during 1950 to 1960, we plot the following graph.

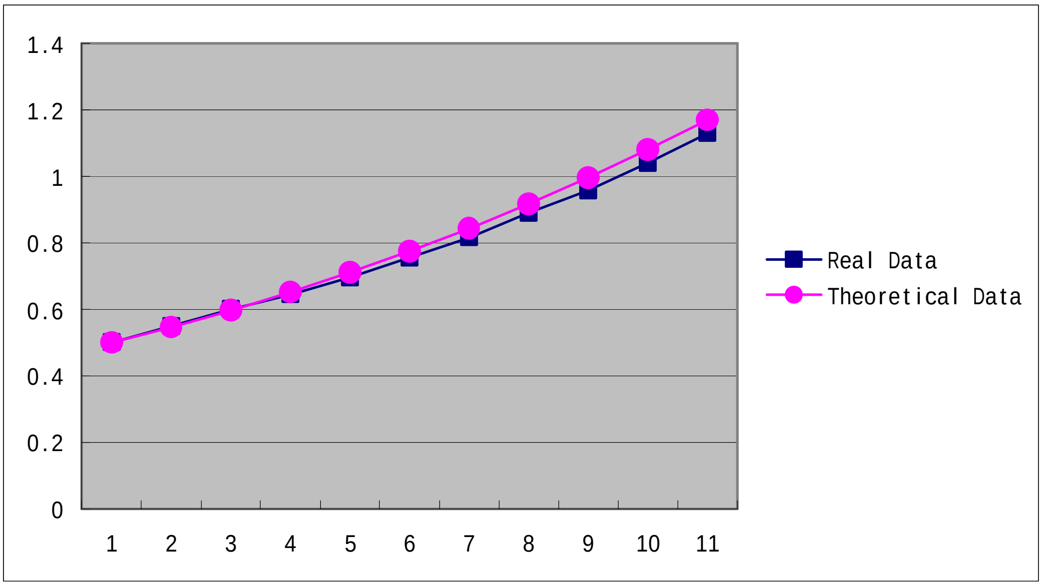


Figure 4

The graph of the solution (4.5.3) has the general appearance shown in the following graph.

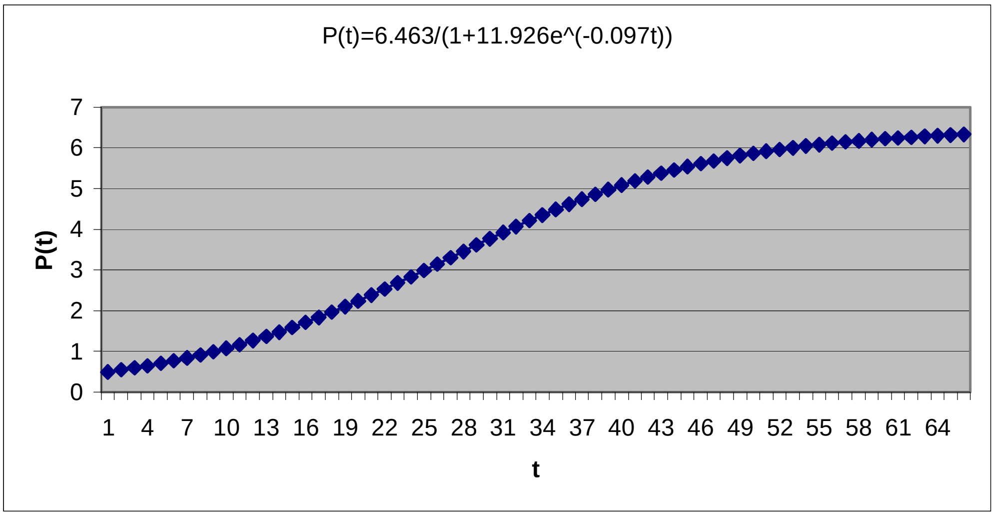
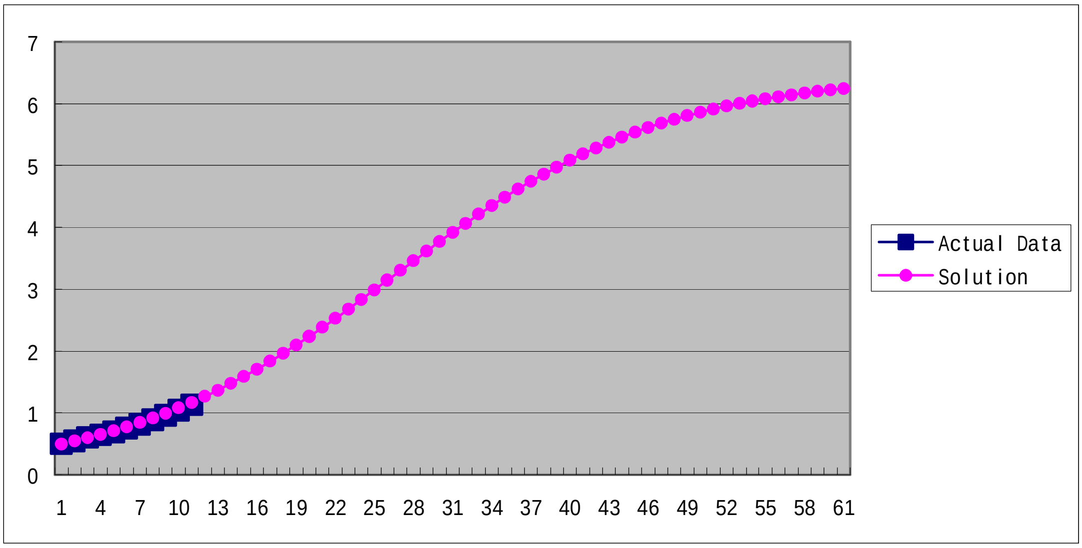


Figure 5

In order to check if the solution curve for the differential equation matches the data of Tibetan Antelope during 1950 to 1960(from t= 0 to t=10), we plot the actual data we have and the graph for the solution together. This is presented in Figure 6.



4

Figure 6

In the Figure 6, we can see the data in the 10 years period which is from 1950 to 1960 are on the plotting of the solution curve.

It implies that the differential equation with r and K that we've calculated estimates the data for Tibetan Antelope from the time period 1950 to 1960.

Chapter 5
Conclusion

A model for the population of Tibetan Antelope from the 1950 to 1960 was constructed. That is, the differential equation that approximately models, this population is

$$P' = 0.097P(1 - \frac{P}{6.463})$$

Also the present model is showed to be logistic. From the model we can predict the population limitation of the Tibetan Antelope. The number is 6.463×10^6.

Moreover, we find the solution for differential equation (4.4.3). Basically it matches the given data. It shows that the mathematical model (4.4.3) we obtain does have potential as a possible logistic growth.

However, this model obviously is not suitable and accountable for the change in population of Tibetan Antelopes after 1960. After 1960, human interferences and influences played a major role in decreasing the population of Tibetan Antelopes. These activities include hunting and killing of Tibetan Antelopes as well as damages to the habitat environment of Tibetan Antelopes. In this case more factors have to be considered in order to create and define a proper model.

The Tibetan Antelope is also among the five doll mascots of the 2008 Summer Olympics to be held in Beijing. The antelope is seen as fully reflecting the spirit of the Olympics. It carries the blessing of health and the strength that comes from harmony with nature.

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