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An Extension of the Jeu de Taquin

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AN EXTENSION OF THE JEU DE TAQUIN

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APPROVAL OF THESIS/DISSERTATION

We, the faculty supervising the work of Joanna McKinney, affirm that the thesis, *An Extension of the Jeu De Taquin*, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the editorial standards of our discipline and the Graduate College of Marshall University. With our signatures, we approve the manuscript for publication.

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ABSTRACT

The Knuth transformations on words, the jeu de taquin moves on tableaux, and the Robinson–Schensted–Knuth algorithm produce the same equivalence classes for words. By observing the connections between these three methods we find and prove there exists connections between the Assaf–Knuth transformations, our extension of the jeu de taquin, and p-RSK. We know there exists an algebraic way to expand Macdonald polynomials in terms of the Schur functions. The form of the expansion implies there should be a combinatorial way to find the expansion. Loehr found a Robinson–Schensted–Knuth like algorithm that works in some cases. By finding an extension of jeu de taquin, we will try to expand the number of cases covered.

CHAPTER 1

INTRODUCTION

Algebraic combinatorics is the study of combinatorial structures that have algebraic applications. Two kinds of combinatorial objects used throughout this paper are words and tableaux. Words are sequences of positive integers. Tableaux are diagrams of squares filled with positive integers. For example 213 and 3412 are words, while $\begin{array}{|c|c} \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$ and $\begin{array}{|c|c|c|c} \cline{1-2} 3 & 4 & \multicolumn{1}{c}{} & \\ \hline \multicolumn{1}{c}{} & 1 & 2 & \multicolumn{1}{|c}{} \\ \cline{2-3} \end{array}$ are tableaux. When given a set of words and a method to show words are equivalent, we can split the set into smaller collections so that any two words in the same collection are equivalent to each other. These collections are called equivalence classes. Equivalence classes can also be found when we start with a set of tableaux. Depending on the method we use to separate the set into equivalence classes, we can look to see if the equivalence classes of the words can be matched with the equivalence classes of the tableaux that correspond to words.

We study a collection of algorithms that extend several classical algorithms which produce equivalence classes of words and tableaux. The Knuth relations give rise to the equivalence relation on words that we study. For the purpose of determining the equivalence classes of words under the Knuth relations we can use the jeu de taquin algorithm or the Robinson–Schensted–Knuth (RSK) algorithm [5]. The jeu de taquin allows us to determine equivalence classes of words by making a tableau from each word and sliding entries in a prescribed manner. The jeu de taquin produces the same equivalence classes as Knuth relations [5]. The Robinson–Schensted–Knuth algorithm takes a word and, using a defined set of rules called an insertion algorithm, creates a pair of tableaux (P, Q). The set of permutations with the same P tableau are in the same Knuth equivalence class. The RSK algorithm has several other applications which are described fully in Sagan [5] and we describe briefly here. The classical RSK algorithm is a key component of the bijective proof of the identity $\Sigma_{\lambda \vdash n}(f^{\lambda})^2 = n!$. This identity states that the size of the set of permutations of n is equal to the set of pairs of standard tableaux of shape λ where λ is all possible partitions of n. By using RSK, we can compare the length of the longest increasing subsequence and longest decreasing subsequence of a permutation to the length of the first row and first column respectively of the P tableau formed by applying RSK to the permutation.

Loehr studied the Assaf–Knuth relations in an effort to identify the combinatorial method to obtain the Schur expansion of modified Macdonald polynomials [3]. Assaf–Knuth relations, when compared to traditional Knuth relations, have two additional transformations which are based on consecutive triple entries and a parameter $p > 0$. Loehr [3] found an extension of the RSK algorithm, called p-RSK, which uses the parameter p, and a newly defined insertion algorithm which includes two special rules. We define a new jeu de taquin on tableaux with properties that parallel the traditional jeu de taquin. We call this algorithm p-jeu de taquin. The p-jeu de taquin algorithm uses the parameter p and includes modified rules from the traditional jeu de taquin algorithm, as well as two additional rules that work with consecutive triple entries.

In order to find the number of standard tableaux of shape λ, we can use the hook length formula, $f^\lambda = \frac{n!}{\Pi_{(i,j)\in\lambda} h_{i,j}}$. A modification of the jeu de taquin aids in the bijective proof of the hook length formula [4]. Motivations for defining the p-jeu de taquin algorithm include completing the combinatorial picture, having another strategy to approach extending the combinatorial methods for the Schur function expansion of $\tilde{H}_\mu$, and giving options for bijectively proving a Macdonald q-analogue of the hook length formula [2].

CHAPTER 2

KNUTH RELATIONS, JEU DE TAQUIN, AND RSK

We are looking at the equivalence classes on words which are provided to us by the traditional versions of the Knuth relations, the jeu de taquin algorithm, and the Robinson–Schensted–Knuth (RSK) algorithm. For these three algorithms we use the conventions of Sagan [5]. We first define each of the algorithms and how they are used. From there, we explain some of the connections between the equivalence classes formed by the three algorithms. Before we begin we review the background information needed to understand the sets of objects the Knuth relations, the jeu de taquin algorithm, and the RSK algorithm act upon.

Definition 1. *A* **word** $w = w_1 \dots w_k$ *is a sequence of positive integers called letters.*

Example 1. An example of a word is $w = 2133521$.

Definition 2. *A word with no repeated letters using* $1, 2, \dots, k$ *exactly once is a* **permutation** *of* $\{1, \dots, k\}$.

Example 2. Given the word $w = 23516748$, we can see w has each letter $1, \dots, 8$ appear exactly once in it. Therefore w is a permutation of $\{1, \dots, 8\}$.

Definition 3. *A word* $w = w_1 \dots w_k$ *of length* $k \leq n$ *is a* **partial permutation** *if every letter is distinct, and* $\{w_1, \dots, w_k\} \subset \{1, \dots, n\}$.

Definition 4. *The* **set of words of length** n , $n \in \mathbb{Z}^+$, *is denoted* W_n.

Definition 5. *A relation* $\sim$ *on set* A *is an* **equivalence relation** *if and only if:*

1. $\sim$ *is reflexive:* $a \sim a$ *for all* $a \in A$,

2. $\sim$ *is symmetric: if* $a \sim b$ *then* $b \sim a$ *for all* $a, b \in A$, *and*

3. $\sim$ *is transitive: If* $a \sim b$, $b \sim c$ *then* $a \sim c$ *for all* $a, b, c \in A$.

The relations we will be looking at in this chapter are the Knuth relations. The Knuth relations, or Knuth transformations, act on W_n. The Knuth relations are a set of rules used to determine if $x, y \in W_n$ are in the same equivalence class.

Definition 6. *Let w_1 and w_2 be words of length n. Let a and b be words where either a or b can be empty. Let $x, y, z \in \mathbb{Z}^+$.*

Let $x < y \leq z$. Then w_1 and w_2 differ by a **Knuth relation of the first kind**, *denoted $K1$, if $w_1 = ayzxb$ and $w_2 = ayxzb$ or vice versa.*

Let $x \leq y < z$. Then w_1 and w_2 differ by a **Knuth relation of the second kind**, *denoted $K2$, if $w_1 = axzyb$ and $w_2 = azxyb$ or vice versa.*

Example 3. An example of two words differing by a $K1$ Knuth relation is $w_1 = 134\mathbf{352}1$ and $w_2 = 134\mathbf{325}1$. An example of two words differing by a $K2$ Knuth relation is $w_2 = 1\mathbf{343}251$ and $w_3 = 1\mathbf{433}251$.

Definition 7. *A word w_1 is* **Knuth equivalent** *to a word w_2, denoted $w_1 \equiv^{\mathrm{K}} w_2$, if and only if w_2 can be obtained from w_1 by applying a sequence of Knuth relations, or if $w_1 = w_2$.*

Example 4. Consider the words $w_1 = 25681374$ and $w_2 = 21563874$. Start with w_1. If we apply a $K1$ Knuth relation to the 681 part of w_1 we obtain 25618374. Applying another $K1$ Knuth relation to 561 provides 25168374. Applying a $K2$ Knuth relation to 837 gives 25163874. Finally, applying a $K1$ Knuth relation to 251 gives $21563874 = w_2$. Since we can obtain w_2 from w_1 by applying a sequence of Knuth relations, we have shown that $w_1 \equiv^{\mathrm{K}} w_2$.

The following theorem can be found in Sagan [5].

Theorem 1. *For each n, Knuth equivalence is an equivalence relation on the set of words W_n.*

Proof. Consider the relation $\equiv^{\mathrm{K}}$ on the set W_n. Let $w, x, y \in W_n$. By definition 7, $w \equiv^{\mathrm{K}} w$. Hence, $\equiv^{\mathrm{K}}$ is reflexive. Let $w \equiv^{\mathrm{K}} x$. Then there exists a sequence of Knuth transformations that allow us to obtain x from w. To get from x to w we can apply the same transformations used, but in opposite order. Hence we have a sequence of Knuth transformations which allows us to obtain w from x. Therefore $x \equiv^{\mathrm{K}} w$. Thus $\equiv^{\mathrm{K}}$ is symmetric. Suppose $w \equiv^{\mathrm{K}} x$ and $x \equiv^{\mathrm{K}} y$. Then there exists a sequence of Knuth transformations that allows us to get from w to x. We will denote this sequence as s_1. There also exists a sequence of Knuth transformations that allows us to get from x to y. We will denote this sequence as s_2. Now we can create a sequence s_3 which is s_1 followed by s_2 which allows us to get from w to y. Thus $w \equiv^{\mathrm{K}} y$. Hence $\equiv^{\mathrm{K}}$ is transitive. Since the relation $\equiv^{\mathrm{K}}$ is reflexive, symmetric, and transitive, $\equiv^{\mathrm{K}}$ is an equivalence relation. □

The jeu de taquin and the RSK algorithms use semi-standard and standard tableaux, which are arrays of positive integers following certain rules.

Definition 8. *A* **partition** λ *of a positive integer* n *is a sequence where* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ *and* $\sum_{i=1}^{k} \lambda_i = n$. *Each* λ_i *is called a* **part** *of* λ.

Example 5. There are seven partitions of 5: $(5), (4,1), (3,2), (3,1,1), (2,2,1)$, $(2,1,1,1)$, and $(1,1,1,1,1)$.

Definition 9. *A* **partition diagram** *of* $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, *denoted* $dg(\mu)$, *is a collection of left-justified boxes with* μ_i *boxes in the* i*th row from the bottom of the diagram.*

These diagrams can be filled with positive integers in a number of ways.

Definition 10. *A* **semi-standard Young tableau** *(SSYT) is a partition diagram filled with positive integers, repetitions allowed, such that each row weakly increases from left to right and each column strictly increases from bottom to top. An example can be found in Figure 2.1.*

$$T_1 = \begin{array}{|c|c|c|c}
\cline{1-2} 3 & 5 & \multicolumn{2}{c}{} \\
\cline{1-3} 2 & 4 & \multicolumn{1}{c|}{4} & \\
\hline 1 & 2 & 2 & \multicolumn{1}{c|}{2} \\
\hline
\end{array} \; .$$

Figure 2.1: Semi-Standard Young Tableau
The semi-standard Young tableau T_1 has partition shape $\mu = (4,3,2)$ and $rw(T_1) = 352441222$.

Definition 11. *A* **standard Young tableau** *is a partition diagram filled with positive integers, repetitions not allowed, such that each row strictly increases from left to right and each column strictly increases from bottom to top. An example can be found in Figure 2.2.*

$$T_2 = \begin{array}{|c|cccc}\hline 9 \\ \hline 5 \\ \hline 3 & 7 & 8 \\ \hline 1 & 2 & 4 & 6 \\ \hline \end{array} \quad .$$

Figure 2.2: Standard Young Tableau
The standard Young tableau T_2 has the partition shape $\mu = (4, 3, 1, 1)$ and $rw(T_2) = 953781246$.

Definition 12. *Let μ and v be integer partitions such that for each $i \geq 1$, $v_i \leq \mu_i$. Then the* **skew diagram** *μ/v is the diagram formed by taking the diagram of μ and erasing the boxes in the diagram of v. An example of a skew diagram can be found in Figure 2.3.*

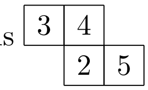
Skew diagrams can be filled with positive integers following the appropriate conventions to obtain skew semi-standard Young tableaux and skew standard Young tableaux. An example of a skew semi-standard Young tableau can be found in Figure 2.4.

$$\mu/v = \begin{array}{ccccccc} \square & \square & \square & & & & \\ & & \square & \square & \square & & \\ & & & & \square & \square & \square \end{array} \quad .$$

Figure 2.3: Skew Partition Diagram
The skew partition diagram μ/v has partition shape $\mu/v = (7, 5, 3)/(4, 2)$.

$$T_3 = \begin{array}{ccccccc} 3 & 4 & 5 & & & & \\ & & 2 & 2 & 4 & & \\ & & & & 1 & 2 & 2 \end{array} \quad .$$

Figure 2.4: Skew Semi-Standard Young Tableau
The skew semi-standard Young tableau, T_3 has partition shape $\mu/v = (7, 5, 3)/(4, 2)$ and $rw(T_3) =$ 345224122.

When given a word w we can create a skew semi-standard Young tableau by first placing the first letter of the word in a cell. If the second letter is greater than or equal to the first letter it gets placed in a cell to the right of the first letter in the same row. If the second letter is less than the first letter then it gets placed below the first letter and starts a new row. Next compare the third letter to the second. This continues until we have placed all the letters in the word. For example if $w = 3425$ then the corresponding skew semi-standard Young tableau is .

Definition 13. *Let T be a tableau. The* **reading word** *of T, denoted $rw(T)$, is the word obtained by listing entries of T row by row left to right starting with the top row.*

Example 6. Consider the tableaux from Figure 2.1, Figure 2.2, and Figure 2.4. The reading words of T_1, T_2, and T_3 are $rw(T_1) = 352441222$, $rw(T_2) = 953781246$, and $rw(T_3) = 345224122$.

Any partition diagram that is not a skew diagram is called a straight shape diagram. Therefore, tableaux T_1 and T_2 in Figure 2.1 and Figure 2.2 respectively are called straight shape tableaux.

In order to turn a skew semi-standard Young tableau into a straight shape semi-standard Young tableau we will use an algorithm called the jeu de taquin, or teasing game.

Definition 14. *An* **inner corner** *of a skew partition diagram μ/v is any cell in partition diagram v where if the cell was added to the μ/v diagram the result would be a valid skew partition diagram. The two inner corners of $(7,5,3)/(4,2)$ can be seen in Figure 2.5.*

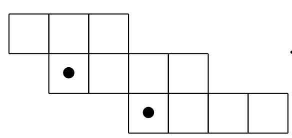
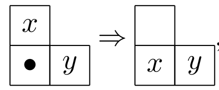
$\mu/v =$  .

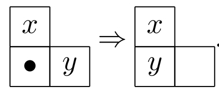
Figure 2.5: Skew Partition Diagram With Inner Corners
The skew partition diagram μ/v has partition shape $\mu/v = (7,5,3)/(4,2)$. The two $\boxed{\bullet}$ symbols represent the inner corners of the skew partition diagram.

Definition 15. *The jeu de taquin algorithm relies on two* **sliding rules**. *Given a skew tableau and an empty cell as shown below, two possible slides can be made. When at the edges of the tableau, either x or y might be empty. Let $\bullet$ be the cell that is going to be filled.*

1. If x and y are letters where $x \leq y$ or y is empty then



2. If x and y are letters where $x > y$ or x is empty then



Algorithm 1. The **jeu de taquin** algorithm starts with an inner corner of a skew tableau and applies the two sliding rules repeatedly until a straight shape tableau is obtained. When performing the jeu de taquin algorithm on a tableau T, the resulting finished straight shape tableau is called $jdt(T)$. This is shown in Figure 2.6. When the initial skew tableau comes from a word w then we use the notation $jdt(w)$ instead.

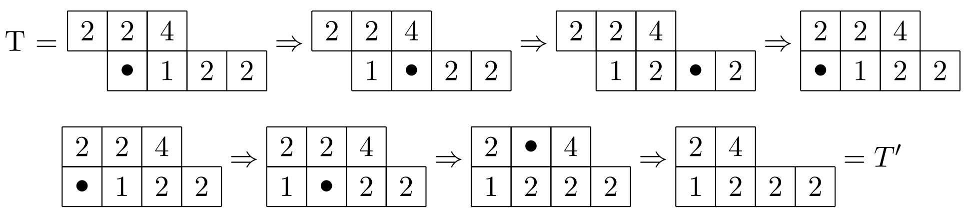


Figure 2.6: Jeu de Taquin Example
An example of the jeu de taquin, where we start with the skew semi-standard Young tableau T and end with a straight shape tableau $T' = jdt(T)$. The cell with symbol $\bullet$ denotes the cell that will be filled as a result of completing one slide move of the jeu de taquin. Starting with T, since $1 < 2$ the 1 will slide to the left one space. Since $2 < 4$ the 2 will slide to the left one space. Since there is no filled cell above the $\bullet$ the 2 will slide to the left to fill the cell. Since $1 < 2$, the 1 will move to the left one space. Since $2 \leq 2$, the 2 will slide down one space providing a space for the 4 to slide to the left.

The following theorem can be found in Sagan [5].

Theorem 2. *All choices of inner corners will result in the same rectified (straight shape) semi-standard tableau.*

We show that the jeu de taquin algorithm on tableaux formed from words and Knuth equivalence on words produce the same equivalence classes. The two tables, Table 2.1 and

Table 2.2, show the relationship between performing the jeu de taquin on a skew tableau and the corresponding word. In Table 2.1 we have $x < y \leq z$ and $rw(T_1) = yzx$. By applying the jeu de taquin algorithm to T_1 we obtain $rw(jdt(T_1)) = rw(T_2) = yxz$. The $K1$ Knuth relation gives $yzx \equiv^{\text{K}} yxz$, so $rw(T_1) \equiv^{\text{K}} rw(jdt(T_1))$. In Table 2.2 we have $x \leq y < z$ and $rw(T_1) = xzy$. After applying the jeu de taquin algorithm to T_1 we obtain $rw(jdt(T_1)) = rw(T_2) = zxy$. The $K2$ Knuth relation gives $xzy \equiv^{\text{K}} zxy$, so $rw(T_1) \equiv^{\text{K}} rw(jdt(T_1))$.

Word	T_1	T_2	$rw(T_2)$
yzx	$\begin{matrix} y & z \\ \square & x \end{matrix}$	$\begin{matrix} y & \\ x & z \end{matrix}$	yxz

Table 2.1: Jeu de Taquin Slide 1
Let $x < y \leq z$. If we take the word yzx and write it as a skew semi-standard Young tableau, we obtain the tableau T_1 in the first column of the table. By applying the jeu de taquin algorithm to T_1 we get $jdt(T_1) = T_2$ where the $rw(T_2) = yxz$. Thus the $rw(T_1) = yzx$ and the $rw(T_2) = yxz$ differ by a $K1$ Knuth relation.

Word	T_1	T_2	$rw(T_2)$
xzy	$\begin{matrix} x & z \\ \square & y \end{matrix}$	$\begin{matrix} z & \\ x & y \end{matrix}$	zxy

Table 2.2: Jeu de Taquin Slide 2
Let $x \leq y < z$. If we take the word xzy and write it as a skew semi-standard Young tableau, we obtain the tableau T_1 in the first column of the table. By applying the jeu de taquin algorithm to T_1 we get $jdt(T_1) = T_2$ which the $rw(T_2) = zxy$. Thus the $rw(T_1) = xzy$ and the $rw(T_2) = zxy$ differ by a $K2$ Knuth relation.

Example 7. From Figure 2.6 we can see that $jdt(T) = T'$. The $rw(T) = 224122$ and the $rw(T') = 241222$. Starting with a $K1$ transformation on 241 in $rw(T)$ we have 2**214**22. From here we can apply another $K1$ transformation on 221 to get **212**422. By applying a $K2$ transformation on 242 we get 21**422**2. Finally, we can apply a $K1$ transformation on 214 to get **241**222 $= rw(jdt(T)) = rw(T')$. Since there is a sequence of Knuth transformations that can get us from $rw(T)$ to $rw(jdt(T))$ then $rw(T) \equiv^{\mathrm{K}} rw(jdt(T))$.

Now that we have shown how the jeu de taquin and Knuth relations are connected, we can prove that the reading word of a skew tableau T is Knuth equivalent to the reading word of the straight shape tableau formed after applying the jeu de taquin to T. In order to prove this, we will need the following lemma found in Sagan [5].

Lemma 1. *Let $a_1 < a_2 < a_3 < \cdots < a_n$. Then,*

1. *If $x < a_1$, then $a_1a_2 \ldots a_nx \equiv^{\mathrm{K}} a_1xa_2 \ldots a_n$.*
2. *If $x > a_n$, then $xa_1a_2 \ldots a_n \equiv^{\mathrm{K}} a_1a_2 \ldots a_{n-1}xa_n$.*

Proof. Let $a_1 < a_2 < a_3 < \cdots < a_n$.

1. Let $x < a_1$. Consider $n = 2$. Then $a_1a_2x \equiv^{\mathrm{K}} a_1xa_2$ by definition of $K1$. Assume the property holds for $n = k - 1$. Thus $a_1a_2 \ldots a_{k-1}x \equiv^{\mathrm{K}} a_1xa_2 \ldots a_{k-1}$. Then we have $a_1a_2 \ldots a_{k-1}a_kx$ where $x < a_1 < \cdots < a_{k-1} < a_k$. Since $x < a_{k-1} < a_k$ we can apply $K1$ to see that $a_1a_2 \ldots a_{k-1}a_kx \equiv^{\mathrm{K}} a_1a_2 \ldots a_{k-1}xa_k$. However by assumption we know $a_1a_2 \ldots a_{k-1}x \equiv^{\mathrm{K}} a_1xa_2 \ldots a_{k-1}$. Thus $a_1a_2 \ldots a_{k-1}a_kx \equiv^{\mathrm{K}} a_1xa_2 \ldots a_{k-1}a_k$. Therefore, by mathematical induction, if $x < a_1$, then $a_1a_2 \ldots a_nx \equiv^{\mathrm{K}} a_1xa_2 \ldots a_n$ for all $n \in \mathbb{N}$, $n \geq 2$.
2. Let $x > a_n$. Consider $n = 2$. Then $xa_1a_2 \equiv^{\mathrm{K}} a_1xa_2$ by definition of $K2$. Assume the property holds for $n = k - 1$. Thus $xa_1a_2 \ldots a_{k-1} \equiv^{\mathrm{K}} a_1a_2 \ldots xa_{k-1}$. Then we have $xa_1a_2 \ldots a_{k-1}a_k$ where $a_1 < \cdots < a_{k-1} < a_k < x$. By assumption we know $xa_1a_2 \ldots a_{k-1} \equiv^{\mathrm{K}} a_1a_2 \ldots xa_{k-1}$. Thus $xa_1a_2 \ldots a_{k-1}a_k \equiv^{\mathrm{K}} a_1a_2 \ldots xa_{k-1}a_k$. Since $a_{k-1} < a_k < x$ we can apply $K2$ to see that $a_1a_2 \ldots xa_{k-1}a_k \equiv^{\mathrm{K}} a_1a_2 \ldots a_{k-1}xa_k$. Therefore, by mathematical induction, if $x > a_n$, then $xa_1a_2 \ldots a_n \equiv^{\mathrm{K}} a_1a_2 \ldots xa_n$ for all $n \in \mathbb{N}$, $n \geq 2$. □

The following theorem can be found in Sagan [5].

Theorem 3. *Let T be a skew tableau and $jdt(T)$ the result of performing jeu de taquin on T. Then $rw(T) \equiv^{\mathrm{K}} rw(jdt(T))$.*

Proof. When performing the jeu de taquin on a tableau, the only rows that are looked at when performing a single move are the row with the empty cell being filled and the row above it. Thus, the proof of this theorem will be by induction on only two row configurations. Also, only one slide move of the jeu de taquin will be taken into consideration for the proof, since performing more than one move is performing the same process over again.

Let T be a skew tableau and T' be the result of performing one of the two rules of the jeu de taquin on T. Consider when T has only one entry. Since jeu de taquin moves are slides, no entries are added or lost while performing one of the rules of the jeu de taquin. Thus, T' has only one entry. Hence, $rw(T) \equiv^{\mathrm{K}} rw(T')$.

Assume that when T has 1 to k entries that $rw(T) \equiv^{\mathrm{K}} rw(T')$. Now consider when T has $k+1$ entries. Let x occupy the cell above the empty cell. A single slide rule of the jeu de taquin algorithm will be applied to fill the cell below the x. Thus T can have two possible configurations. The first is the configuration in Figure 2.7 and the second is the configuration in Figure 2.8.

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline y_1 & \cdots & y_n & x & v_1 & \cdots & v_m \\ \hline u_1 & \cdots & u_n & & z_1 & \cdots & z_m \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline y_2 & \cdots & y_n & x & v_1 & \cdots & v_m \\ \hline u_2 & \cdots & u_n & & z_1 & \cdots & z_m \\ \hline \end{array}$$

Figure 2.7: Configuration 1 for Theorem 3

In Figure 2.7 we have T which is configuration 1 for Theorem 3. For this configuration there needs to be the same number of y_i and u_i, as well as the same number of v_i and z_i. The second tableau T_2 is the result of removing the first column of T.

$$\begin{array}{|c|c|c|c|c}\cline{1-4} l & w_1 & x & w_2 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & w_3 & & w_4 & \multicolumn{1}{c|}{h} \\ \cline{2-5} \end{array}$$

Figure 2.8: Configuration 2 for Theorem 3

Each w_i where $i \in \{1, 2, 3, 4\}$ is a word. Since Figure 2.8 is a skew tableau, the length of w_3 must be less than or equal to the length of w_1. Similarly, the length of w_2 has to be less than or equal to the length of w_4. Notice either l or h can be empty.

First we will consider the configuration in Figure 2.7. If x does not slide down into the empty cell then z_1 on the bottom row slides to the left resulting in $rw(T) = rw(T')$ and thus $rw(T) \equiv^{\mathrm{K}} rw(T')$. Now consider when x slides down into the empty cell. Let T, with $k+1$ entries have the configuration in Figure 2.7. If we remove the first column of T we have T_2 from Figure 2.7 which has $k-1$ entries. By our induction hypothesis $rw(T_2) \equiv^{\mathrm{K}} rw(T_2')$. Thus,

$$y_2 \ldots y_n x v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m \equiv^{\mathrm{K}} y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n x z_1 \ldots z_m.$$

Now consider the reading word of T. Notice $rw(T) = y_1 \dots y_n x v_1 \dots v_m u_1 \dots u_n z_1 \dots z_m$. Since u_1 is less than all letters that are written before it, that is $u_1 < y_1 < \cdots < y_n < x < v_1 < \dots v_m$, and the letters are strictly increasing till u_1 in $rw(T)$, by Lemma 1, we have

$$y_1 \dots y_n x v_1 \dots v_m u_1 \dots u_n z_1 \dots z_m \equiv^{\mathrm{K}} y_1 u_1 y_2 \dots y_n x v_1 \dots v_m u_2 \dots u_n z_1 \dots z_m.$$

Since

$$y_2 \dots y_n x v_1 \dots v_m u_2 \dots u_n z_1 \dots z_m \equiv^{\mathrm{K}} y_2 \dots y_n v_1 \dots v_m u_2 \dots u_n x z_1 \dots z_m$$

then

$$y_1 u_1 y_2 \dots y_n x v_1 \dots v_m u_2 \dots u_n z_1 \dots z_m \equiv^{\mathrm{K}} y_1 u_1 y_2 \dots y_n v_1 \dots v_m u_2 \dots u_n x z_1 \dots z_m.$$

By applying Lemma 1 again we get

$$y_1 u_1 y_2 \dots y_n v_1 \dots v_m u_2 \dots u_n x z_1 \dots z_m \equiv^{\mathrm{K}} y_1 \dots y_n v_1 \dots v_m u_1 u_2 \dots u_n x z_1 \dots z_m.$$

Therefore,

$$rw(T) = y_1 \dots y_n x v_1 \dots v_m u_1 \dots u_n z_1 \dots z_m \equiv^{\mathrm{K}} y_1 \dots y_n v_1 \dots v_m u_1 u_2 \dots u_n x z_1 \dots z_m = rw(T').$$

Hence, $rw(T) \equiv^{\mathrm{K}} rw(T')$ when T has the configuration of Figure 2.7.

Now consider the configuration in Figure 2.8. Each w_i where $i \in \{1, 2, 3, 4\}$ is a word. Since the tableau in Figure 2.8 is a skew tableau, the length of w_3 must be less than or equal to the length of w_1. Similarly, the length of w_2 has to be less than or equal to the length of w_4. Let T, with $k+1$, entries have the configuration in Figure 2.8. Without loss of generality let l be a filled cell. Notice if we remove l and its corresponding cell we get a valid skew tableau called T_2 which has k entries. Notice $rw(T) = l \cdot rw(T_2)$. Since T_2 has k entries, by induction $rw(T_2) \equiv^{\mathrm{K}} rw(T_2')$. Thus $rw(T) = l \cdot rw(T_2) \equiv^{\mathrm{K}} l \cdot rw(T_2')$. Since l is the first letter in $rw(T)$ and it is not in a cell directly above or to the right of the cell being filled, the position of l will not change from T to T'. Therefore $l \cdot rw(T_2') = rw(T')$. Hence $rw(T) \equiv^{\mathrm{K}} rw(T')$ when T has $k+1$ entries and has the configuration of Figure 2.8 .

Since $rw(T) \equiv^{\mathrm{K}} rw(T')$ for both Figure 2.7 and Figure 2.8 then $rw(T) \equiv^{\mathrm{K}} rw(T')$ for all valid skew tableaux. □

The following theorem can be found in Sagan [5].

Theorem 4. *Let w_1 and w_2 be words such that $w_1 \equiv^{\mathrm{K}} w_2$. Then $jdt(w_1) = jdt(w_2)$.*

So far, we know two words are in the same equivalence classes if they are Knuth equivalent or if one is the $rw(T)$ and the other the $rw(jdt(T))$. The final way to tell if two words are Knuth equivalent or are in the same equivalence class, is by using the Robinson–Schensted–Knuth (RSK) algorithm. Before we can talk about the RSK algorithm we need to define the insertion algorithm.

Algorithm 2. Given a tableau T and a positive integer x, insert x into T, denoted $T \leftarrow x$, in the following way:

1. Starting with the bottom row, let y denote the smallest (left most) letter in the row such that $y > x$. Replace y with x.

2. If no such y value exists append x to the end of the current row in a new box.

3. Now insert y into the next row following the same procedure.

4. Repeat until step 2 occurs.

Example 8. We want to insert $\boxed{1}$ into

$$T = \begin{array}{|c|c|c|c|} \hline 4 \\ \cline{1-1} 3 \\ \cline{1-3} 2 & 3 & 4 \\ \hline 1 & 2 & 2 & 3 \\ \hline \end{array} .$$

The process to do so is laid out in Table 2.3.

Inserted	Row	New Tableau	Bumped Entry
$\boxed{1}$	1	4 3 2 3 4 1 **1** 2 3	$\boxed{2}$
$\boxed{2}$	2	4 3 2 **2** 4 1 1 2 3	$\boxed{3}$
$\boxed{3}$	3	4 3 **3** 2 3 4 1 1 2 3	Nothing, the algorithm has ended.

Table 2.3: Insertion Algorithm Example
Given T from Example 8, the insertion algorithm is shown while beginning by inserting 1 into the tableau.

Given a word $w_1 w_2 w_3 \dots w_k$ we can use insertion to create a tableau. That is, $(((\phi \leftarrow w_1) \leftarrow w_2) \cdots \leftarrow w_k)$. To invert row insertion we need to know the order the boxes were added. This brings us to the RSK algorithm.

Algorithm 3. The **Robinson–Schensted–Knuth** algorithm (RSK) starts with a word $w = w_1 w_2 \dots w_k$, and forms P a SSYT and Q a SYT in the following manner:

$$P_1 = (\phi \leftarrow w_1) = \boxed{w_1} \text{ and } Q_1 = \boxed{1},$$

then $P_j = (P_{j-1} \leftarrow w_j)$ and Q_j is created from Q_{j-1} by placing j in the cell added to P_j.

The tableau Q is always a standard Young tableau and P is a standard Young tableau when w

P_i	Q_i
$P_1 = \begin{array}{|c|}\hline 3 \\ \hline \end{array}$	$Q_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$
$P_2 = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array}$	$Q_2 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$
$P_3 = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$	$Q_3 = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$
$P_4 = \begin{array}{|c|c}\cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$	$Q_4 = \begin{array}{|c|c}\cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$

Table 2.4: Constructing P and Q Tableaux by RSK
Given the word $w = 3212$ we construct the corresponding P tableau and Q tableau by using the RSK algorithm.

is a permutation.

Example 9. We construct the tableaux that corresponds to $w = 3212$ using the RSK algorithm in Table 2.4.

When given a word w, if we perform the RSK algorithm on w, the P tableau will be the same as $jdt(w)$. Therefore, if two words are Knuth equivalent, then the words have the same P tableau under RSK, since the words would have the same ending straight shape tableaux when the jeu de taquin algorithm is applied to them. The following theorem can be found in Sagan [5].

Theorem 5. *Let $w = w_1 \ldots w_n$ be a permutation and $w^{-1} = v_1 \ldots v_n$ be the inverse permutation of w such that $v_j = i$ if and only if $w_i = j$. Let P and Q be the tableaux formed after applying RSK to w. Then $P = jdt(w)$ and $Q = jdt(w^{-1})$.*

Example 10. Let $w = 532164$. Then $w^{-1} = 432615$. When we perform the RSK algorithm on w we obtain the P and Q tableaux in Table 2.5. The P tableau that corresponds to w after performing RSK is the same as the straight shape tableau formed when the jeu de taquin is applied to w as seen in Figure 2.9. The Q tableau that corresponds to w after performing RSK is

5	
3	
2	
1	6
•	4

5	
3	
2	6
1	4

Figure 2.9: Example of $jdt(w)$
The jeu de taquin algorithm is performed on the skew tableau that corresponds to the word $w = 532164$. Since there are four decreases in w there are five rows in the skew tableau that has the reading word w.

4		
3		
2	6	
•	1	5

4	
3	
2	6
1	5

Figure 2.10: Example of $jdt(w^{-1})$
Given the word $w = 532164$, we perform the jeu de taquin algorithm on the skew tableau that corresponds to the word $w^{-1} = 432615$. Since there are three decreases in w^{-1}, there are four rows in the skew tableau that has the reading word w^{-1}.

the same as the straight shape tableau formed when the jeu de taquin is applied to w^{-1} as seen in Figure 2.10.

P_i	Q_i
$P_1 = \begin{array}{l} 5 \end{array}$	$Q_1 = \begin{array}{l} 1 \end{array}$
$P_2 = \begin{array}{l} 5 \\ 3 \end{array}$	$Q_2 = \begin{array}{l} 2 \\ 1 \end{array}$
$P_3 = \begin{array}{l} 5 \\ 3 \\ 2 \end{array}$	$Q_3 = \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$
$P_4 = \begin{array}{l} 5 \\ 3 \\ 2 \\ 1 \end{array}$	$Q_4 = \begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \end{array}$
$P_5 = \begin{array}{ll} 5 & \\ 3 & \\ 2 & \\ 1 & 6 \end{array}$	$Q_5 = \begin{array}{ll} 4 & \\ 3 & \\ 2 & \\ 1 & 5 \end{array}$
$P_6 = \begin{array}{ll} 5 & \\ 3 & \\ 2 & 6 \\ 1 & 4 \end{array}$	$Q_6 = \begin{array}{ll} 4 & \\ 3 & \\ 2 & 6 \\ 1 & 5 \end{array}$

Table 2.5: Example of RSK
The steps of RSK with word $w = 532164$. The result of the RSK are the two tableaux P and Q.

CHAPTER 3

ASSAF–KNUTH RELATIONS, P-JDT, AND P-RSK

The Assaf–Knuth relations on permutations are an extension of the Knuth relations. We will be using the Assaf–Knuth relations as defined by Loehr [3]. The dual of the relations defined by Loehr was introduced by Assaf [1]. In order to find a modified version of the jeu de taquin that corresponds with the Assaf–Knuth relations we must account for the additional relations.

Definition 16. *Let p be a fixed value such that $p \in \mathbb{Z}^+$. Let w_1 and w_2 be partial permutations. Let y and z be partial permutations where either y or z can be empty. Let $a, b, c \in \mathbb{Z}^+$ such that $a < b < c$.*

1. *If $p > a$ or a, b, c are not consecutive, then w_1 and w_2 differ by an* **Assaf–Knuth relation of the first kind***, denoted $K1_p$, if $w_1 = ybcaz$ and $w_2 = ybacz$ or vice versa.*

2. *If $p > a$ or a, b, c are not consecutive, then w_1 and w_2 differ by an* **Assaf–Knuth relation of the second kind***, denoted $K2_p$, if $w_1 = yacbz$ and $w_2 = ycabz$ or vice versa.*

3. *If $p \leq a$ and a, b, c are consecutive, then w_1 and w_2 differ by an* **Assaf–Knuth relation of the third kind***, denoted $A1_p$, if $w_1 = yacbz$ and $w_2 = ycabz$ or vice versa.*

4. *If $p \leq a$ and a, b, c are consecutive, then w_1 and w_2 differ by an* **Assaf–Knuth relation of the fourth kind***, denoted $A2_p$, if $w_1 = yacbz$ and $w_2 = ybacz$ or vice versa.*

Definition 17. *We say a permutation w_1 is* **p-Assaf–Knuth equivalent** *to a permutation w_2, denoted $w_1 \equiv_{\mathrm{AK_p}} w_2$, if and only if w_2 can be obtained from w_1 by a sequence of Assaf–Knuth relations where p is fixed or $w_1 = w_2$.*

Since the Assaf–Knuth relations have a parameter p which helps determine which of the Assaf–Knuth relations we may use, examples of p-Assaf–Knuth equivalent words are provided for several values of p.

Example 11. Let $p = 1$. Consider the permutations $w_1 = 251348679$ and $w_2 = 214573896$. Start with w_1. By applying a $K1_1$ transformation to 251 in w_1 we obtain **215**348679. Applying an $A1_1$

transformation to 534 results in 21**453**8679. Applying another $A1_1$ transformation to 867 results in 21453**786**9. Using a $K1_1$ transformation on 869 provides 214537**896**. Finally, we can apply $K1_1$ on 537 to get 214**573**896 $= w_2$. Since we can obtain w_2 from w_1 by applying a sequence of Assaf–Knuth relations, we have shown that $w_1 \equiv_{\mathrm{AK}_1} w_2$.

Example 12. Let $p = 4$. Consider the permutations $w_1 = 251348679$ and $w_2 = 213574896$. Start with w_1. By applying a $K1_4$ transformation to 251 in w_1, we obtain **215**348679. While looking at 534 our least element 3 is less than $p = 4$. Therefore, we will use a $K2_4$ transformation on 534 to obtain 21**354**8679. Since $6 > 4 = p$, we can apply an $A1_4$ transformation to 867 which results in 213547**86**9. Using a $K1_4$ transformation on 869 provides 213547**896**. Finally, applying $K1_4$ on 547 results in 213**574**896 $= w_2$. Since we can obtain w_2 from w_1 by applying a sequence of Assaf–Knuth relations, we have shown that $w_1 \equiv_{\mathrm{AK}_4} w_2$.

Example 11 and Example 12 both start with $w_1 = 251348679$. After applying Assaf–Knuth transformations to the same locations and order to w_1 in both examples, we find that the permutation w_2 we obtain is different. In Example 11 when $p = 1$ we obtain $w_2 = 214573896$, and in Example 12 when $p = 4$ we obtain $w_2 = 213574896$. If we fix p to be different values before applying the Assaf–Knuth relations to w_1, the permutations that are p-Assaf–Knuth equivalent to w_1 change. If $p > n - 2$, then only $K1_p$ and $K2_p$ Assaf–Knuth relations will be used. Since $K1_p$ and $K2_p$ are the same moves as the traditional Knuth relations $K1$ and $K2$, the equivalence classes formed will be the same as the traditional Knuth equivalence classes when $p > n - 2$.

The Knuth equivalence is an equivalence relation on the set of words of length n. Since the Assaf–Knuth relations are performed on partial permutations, we show that p-Assaf–Knuth equivalence is an equivalence relation on the set of permutations of length n. The set of permutations of length n is denoted S_n.

Theorem 6. *The p-Assaf–Knuth equivalence is an equivalence relation on the set of permutations S_n.*

Proof. Consider the relation $\equiv_{\mathrm{AK_p}}$ on the set of S_n. Let p be a fixed positive integer. Let $w, x, y \in S_n$. By Definition 17, $w \equiv_{\mathrm{AK_p}} w$. Hence $\equiv_{\mathrm{AK_p}}$ is reflexive. Let $w \equiv_{\mathrm{AK_p}} x$. Then there exists a sequence of Assaf–Knuth transformations that allow us to obtain x from w. To get from

x to w, we can apply the same transformations used, but in opposite order. Hence we have a sequence of Assaf–Knuth transformations which allow us to obtain w from x. Therefore $x \equiv_{\mathrm{AK_p}} w$. Thus $\equiv_{\mathrm{AK_p}}$ is symmetric. Consider when $w \equiv_{\mathrm{AK_p}} x$ and $x \equiv_{\mathrm{AK_p}} y$. Then there exists a sequence of Assaf–Knuth transformations that allow us to get from w to x. We will denote this sequence as s_1. There also exists a sequence of Assaf–Knuth transformations that allow us to get from x to y. We will denote this sequence as s_2. Now we can create a sequence s_3 which is s_1 followed by s_2 which allows us to get from w to y. Thus $w \equiv_{\mathrm{AK_p}} y$. Hence $\equiv_{\mathrm{AK_p}}$ is transitive. Since the relation $\equiv_{\mathrm{AK_p}}$ is reflexive, symmetric and transitive, $\equiv_{\mathrm{AK_p}}$ is an equivalence relation. □

We construct a jeu de taquin type process that incorporates all of the Assaf–Knuth relations. As a result, p is an important factor of our new jeu de taquin. Thus our newly defined jeu de taquin is called p-jeu de taquin. The p-jeu de taquin algorithm relies on two slide rules and two swapping rules.

Definition 18. *The* **sliding moves** *needed for the p-jeu de taquin are the following. Let $p > 0$ and $\bullet$ be the cell that is going to be filled.*

1. *When given $\begin{array}{|c|c|}\hline b & c \\ \hline \bullet & a \\ \hline\end{array}$ where c is empty with $a < b$, or b and c are empty, or $a < b < c$ and not consecutive, or $p > a$, perform the following slide,*

$$\begin{array}{|c|c|}\hline b & c \\ \hline \bullet & a \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|}\hline b & c \\ \hline a \\ \cline{1-1}\end{array},$$

2. *When given $\begin{array}{|c|c|}\hline a & c \\ \hline \bullet & b \\ \hline\end{array}$ where c is empty with $a < b$, or b and c are empty, or $a < b < c$ and not consecutive, or $p > a$, perform the following slide,*

$$\begin{array}{|c|c|}\hline a & c \\ \hline \bullet & b \\ \hline\end{array} \Rightarrow \begin{array}{c|c|}\cline{2-2} & c \\ \hline \multicolumn{1}{|c|}{a} & b \\ \hline\end{array}.$$

Definition 19. *The* **swapping moves** *needed for the p-jeu de taquin are the following. Let $p > 0$ and $\bullet$ be the cell that is going to be filled.*

1. *When given $\begin{array}{|c|c|}\hline b & c \\ \hline \bullet & a \\ \hline\end{array}$ where $p \leq a < b < c$ and a, b, c are consecutive and c is not empty,*

perform the following steps,

(a) *Swap a and b giving* $\begin{array}{|c|c|}\hline a & c \\ \hline \bullet & b \\ \hline \end{array}$.

(b) *Since* $a < b$*, slide a into the empty slot, giving* $\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} \\ \hline \end{array}$.

2. *When given* $\begin{array}{|c|c|}\hline a & c \\ \hline \bullet & b \\ \hline \end{array}$ *where* $p \leq a < b < c$ *and* a, b, c *are consecutive and c is not empty, perform the following steps,*

(a) *Swap a and b giving* $\begin{array}{|c|c|}\hline b & c \\ \hline \bullet & a \\ \hline \end{array}$.

(b) *Since* $a < b$*, slide a into the empty slot, giving* $\begin{array}{cc} \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{c} \\ \hline \multicolumn{1}{|c|}{a} & \\ \cline{1-1} \end{array}$.

Definition 20. *Let* $p > 0$*. A single move of the p-jeu de taquin consists of a slide move, or one or more swap moves followed by a slide move. We choose an inner corner of a skew tableau and apply both the sliding rules and swapping rules repeatedly by performing one p-jeu de taquin move at a time until we obtain a straight shape tableau. When performing the p-jeu de taquin on a tableau T, the resulting finished straight shape tableau is called* $jdt_p(T)$*.*

We will now look at an example before proving characteristics of the p-jeu de taquin.

Example 13. Let $p = 1$. Consider the word $w = 34625$. In Figure 3.1 we have that $p = 1$ and start with the tableau T such that $rw(T) = w$. After performing the 1-jeu de taquin on T the resulting straight shape tableau $jdt_1(T)$ has the reading word 46235. The corresponding sequence of Assaf–Knuth transformations that get us from w to 46235 is $K1_1, A1_1, K2_1, K2_1$. Now let $p = 3$. In Figure 3.2 we have that $p = 3$ and start with the tableau T such that $rw(T) = w$. After performing the 3-jeu de taquin on T the resulting straight shape tableau $jdt_3(T)$ has the reading word 35246. The corresponding sequence of Assaf–Knuth transformations that get us from w to 35246 is $K2_3, K1_3, A2_3, K2_3$.

To help show the connections between the Assaf–Knuth transformations, and the p-jeu de taquin algorithm, we address each Assaf–Knuth transformation individually in Tables 3.1, 3.2, 3.3, and 3.4. The starting tableau in Table 3.1 has the reading word bca and the ending tableau after the p-jeu de taquin has the reading word bac. Notice bca and bac differ by an Assaf–Knuth

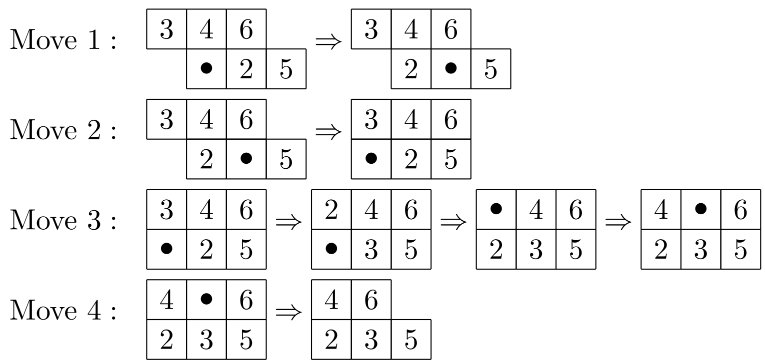


Figure 3.1: p-Jeu de Taquin where $p = 1$
An example of the p-jeu de taquin, where the p-jeu de taquin is applied to a tableau T whose reading word corresponds to $w = 34625$ and $p = 1$. Since $2 < 4$ and $5 < 6$ the 2 and 5 slide to the left under the 4 and 6 respectively. Since $2, 3, 4$ are consecutive and $2 \geq 1 = p$ we apply swap move number 1 to the $3, 4, 2$. Finally, $4 > 2$ and $6 > 3$, so the 4 and 6 slide to the left above the 2 and 3, respectively.

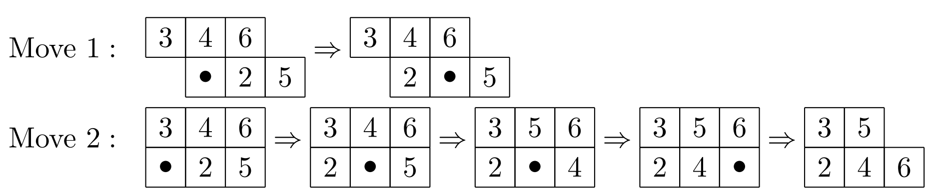


Figure 3.2: p-Jeu de Taquin where $p = 3$
An example of the p-jeu de taquin, where the p-jeu de taquin is applied to a tableau T whose reading word corresponds to $w = 34625$ and $p = 3$. Since $2 < 4$ and $5 < 6$ the 2 and 5 slide to the left under the 4 and 6 respectively. Since $2, 3, 4$ are consecutive and $2 < 3 = p$ we apply slide move number 1 to the $3, 4, 2$. Now $4, 5, 6$ and $4 \geq 3 = p$ so we apply swap move number 2 to the $4, 5, 6$. Finally, since there are no filled cells to the right of the open cell the 6 can slide down.

T_1	$rw(T_1)$	T_2	$rw(T_2)$
$\begin{array}{\|c\|c\|}\hline b & c \\ \hline & a \\ \hline\end{array}$	bca	$\begin{array}{\|c\|c}\hline b & \\ \hline a & \multicolumn{1}{c\|}{c} \\ \hline\end{array}$	bac

Table 3.1: p-Jeu de Taquin Slide 1
Let $a < b < c$ where $a + 1 \neq b$ or $a < p$. This is the chart that shows the connection between Assaf–Knuth transformation $K1_p$ and the p-jeu de taquin.

T_1	$rw(T_1)$	T_2	$rw(T_2)$
$\begin{array}{\|c\|c\|}\hline a & c \\ \hline & b \\ \hline\end{array}$	acb	$\begin{array}{\|c\|c}\hline c & \\ \hline a & \multicolumn{1}{c\|}{b} \\ \hline\end{array}$	cab

Table 3.2: p-Jeu de Taquin Slide 2
Let $a < b < c$ where $a + 1 \neq b$ or $a < p$. This is the chart that shows the connection between Assaf–Knuth transformation $K2_p$ and the p-jeu de taquin.

$K1_p$ relation. Thus Table 3.1 shows the connection between the p-jeu de taquin and the Assaf–Knuth $K1_p$ transformation. The starting tableau in Table 3.2 has the reading word acb and the straight shape tableau after the p-jeu de taquin has the reading word cab. Since acb and cab differ by an Assaf–Knuth $K2_p$ relation, Table 3.2 shows the connection between the p-jeu de taquin and the Assaf–Knuth $K2_p$ transformation. The starting tableau in Table 3.3 has the reading word bca and the ending tableau after the p-jeu de taquin has the reading word cab. Since the two reading words differ by an Assaf–Knuth $A1_p$ relation, Table 3.3 shows the connection between the p-jeu de taquin and Assaf–Knuth transformation $A1_p$. Finally, Table 3.4 gives the two reading words acb and bac. The Assaf–Knuth transformation $A2_p$ makes these two words Assaf–Knuth equivalent. Therefore, Table 3.3 shows the connection between the p-jeu de taquin and Assaf–Knuth transformation $A2_p$.

Since each of the scenarios of Assaf–Knuth relations are covered by the p-jeu de taquin we can now look to see if $rw(T)$ and the reading word of $rw(jdt_p(T))$ are Assaf–Knuth equivalent. The following lemma will assist in the proof of the equivalence $rw(T) \equiv_{\mathrm{AK_p}} rw(jdt_p(T))$.

Lemma 2. *Let $a_1 < a_2 < a_3 < \cdots < a_n$ and $p > 0$.*

1. *If $x < a_1$, and x, a_1, and a_2 are not consecutive, then $a_1 a_2 \ldots a_n x \equiv_{\mathrm{AK_p}} a_1 x a_2 \ldots a_n$.*

T_1	$rw(T_1)$	T_2	T_3	$rw(T_3)$
$\begin{array}{\|c\|c\|}\hline b & c \\ \hline & a \\ \hline\end{array}$	bca	$\begin{array}{\|c\|c\|}\hline a & c \\ \hline & b \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline c & \\ \hline a & b \\ \hline\end{array}$	cab

Table 3.3: p-Jeu de Taquin Swap 1
Let $a < b < c$ where a, b, and c are consecutive and $a \geq p$. This is the chart that shows the connection between Assaf–Knuth transformation $A1_p$ and the p-jeu de taquin.

T_1	$rw(T_1)$	$rw(T_2)$	T_3	$rw(T_3)$
$\begin{array}{\|c\|c\|}\hline a & c \\ \hline & b \\ \hline\end{array}$	acb	$\begin{array}{\|c\|c\|}\hline b & c \\ \hline & a \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline b & \\ \hline a & c \\ \hline\end{array}$	bac

Table 3.4: p-Jeu de Taquin Swap 2
Let $a < b < c$ where a, b, and c are consecutive and $a \geq p$. This is the chart that shows the connection between Assaf–Knuth transformation $A2_p$ and the p-jeu de taquin.

2. If $x > a_n$, and x, a_n, and a_{n-1} are not consecutive, then $xa_1a_2 \ldots a_n \equiv_{\mathrm{AK_p}} a_1a_2 \ldots a_{n-1}xa_n$.

Proof. Let $a_1 < a_2 < a_3 < \cdots < a_n$ and $p > 0$. Assume x, a_1, and a_2 are not consecutive, and x, a_n, and a_{n-1} are not consecutive. Therefore, only the Assaf–Knuth transformations $K1_p$ and $K2_p$ are used. Thus, we do not need to worry about the value of p in Lemma 2 since p is used to determine if moves $A1_p$ and $A2_p$ should be used on consecutive triples or if $K1_p$ and $K2_p$ should be used on the consecutive triples.

1. Consider $n = 2$. Let $x < a_1$ and the letters x, a_1, and a_2 not be consecutive. Then $a_1a_2x \equiv_{\mathrm{AK_p}} a_1xa_2$ by definition of $K1_p$. Assume property 1 holds for $n = k-1$, that is, $a_1a_2 \ldots a_{k-1}x \equiv_{\mathrm{AK_p}} a_1xa_2 \ldots a_{k-1}$. Consider $a_1a_2 \ldots a_{k-1}a_kx$ where $x < a_1 < \cdots < a_{k-1} < a_k$. Since $x < a_{k-1} < a_k$ we can apply $K1_p$ to see that $a_1a_2 \ldots a_{k-1}a_kx \equiv_{\mathrm{AK_p}} a_1a_2 \ldots a_{k-1}xa_k$. However, by assumption, we know $a_1a_2 \ldots a_{k-1}x \equiv_{\mathrm{AK_p}} a_1xa_2 \ldots a_{k-1}$. Thus $a_1a_2 \ldots a_{k-1}a_kx \equiv_{\mathrm{AK_p}} a_1xa_2 \ldots a_{k-1}a_k$. Therefore, by mathematical induction, if $x < a_1$ and x, a_1, and a_2 are not consecutive, then $a_1a_2 \ldots a_nx \equiv_{\mathrm{AK_p}} a_1xa_2 \ldots a_n$ for all $n \in \mathbb{N}$, $n \geq 2$.

2. Consider $n = 2$. Let $x > a_2$ and the letters x, a_2, and a_1 not be consecutive. Then

$xa_1a_2 \equiv_{\mathrm{AK_p}} a_1xa_2$ by definition of $K2_p$. Assume property 2 holds for $n = k-1$, that is, $xa_1a_2 \ldots a_{k-1} \equiv_{\mathrm{AK_p}} a_1a_2 \ldots xa_{k-1}$. Consider $xa_1a_2 \ldots a_{k-1}a_k$ where $a_1 < \cdots < a_{k-1} < a_k < x$. By assumption we know $xa_1a_2 \ldots a_{k-1} \equiv_{\mathrm{AK_p}} a_1a_2 \ldots xa_{k-1}$. Thus $xa_1a_2 \ldots a_{k-1}a_k \equiv_{\mathrm{AK_p}} a_1a_2 \ldots xa_{k-1}a_k$. Since $a_{k-1} < a_k < x$ we can apply $K2_p$ to see that $a_1a_2 \ldots xa_{k-1}a_k \equiv_{\mathrm{AK_p}} a_1a_2 \ldots a_{k-1}xa_k$. Therefore by mathematical induction, if $x > a_n$ and x, a_1, and a_2 are not consecutive, then $xa_1a_2 \ldots a_n \equiv_{\mathrm{AK_p}} a_1a_2 \ldots xa_n$ for all $n \in \mathbb{N}$. □

Lemma 2 will be used in most cases to show that $rw(T) \equiv_{\mathrm{AK_p}} rw(jdt_p(T))$. There is one case where Lemma 2 alone is not enough to prove that $rw(T) \equiv_{\mathrm{AK_p}} rw(jdt_p(T))$. For this scenario we will need the following lemma.

Lemma 3. *Let $p > 0$, $a \geq p$ and $n \geq 1$. Then* $a, a+2, \ldots, a+2n, a+1, a+3, \ldots, a+2n-1 \equiv_{\mathrm{AK_p}} a+1, a+3, \ldots, a+2n-1, a, a+2, \ldots, a+2n.$

Proof. Let $p > 0$, $a \geq p$ and $n \geq 1$.

Consider when $n = 1$. Then $a, a+2, a+1 \equiv_{\mathrm{AK_p}} a+1, a, a+2$ by definition of $A2_p$.

Suppose this property holds when $n = k-1$, that is,

$$\begin{aligned} &a, a+2, \ldots, a+2(k-1), a+1, a+3, \ldots, a+2(k-1)-1 \\ &\qquad \equiv_{\mathrm{AK_p}} a+1, a+3, \ldots, a+2(k-1)-1, a, a+2, \ldots, a+2(k-1). \end{aligned}$$

Consider when $n = k$. Then we have, $a, a+2, \ldots, a+2k, a+1, a+3, \ldots, a+2k-1$. Since $a+2k$, $a+2(k-1)-1$, and $a+2k-1$ are not consecutive, and

$$a+1 < a+3 < \cdots < a+2k-1 < a+2k,$$

we can apply Lemma 2 part two to obtain

$$v = a, a+2, \ldots, a+2(k-1), a+1, a+3, \ldots, a+2(k-1)-1, a+2k, a+2k-1.$$

By induction

$$v \equiv_{\mathrm{AK_p}} a+1, a+3, \ldots, a+2(k-1)-1, a, a+2, \ldots, a+2(k-1), a+2k, a+2k-1.$$

Now we can apply the Assaf–Knuth relation $A2_p$ to get

$$a+1, a+3, \ldots, a+2(k-1)-1, a, a+2, \ldots, a+2(k-2), \boldsymbol{a+2k-1}, a+2(k-1), a+2k.$$

Since $a+2k-1$, $a+2(k-1)$, and $a+2(k-2)$ are not consecutive, and
$a < a+2 < \cdots < a+2(k-1) < a+2k-1$, we can apply Lemma 2 part two to obtain

$$a+1, a+3, \ldots, a+2k-1, a, a+2, \ldots, a+2k.$$

Thus,

$$a, a+2, \ldots, a+2k, a+1, a+3, \ldots, a+2k-1 \equiv_{\mathrm{AK_p}} a+1, a+3, \ldots, a+2k-1, a, a+2, \ldots, a+2k.$$

Therefore, by mathematical induction, if $p > 0$, $a \geq p$ and $n \geq 1$, then
$a, a+2, \ldots, a+2n, a+1, a+3, \ldots, a+2n-1 \equiv_{\mathrm{AK_p}} a+1, a+3, \ldots, a+2n-1, a, a+2, \ldots, a+2n.$ □

Theorem 7. *Let $p > 0$ and T be a skew tableau. Then $rw(T) \equiv_{\mathrm{AK_p}} rw(jdt_p(T))$.*

Proof. When performing the p-jeu de taquin on a tableau, only two rows are considered for each move, the row with the empty cell being filled and the row above it. Thus, the proof of this theorem will be by induction on two row configurations. Only one p-jeu de taquin move will be taken into consideration for the proof since performing more than one move is performing the same process over again.

Let $p > 0$. Let T be a skew tableau and T' the result of performing one p-jeu de taquin move on T. Consider when T has only one entry. Since the p-jeu de taquin moves are either one slide or one or more swap moves followed by a slide, no entries are added or lost while performing the p-jeu de taquin. Thus, T' also has only one entry. Hence, $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$.

Assume that when T has 1 to k entries that $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$. Consider when T has $k+1$

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline y_1 & \cdots & y_n & x & v_1 & \cdots & v_m \\ \hline u_1 & \cdots & u_n & & z_1 & \cdots & z_m \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline y_2 & \cdots & y_n & x & v_1 & \cdots & v_m \\ \hline u_2 & \cdots & u_n & & z_1 & \cdots & z_m \\ \hline \end{array}$$

Figure 3.3: Configuration 1 for Theorem 7
In Figure 3.3 we have configuration one for Theorem 7. For this configuration there needs to be the same number of y_i and u_i, as well as same number of v_i and z_i. The second tableau T_2 is the result of removing the first column of T.

$$\begin{array}{ccccc} \boxed{l} & \boxed{w_1} & \boxed{x} & \boxed{w_2} & \\ & \boxed{w_3} & \boxed{} & \boxed{w_4} & \boxed{h} \end{array}$$

Figure 3.4: Configuration 2 for Theorem 7
Each w_i where $i \in \{1, 2, 3, 4\}$ is a word. Since Figure 3.4 is a skew tableau, the length of w_3 must be less than or equal to the length of w_1. Similarly, the length of w_2 has to be less than or equal to the length of w_4. Notice either l or h can be empty.

entries. Let x be the indicated cell above the inner corner in which the p-jeu de taquin will be performed on. Thus T can have two possible configurations. The first is in Figure 3.3 and the second is in Figure 3.4.

Let T, with $k+1$ entries, have the configuration in Figure 3.3. For this shape we have to look at the following cases.

Case 1: Let x, z_1 and v_1 be not consecutive or $x < p$. Suppose $z_1 < x$ and thus $z_1 < p$. Therefore, x remains where it is and z_1 on the bottom row slides to the left resulting in $rw(T) = rw(T')$. Thus $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$. Consider when $x < z_1$. Then x slides down giving $rw(T') = y_1 \ldots y_n v_1 \ldots v_m u_1 u_2 \ldots u_n x z_1 \ldots z_m$. If we remove the first column of T, the result is T_2 from Figure 3.3, which has $k-1$ entries. By assumption we know $rw(T_2) \equiv_{\mathrm{AK_p}} rw(T_2')$. Thus,

$$y_2 \ldots y_n x v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m \equiv_{\mathrm{AK_p}} y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n x z_1 \ldots z_m.$$

Consider the reading word of T, $rw(T) = y_1 \ldots y_n x v_1 \ldots v_m u_1 \ldots u_n z_1 \ldots z_m$. Notice u_1 is less than all letters that are written before it, that is

$$u_1 < y_1 < \cdots < y_n < x < v_1 < \cdots < v_m,$$

and the letters are strictly increasing till u_1. Also, since $u_1 < y_1 < y_2$ and $u_1 < u_2 < y_2$, we have

$y_2 - u_1 > 2$ and thus u_1, y_1 and y_2 can not be a consecutive triple. Therefore, by Lemma 2 part 1, we have $rw(T) \equiv_{\mathrm{AK_p}} y_1\boldsymbol{u_1}y_2 \ldots y_n x v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m$. By induction,

$$y_1 u_1 y_2 \ldots y_n \boldsymbol{x} v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m \equiv_{\mathrm{AK_p}} y_1 u_1 y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n \boldsymbol{x} z_1 \ldots z_m.$$

By applying Lemma 2 again we get

$$y_1 \boldsymbol{u_1} y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n x z_1 \ldots z_m \equiv_{\mathrm{AK_p}} y_1 \ldots y_n v_1 \ldots v_m \boldsymbol{u_1} u_2 \ldots u_n x z_1 \ldots z_m.$$

Therefore, $rw(T) \equiv_{\mathrm{AK_p}} y_1 \ldots y_n v_1 \ldots v_m u_1 u_2 \ldots u_n x z_1 \ldots z_m = rw(T')$. Hence $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$.

Case 2: Let $p \leq x$. Let $z_1 < x < v_1$ such that z_1, x and v_1 are consecutive numbers. If we remove the first column of T, the result is T_2 from Figure 3.3, which has $k-1$ entries. By assumption $rw(T_2) \equiv_{\mathrm{AK_p}} rw(T_2')$. By applying one move of the p-jeu de taquin we would have to apply a swap move followed by a slide, that is apply swap 1 which causes x and z_1 to switch places and z_1 to drop down, followed by the slide move which causes the v_1 to slide to the left. Therefore, $T_2' =$

y_2	$\cdots$	y_n	v_1	v_2	$\cdots$	v_m
u_2	$\cdots$	u_n	z_1	x	$\cdots$	z_m

, which results in

$$y_2 \ldots y_n x v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m \equiv_{\mathrm{AK_p}} y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n z_1 x z_2 \ldots z_m.$$

Consider the reading word of T, $rw(T) = y_1 \ldots y_n x v_1 \ldots v_m u_1 \ldots u_n z_1 \ldots z_m$. Notice u_1 is less than all letters that are written before it, that is $u_1 < y_1 < \cdots < y_n < x < v_1 < \cdots < v_m$, and the letters are strictly increasing till u_1. Also, since $u_1 < y_1 < y_2$ and $u_1 < u_2 < y_2$, we have $y_2 - u_1 > 2$ and thus u_1, y_1 and y_2 can not be a consecutive triple. Therefore, by Lemma 2 part 1, we have $rw(T) \equiv_{\mathrm{AK_p}} y_1 \boldsymbol{u_1} y_2 \ldots y_n x v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m$. By induction

$$y_1 u_1 y_2 \ldots y_n \boldsymbol{x} v_1 \ldots v_m u_2 \ldots u_n z_1 \ldots z_m \equiv_{\mathrm{AK_p}} y_1 u_1 y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n z_1 \boldsymbol{x} z_2 \ldots z_m.$$

By applying Lemma 2 part 1 again we obtain

$$y_1 \boldsymbol{u_1} y_2 \ldots y_n v_1 \ldots v_m u_2 \ldots u_n x z_1 \ldots z_m \equiv_{\mathrm{AK_p}} y_1 \ldots y_n v_1 \ldots v_m \boldsymbol{u_1} u_2 \ldots u_n z_1 x z_2 \ldots z_m.$$

Therefore, $rw(T) \equiv_{\mathrm{AK_p}} y_1 \dots y_n v_1 \dots v_m u_1 u_2 \dots u_n z_1 x z_2 \dots z_m = rw(T')$. Hence, $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$.

Case 3: The only time where it is possible to have more than one swap is when z_1, x, and v_1 are consecutive and $x < z_1 < v_1$. This case will address this scenario. For clarity we use a modified version of the configuration in Figure 3.3. Instead T will be

y_1	$\dots$	y_n	x	$x+2$	$\dots$	$x+2s$	v_1	$\dots$	v_m
u_1	$\dots$	u_n		$x+1$	$\dots$	$x+2s-1$	z_1	$\dots$	z_m

.

Notice that if $s = 1$, then there will only be one swap that occurs before a slide move. If $s > 1$, then more than one swap will occur in a single move before the slide move happens. First we consider what happens when $m = 0$ and then what happens when $n = 0$.

Let $p \leq x$, $s > 0$ and $m = 0$. Then T has the following shape:

$T =$

y_1	y_2	$\dots$	y_n	x	$x+2$	$\dots$	$x+2s$
u_1	u_2	$\dots$	u_n		$x+1$	$\dots$	$x+2s-1$

.

Performing one move of the p-jeu de taquin on T results in

$T' =$

y_1	y_2	$\dots$	y_n	$x+1$	$\dots$	$x+2s-1$	
u_1	u_2	$\dots$	u_n	x	$x+2$	$\dots$	$x+2s$

.

Then

$$rw(T) = y_1 y_2 \dots y_n x(x+2) \dots (x+2s) u_1 u_2 \dots u_n (x+1) \dots (x+2s-1).$$

Since $u_1 < y_1 < \dots < y_n < x < x+2 < \dots < (x+2s)$, we can apply Lemma 2 part 1 to obtain

$$rw(T) \equiv_{\mathrm{AK_p}} y_1 \boldsymbol{u_1} y_2 \dots y_n x(x+2) \dots (x+2s) u_2 \dots u_n (x+1) \dots (x+2s-1).$$

Since $u_i < y_i < \dots < y_n < x < x+2 < \dots < (x+2s)$, we can apply Lemma 2 part 1 repeatedly to obtain

$$rw(T) \equiv_{\mathrm{AK_p}} y_1 \boldsymbol{u_1} y_2 \boldsymbol{u_2} \dots y_n \boldsymbol{u_n} x(x+2) \dots (x+2s)(x+1) \dots (x+2s-1).$$

By applying Lemma 3 we obtain

$$rw(T) \equiv_{\mathrm{AK_p}} y_1 u_1 y_2 u_2 \ldots y_n u_n (x+1) \ldots (x+2s-1) x (x+2) \ldots (x+2s).$$

Since $u_n < y_n < x+1 < \cdots < (x+2s-1)$, we can apply Lemma 2 part 1 which results in

$$rw(T) \equiv_{\mathrm{AK_p}} y_1 u_1 y_2 u_2 \ldots y_{n-1} u_{n-1} y_n (x+1) \ldots (x+2s-1) \boldsymbol{u_n} x (x+2) \ldots (x+2s).$$

Since $u_i < y_i < x+1 < \cdots < (x+2s-1)$, we can apply Lemma 2 part 1 repeatedly which results in

$$rw(T) \equiv_{\mathrm{AK_p}} y_1 \ldots y_n (x+1) \ldots (x+2s-1) \boldsymbol{u_1} \ldots \boldsymbol{u_n} x (x+2) \ldots (x+2s) = rw(T').$$

Thus $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$ for all $n > 0$.

Now we will consider what happens when $n = 0$. Let $p \leq x$, $s > 0$ and $n = 0$. Then T has the following shape

$T =$

x	$x+2$	$\ldots$	$x+2s$	v_1	$\ldots$	v_m
	$x+1$	$\ldots$	$x+2s-1$	z_1	$\ldots$	z_m

Performing one move of the p-jeu de taquin on T results in

$T' =$

$x+1$	$\ldots$	$x+2s-1$	v_1	$\ldots$	v_m	
x	$x+2$	$\ldots$	$x+2s$	z_1	$\ldots$	z_m

Then $rw(T) = x(x+2)\ldots(x+2s)v_1 \ldots v_m (x+1) \ldots (x+2s-1) z_1 \ldots z_m$. Since $(x+1) < \cdots < (x+2s-1) < z_1 < \cdots < z_m < v_m$, we can apply Lemma 2 part 2 to obtain

$$rw(T) \equiv_{\mathrm{AK_p}} x(x+2)\ldots(x+2s) v_1 \ldots v_{m-1} (x+1) \ldots (x+2s-1) z_1 \ldots z_{m-1} \boldsymbol{v_m} z_m.$$

Since $(x+1) < \cdots < (x+2s-1) < z_1 < \cdots < z_i < v_i$, we can repeatedly apply Lemma 2 part 2 to obtain

$$rw(T) \equiv_{\mathrm{AK_p}} x(x+2)\ldots(x+2s)(x+1)\ldots(x+2s-1) \boldsymbol{v_1} z_1 \boldsymbol{v_2} z_2 \ldots \boldsymbol{v_m} z_m.$$

By applying Lemma 3 we obtain

$$rw(T) \equiv_{\mathrm{AK_p}} (x+1)\dots(x+2s-1)x(x+2)\dots(x+2s)v_1z_1v_2z_2\dots v_mz_m.$$

Since $x < x+2 < \dots < x+2s < z_1 < v_1$, we can apply Lemma 2 part 2 to obtain

$$rw(T) \equiv_{\mathrm{AK_p}} (x+1)\dots(x+2s-1)\boldsymbol{v_1}x(x+2)\dots(x+2s)z_1v_2z_2\dots v_mz_m.$$

Since $x < x+2 < \dots < x+2s < z_i < v_i$, we can repeatedly apply Lemma 2 part 2 to obtain

$$rw(T) \equiv_{\mathrm{AK_p}} (x+1)\dots(x+2s-1)\boldsymbol{v_1}\dots\boldsymbol{v_m}x(x+2)\dots(x+2s)z_1\dots z_m = rw(T').$$

Thus $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$ for all $m > 0$.

Therefore, no matter how many columns are before x and how many columns are after $(x+2s)$, $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$.

Now consider the configuration in Figure 3.4. Each w_i where $i \in \{1,2,3,4\}$ is a word. Since Figure 3.4 is a skew tableau, the length of w_3 must be less than or equal to the length of w_1. Similarly, the length of w_2 has to be less than or equal to the length of w_4. Let T, with $k+1$, entries have the configuration in Figure 3.4. Without loss of generality, let l be a filled cell. By removing l and its corresponding cell we obtain a valid skew tableau called T_2 which has k entries. Notice $rw(T) = l \cdot rw(T_2)$. Since T_2 has k entries, by induction $rw(T_2) \equiv_{\mathrm{AK_p}} rw(T_2')$. Thus $rw(T) = l \cdot rw(T_2) \equiv_{\mathrm{AK_p}} l \cdot rw(T_2')$. Since l is the first letter in $rw(T)$ and it is not in a cell directly above or to the right of the cell being filled, the position of l will not change from T to T'. Therefore $l \cdot rw(T_2') = rw(T')$. Hence $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$ when T has $k+1$ entries and has configuration from Figure 3.4.

Since $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$ for both configuration 3.3 and configuration 3.4, $rw(T) \equiv_{\mathrm{AK_p}} rw(T')$ for all valid skew tableau. □

Corollary 1. *Let $p > 0$ and w be a permutation. Then $w \equiv_{\mathrm{AK_p}} rw(jdt_p(w))$.*

Proof. Let $p > 0$ and w be a permutation. Construct a skew standard Young tableau from w and apply the p-jeu de taquin algorithm to the skew tableau. Then, by Theorem 7, $w \equiv_{\mathrm{AK_p}} rw(jdt_p(w))$. □

Lemma 4. *Let $p > 0$. If T is a standard tableau and $w = rw(T)$ then $jdt_p(w) = T$.*

Proof. Let $p > 0$ and T be a standard tableau. Thus each row of T is increasing from left to right and each column is increasing from bottom to top. Let $w = rw(T)$. Construct a skew tableau J from w. Recall a new row is formed in the skew tableau only when there is a descent in w. The only time there is a descent in w is when a row from T ends and the next row from T begins. Thus the skew tableau will have the same rows as T. Now we can apply the p-jeu de taquin to the skew tableau. Let

a_1	a_2	$\cdots$	a_k			
			b_1	b_2	$\cdots$	b_j

where $k \leq j$ be two rows of our skew tableau. Since the rows of J have the same entries as T we know $b_1 < a_1 < a_2 < \cdots < a_k$ and $b_2 < a_2 < \cdots < a_k$. Thus b_1 will not form a consecutive triple with any letter from the top row since $b_1 < b_2 < a_2$. Therefore, the only p-jeu de taquin moves that will be applied are slide moves. More specifically only horizontal slide moves will be performed. Hence,

a_1	a_2	$\cdots$	a_k		
b_1			b_2	$\cdots$	b_j

We can then repeat this process and reasoning until we obtain a rectified straight shape tableau. Since only horizontal slides are needed $jdt_p(w) = T$. □

Corollary 2. *Let $p > 0$ and w, v be permutations. Then $w \equiv_{\mathrm{AK_p}} v$ if and only if $jdt_p(w) = jdt_p(v)$.*

Proof. Let $p > 0$ and w, v be permutations.

Assume $jdt_p(w) = jdt_p(v)$. Then by Corollary 1, $w \equiv_{\mathrm{AK_p}} rw(jdt_p(w)) = rw(jdt_p(v)) \equiv_{\mathrm{AK_p}} v$. Thus $w \equiv_{\mathrm{AK_p}} v$.

Assume $w \equiv_{\mathrm{AK_p}} v$. Note $jdt_p(w)$ and $jdt_p(v)$ are standard tableaux. Loehr [3] shows that the reading words of distinct standard tableaux are not p-Assaf–Knuth equivalent. By Theorem 7 $w \equiv_{\mathrm{AK_p}} rw(jdt(w))$ and $v \equiv_{\mathrm{AK_p}} rw(jdt(v))$. Therefore $rw(jdt(w)) \equiv_{\mathrm{AK_p}} rw(jdt(v))$. Thus $jdt_p(w) = jdt_p(v)$. Therefore, if $w \equiv_{\mathrm{AK_p}} v$ then $jdt_p(w) = jdt_p(v)$. □

Loehr [3] defines the p-RSK algorithm which is a variant of the Robinson–Schensted–Knuth algorithm. Loehr's variant has a modified insertion algorithm.

Algorithm 4. Let $p > 0$. Given a tableau T and a positive integer x, insert x into T, denoted $T \leftarrow x$, in the following way: Start by inserting x into the bottom row of T using the following rules to determine which y value is bumped from the row. Replace y with x. Let S be the row in T that x is being inserted.

1. Special Rule 1: If $p \leq x$ and both $x+1$ and $x+2$ are in S then $y = x+2$ is bumped to the next row.

2. Special Rule 2: If $p < x$ and both $x+1$ and $x-1$ are in S and $x+2$ is not in S, let b be equal the smallest entry in S such that $b, b+1, \ldots, a-1$ are in S then $y = \max(b+1, p+1)$ is bumped to the next row.

3. Default Rule: If the two special rules do not apply then let y denote the smallest (left most) letter in S such that $y > x$.

If no such y value exists, append x to the end of the current row in a new box. If y does exist, insert y into the next row following the same procedure. Repeat this process till no y value is bumped.

Thus the p-RSK algorithm is the following algorithm.

Algorithm 5. The p-RSK algorithm is: Given a permutation $w = w_1 w_2 \ldots w_k$ and $p > 0$, form two standard Young tableaux P and Q in the following manner: $P_1 = \phi \leftarrow w_1 = \boxed{w_1}$ and $Q_1 = \boxed{1}$, and $P_j = P_{j-1} \leftarrow w_j$ and Q_j is created from Q_{j-1} by placing j in the cell added to P_j.

With the p-RSK algorithm the P and Q tableaux are both standard tableaux.

Example 14. We are going to construct the p-RSK tableaux P and Q that corresponds to the permutation $w = 512643$ when $p = 1$ and when $p = 3$. The corresponding tableaux that are found using the p-RSK algorithm are in Table 3.5 and Table 3.6 respectively. In Table 3.5, when inserting 3 into P_5 we apply special rule 2 since both 2 and 4 are in the first row of P_5 and $3 > 1$. The b value of the row would be 1 since 1 and 2 are in the first row. Thus $y = \max(2, 1) = 2$.

Therefore, 2 is bumped to the second row and 3 is placed in the first row. In Table 3.6, when inserting 3 into P_5 we do not have to apply special rule 2 since $3 \not> 3$. Therefore, we use our default rule and bump 4 to the second row. Since 5 and 6 are in the second row and $4 \geq 3$, we apply special rule 1 and bump 6 to the next row.

$P_1 = \begin{ytableau} 5 \end{ytableau}$	$Q_1 = \begin{ytableau} 1 \end{ytableau}$
$P_2 = \begin{ytableau} 5 \\ 1 \end{ytableau}$	$Q_2 = \begin{ytableau} 2 \\ 1 \end{ytableau}$
$P_3 = \begin{ytableau} 5 \\ 1 & 2 \end{ytableau}$	$Q_3 = \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$
$P_4 = \begin{ytableau} 5 \\ 1 & 2 & 6 \end{ytableau}$	$Q_4 = \begin{ytableau} 2 \\ 1 & 3 & 4 \end{ytableau}$
$P_5 = \begin{ytableau} 5 & 6 \\ 1 & 2 & 4 \end{ytableau}$	$Q_5 = \begin{ytableau} 2 & 5 \\ 1 & 3 & 4 \end{ytableau}$
$P_6 = \begin{ytableau} 5 \\ 2 & 6 \\ 1 & 3 & 4 \end{ytableau}$	$Q_6 = \begin{ytableau} 6 \\ 2 & 5 \\ 1 & 3 & 4 \end{ytableau}$

Table 3.5: Constructing P and Q Tableaux by p-RSK, $p = 1$
Given the word $w = 512643$ and $p = 1$ we construct the corresponding P tableau and Q tableau by using the p-RSK algorithm.

$$P_1 = \begin{array}{|c|}\hline 5 \\ \hline \end{array} \qquad Q_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$P_2 = \begin{array}{|c|}\hline 5 \\ \hline 1 \\ \hline \end{array} \qquad Q_2 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$P_3 = \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \qquad Q_3 = \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}$$

$$P_4 = \begin{array}{|c|c|c|}\cline{1-1} 5 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 6 \\ \hline \end{array} \qquad Q_4 = \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

$$P_5 = \begin{array}{|c|c|c|}\cline{1-2} 5 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} \qquad Q_5 = \begin{array}{|c|c|c|}\cline{1-2} 2 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

$$P_6 = \begin{array}{|c|c|c|}\cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad Q_6 = \begin{array}{|c|c|c|}\cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

Table 3.6: Constructing P and Q Tableaux by p-RSK, $p = 3$
Given the word $w = 512643$ and $p = 3$ we construct the corresponding P tableau and Q tableau by using the p-RSK algorithm.

Loehr [3] shows that reading word of the P tableau of permutation w is p-Assaf-Knuth equivalent to w, that means $w \equiv_{\mathrm{AK_p}} rw(P(w))$. Thus the P tableau of permutation w as a result of p-RSK is equal to the the straight shape tableau $jdt_p(w)$. Also, the Q tableau of permutation w as a result of p-RSK is equal to the the straight shape tableau $jdt_p(w^{-1})$.

To demonstrate that the Assaf–Knuth relations, the p-jeu de taquin algorithm, and the p-RSK algorithm produce the same equivalence classes on words we will provide an example. In this example we first show that the two words are p-Assaf–Knuth equivalent by applying Lemma 3 and Assaf-Knuth relations. Then we demonstrate that by using the p-RSK algorithm the words will produce the same P tableau. Finally, we apply the p-jeu de taquin algorithm to the two words to show that we obtain the same standard Young tableau.

$P_1 = \begin{matrix} 1 \end{matrix}$	$Q_1 = \begin{matrix} 1 \end{matrix}$
$P_2 = \begin{matrix} 1 & 2 \end{matrix}$	$Q_2 = \begin{matrix} 1 & 2 \end{matrix}$
$P_3 = \begin{matrix} 1 & 2 & 4 \end{matrix}$	$Q_3 = \begin{matrix} 1 & 2 & 3 \end{matrix}$
$P_4 = \begin{matrix} 1 & 2 & 4 & 6 \end{matrix}$	$Q_4 = \begin{matrix} 1 & 2 & 3 & 4 \end{matrix}$
$P_5 = \begin{matrix} 2 & & & \\ 1 & 3 & 4 & 6 \end{matrix}$	$Q_5 = \begin{matrix} 5 & & & \\ 1 & 2 & 3 & 4 \end{matrix}$
$P_6 = \begin{matrix} 2 & 4 & & \\ 1 & 3 & 5 & 6 \end{matrix}$	$Q_6 = \begin{matrix} 5 & 6 & & \\ 1 & 2 & 3 & 4 \end{matrix}$

Table 3.7: p-RSK of w_1
Given the word $w_1 = 124635$ and $p = 1$ we construct the corresponding P tableau and Q tableau by using the p-RSK algorithm.

Example 15. Let $w_1 = 124635$ and $w_2 = 132546$. Let $p = 1$. By Lemma 3, $24635 \equiv_{\text{AK}_1} 35246$. Thus, $124635 \equiv_{\text{AK}_1} 135246$. By applying the Assaf–Knuth transformation $K1_1$ on 352 we obtain $w_1 \equiv_{\text{AK}_1} 132546 = w_2$. Since $w_1 \equiv_{\text{AK}_1} w_2$, when we apply 1-RSK to w_1 and w_2 we will obtain the same P tableau. Table 3.7 shows 1-RSK being applied to w_1 and Table 3.8 shows 1-RSK being applied to w_2. Notice Table 3.7 and Table 3.8 have the same ending P tableaux. Since $w_1 \equiv_{\text{AK}_1} w_2$, when we apply the 1-jeu de taquin algorithm to w_1 and w_2 we will obtain the same straight shape standard Young tableau. In Figure 3.5 and Figure 3.6 we start with tableaux T_1 and T_2 that correspond to w_1 and w_2 respectively. Then the 1-jeu de taquin is applied to find that $jdt_1(T_1) = jdt_1(T_2)$ or $jdt_1(w_1) = jdt_1(w_2)$.

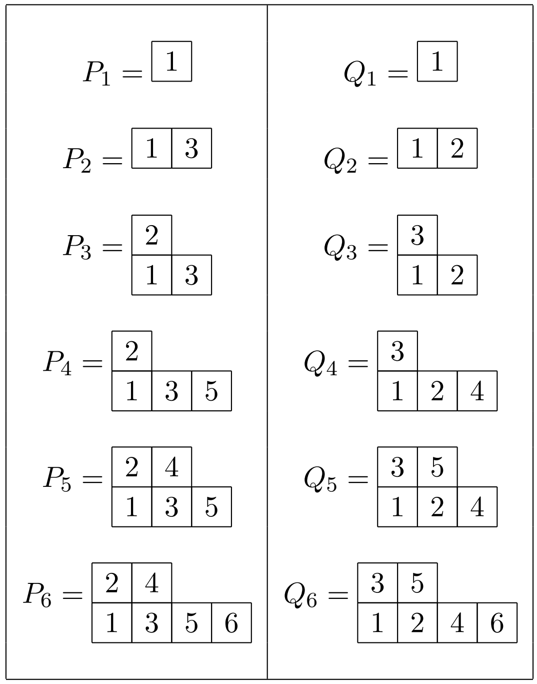


Table 3.8: p-RSK of w_2
Given the word $w_2 = 132546$ and $p = 1$ we construct the corresponding P tableau and Q tableau by using the p-RSK algorithm.

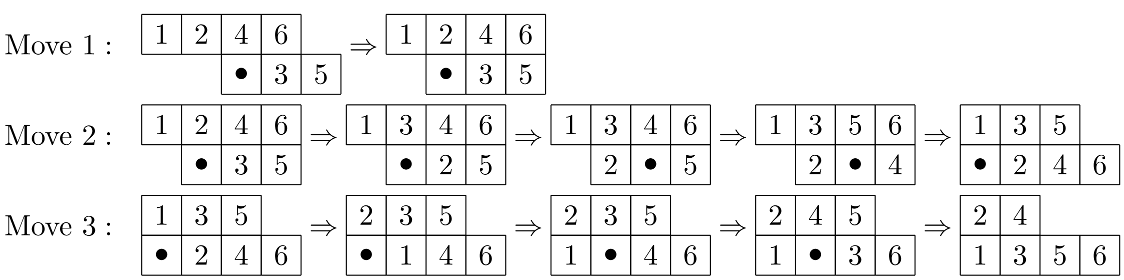


Figure 3.5: Example of $jdt_p(w_1)$
Let $p = 1$. Given $w_1 = 124635$ we can construct the skew tableau T_1. Since $3 < 4$ and $5 < 6$ the 3 and 5 will move to the left. Since $2, 3, 4$ are consecutive and $2 \geq p = 1$, we need to swap 2 and 3. Since $2 < 3$ the 2 will slide under the 3. Since $4, 5, 6$ are consecutive, 4 can not drop down since we have to swap it with the 5. Since $4 < 5$ the 4 will slide under the 5. Since $1, 2, 3$ are consecutive and $1 \geq p = 1$, we need to swap 1 and 2. Since $1 < 2$ the 1 will slide under the 2. Since $3, 4, 5$ are consecutive, 3 can not drop down since we have to swap it with the 4. Since $3 < 4$ the 3 will slide under the 4.

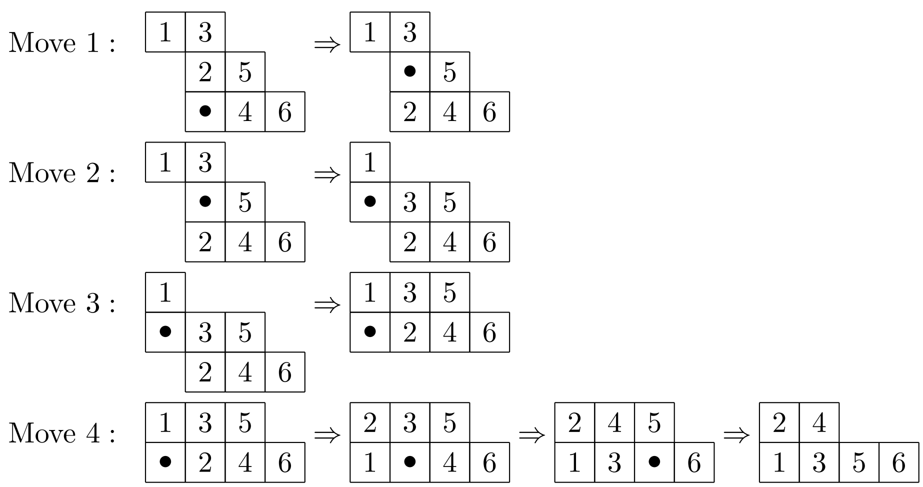


Figure 3.6: Example of $jdt_p(w_2)$
Given tableau T_2 and $p = 1$ we can find $jdt_p(T_2)$. Since $2 < 4$ the 2 will slide down to fill the cell with $\bullet$. Since $3 < 5$ and there is nothing to the left of the $\bullet$, the 1 and 3 will slide down. Notice $1, 2, 3$ are consecutive and $1 \geq p = 1$. Thus we need to swap 1 and 2. Since $1 < 2$ the 1 will slide under the 2. Notice $3, 4, 5$ are consecutive, but 3 can not drop down since we have to swap it with the 4. Since $3 < 4$ the 3 will slide under the 4. There is nothing to the right of 5 and $5 < 6$, so the 5 will slide.

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APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

MARSHALL UNIVERSITY®
w w w . m a r s h a l l . e d u

Office of Research Integrity

September 11, 2018

Joanna McKinney
Smith Hall Room 523
Math Department

Dear Ms. McKinney:

This letter is in response to the submitted thesis abstract entitled *"An Extension of Jeu-De-Taquin."* After assessing the abstract, it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction, it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director

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APPENDIX B

VITA

Joanna McKinney

Education

- Master of Arts. Marshall University, Expected December 2018.
  Thesis Advisor: Elizabeth Niese.
- Bachelor of Science. State University of New York College at Oswego, December 2016.
  GPA 3.52.

Publications

- *An Extension of the Jeu de taquin.* Master's thesis, Marshall University, Expected December 2018.

Accomplishments

- The John Walcott Award
  Award is presented to a student for consistent excellence in Math courses required for the Mathematics Adolescence Education major. SUNY Oswego, Spring 2016.
- NOYCE Fellowship
  The Noyce Fellowship is a national fellowship which is administered by local institutions of higher education in order to encourage students with a strong STEM background to consider a career in STEM education. SUNY Oswego, Fall 2015-Fall 2016.

Professional Experiences

- Graduate Assistant, Marshall University, Spring 2017 – Fall 2018
  Primary Instructor:
  Fall 2017 MTH130 (two sections)
  Spring 2018 MTH102
  Fall 2018 MTH 127
  Secondary Instructor:
  Spring 2017 MTH 100 and MTH 102
- Student Teacher, Soule Road Middle School, 10/24/2016 – 12/15/2016
- Student Teacher, LaFayette Big Picture High School, 9/7/2016 – 10/20/2016
- Research Experience for Undergraduates, Marshall University, Summer 2016
- Teaching Assistant for PHL 295, SUNY Oswego, Fall 2014

Presentations & Posters

- (q, t) Symmetry in Macdonald Polynomials (poster), Joint Mathematics Meetings of the AMS and MAA in Atlanta, GA January 2017.

- My Student Teaching Experience (presentation), SUNY Oswego, December 2016.
- MacDonald Polynomials for Fillings of Integer Partition Diagrams (presentation), MAA Seaway Section Conference at RIT, Fall 2016.
- Algebraic Combinatorics at Marshall REU 2016 (presentation), SUNY Oswego Mathematics Department, Fall 2016.
- Karl Weierstrass: "father of modern analysis" (poster), SUNY Oswego QUEST, Spring 2015.

Additional Skills

- Software: Excel, R, Geometer's Sketchpad, SageMathCloud
- Programming: Java, LaTeX, Python (Sage)
- Languages: German

Conferences Attended

- Cumberland Conference, Marshall University, May 2018
- UNC Greensboro Conference, UNC, Fall 2017
- Joint Mathematics Meetings of the AMS and MAA, Atlanta, GA, Spring 2017
- MAA Seaway Section Conference, RIT, Fall 2016
- MAA Seaway Section Conference, SUNY Geneseo, Spring 2016
- MAA Seaway Section Conference, St Lawrence University, Fall 2015
- MAA Seaway Section Conference, Colgate University, Spring 2015
- MAA Seaway Section Conference, Alfred University, Fall 2014