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On monochromatic sets with nondecreasing diameter

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ON MONOCHROMATIC SETS WITH NONDECREASING DIAMETER

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We, the faculty supervising the work of Adam Thomas O'Neal, affirm that the thesis, *On Monochromatic Sets with Nondecreasing Diameter*, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the editorial standards of our discipline and the Graduate College of Marshall University. With our signatures, we approve the manuscript for publication.

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ABSTRACT

Our problem comes from the field of combinatorics known as Ramsey theory. Ramsey theory, in a general sense, is about identifying the threshold for which a family of objects, associated with a particular parameter, goes from never or sometimes satisfying a certain property to always satisfying that property. Research in Ramsey theory has applications in design theory and coding theory.

For integers m, r, and t, we say that a set of n integers colored with r colors is (m, r, t)-permissible if there exist t monochromatic subsets $B_1, B_2, \ldots, B_t$ such that (a) $|B_1| = |B_2| = \cdots = |B_t| = m$, (b) the largest element in B_i is less than the smallest element in B_{i+1} for $1 \leq i \leq t-1$, and (c) the diameters of the subsets are nondecreasing.

We define $f(m, r, t)$ to be the smallest integer n such that every string of length n is (m, r, t)-permissible. In this thesis, we first look at some preliminary results for values of $f(m, r, t)$, specifically when each individual parameter is 1 as the others vary. We then show that $f(m, r, t)$ exists for all possible positive parameters. We proceed by determining $f(2, 2, t)$ for all positive integers t. We conclude by considering colorings with more than two colors and monochromatic sets that have more than 2 elements, as well as investigating an enumeration of the number of ways a string could be realized as (m, r, t)-permissible.

CHAPTER 1

INTRODUCTION AND DEFINITIONS

In this thesis, we study a Ramsey-type problem, originally presented by Bialostocki et al. [2] in 1995; for more information on Ramsey theory, see the book by Graham, Rothschild, and Spencer [6]. We introduce this problem and explore the notion of *permissibility*. We begin with some necessary definitions. For integers a, b, and n, we use the notation $[n]$ to refer to the set $\{1, 2, \ldots, n\}$ and $[a, b]$ to refer to the set $\{a, a+1, \ldots, b\}$.

Definition 1.1. Let n and r be positive integers and R a set of r elements. Then an **integer coloring**, or r-coloring of $[n]$, is a function from $[n]$ to R; that is, a function which assigns to each integer a particular "color" from R. In this and the following definitions, we use the following example: Let $n = 15$ and $R = \{a, b\}$. Define $\Delta : [15] \to R$ as the 2-coloring given below.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\Delta(x)$	b	b	a	b	a	a	b	a	b	b	a	a	a	b	a

(1.1)

We often represent an integer coloring as a **string**. In this example, that string is $\Delta = bbabaababbaaaba$. We abbreviate using exponents. In this example, the abbreviation is $\Delta = b^2aba(ab)^2ba^3ba$.

Furthermore, if $A \subseteq [n]$, then $\Delta(A)$ is the set of all colors used to color the elements of A. For example, let $A = \{1, 4, 6\}$ and $B = \{1, 2, 7\}$, then $\Delta(A) = \{a, b\}$ and $\Delta(B) = \{b\}$.

Definition 1.2. Let n and r be positive integers, R a set of r elements, and $\Delta : [n] \to R$ an r-coloring of $[n]$. Let $A, B \subseteq [n]$. We say that a set A is **monochromatic** with respect to Δ if every element in A is mapped to the same color in R by Δ. That is, $|\{\Delta(x) \mid x \in A\}| = 1$. The **diameter** of A, denoted as $\operatorname{diam}(A)$, is the difference of its largest and smallest elements. We say that A **precedes** B, denoted as $A <_p B$, if $\max(A) < \min(B)$; this matches notation used by Bialostocki et al. [2]. We also say A and B are **non-overlapping** if either $A <_p B$ or $B <_p A$. Let Δ be the coloring giving in (1.1). Let $A = \{3, 5, 6\}$ and $B = \{8, 12\}$. Here, $\Delta(x) = a$ for each

$x \in A$, so A is monochromatic with respect to Δ and $\text{diam}(A) = \max(A) - \min(A) = 3$. Since $\max(A) = 6$ and $\min(B) = 8$, we have that $A <_p B$.

We now introduce the notion of permissibility, a major focus of this thesis.

Definition 1.3. Let m, r, t, and n be positive integers. Let Δ be an r-coloring of $[n]$. We say a collection of t m-subsets of $[n]$, $B_1, B_2, \ldots, B_t$, is **permissible with respect to $\boldsymbol{\Delta}$**, or simply **permissible**, if

1. for each $i \in [t]$, B_i is monochromatic with respect to Δ,
2. $\{B_1, B_2, \ldots, B_t\}$ are ordered by precedence; that is, if $i < j$, $B_i <_p B_j$, and
3. the diameters of $B_1, B_2, \ldots, B_t$ are nondecreasing; that is, if $i < j$, $\text{diam}(B_i) \leq \text{diam}(B_j)$.

We say that Δ is **$(\boldsymbol{m, r, t})$-permissible** if there exists a permissible collection of t sets in $[n]$ with respect to Δ. For a positive integer d, we say that Δ is **$(\boldsymbol{m, r, t; d})$-permissible** if there exists a permissible collection t sets in $[n]$ with respect to Δ, all of which have the same diameter, d. That is, $\text{diam}(B_i) = d$ for all $i \in [t]$. Throughout this thesis, we use m to denote the size of the sets in a collection, r to denote the number of colors being used in a coloring, and t to denote the number of sets being taken in a collection. Again, this matches notation used by Bialostocki et al. [2].

Example 1.4. Let $m = 3$, $r = 2$, $t = 3$, $n = 15$ and $R = \{a, b\}$. Let $\Delta : [n] \to R$ be defined as in (1.1). Let $B_1 = \{1, 2, 4\}$, $B_2 = \{5, 6, 8\}$, and $B_3 = \{11, 13, 15\}$. We know the following about these sets with respect to Δ:

i	1	2	3
$\Delta(B_i)$	$\{b\}$	$\{a\}$	$\{a\}$
$\text{diam}(B_i)$	3	3	4

Since $|\Delta(B_i)| = 1$ for each $i \in [3]$, each of the sets is monochromatic with respect to Δ. Clearly, $B_1 <_p B_2 <_p B_3$ and $\text{diam}(B_1) \leq \text{diam}(B_2) \leq \text{diam}(B_3)$. So B_1, B_2, and B_3 form a collection of 3 permissible sets which demonstrate that Δ is $(3, 2, 3)$-permissible. In fact, by letting $B_3 = \{12, 13, 15\}$, we show that Δ is $(3, 2, 3; 3)$-permissible.

Definition 1.5. Let m, r, and t be positive integers. We define the function $f(m,r,t)$ to be the least positive integer n such that all r-colorings of $[n]$ are (m,r,t)-permissible. Similarly, we define the function $s(m,r,t)$ to be the least positive integer n such that all r-colorings of $[n]$ are $(m,r,t;d)$-permissible for some positive integer d. We establish the $f(m,r,t)$ and $s(m,r,t)$ are well-defined in Chapter 2. Note that $f(m,r,t) \leq s(m,r,t)$ because if an r-coloring is $(m,r,t;d)$-permissible for some positive integer d, then it is necessarily (m,r,t)-permissible.

Considering the function $f(m,r,t)$, we arrive at the following question:

Question. *Let m, r, and t be positive integers. Does there exist a positive integer n such that all r-colorings of $[n]$ are (m,r,t)-permissible? If so, what is the minimum value for n?*

The existence of such an integer is known. Bialostocki et al. [2] prove the existence of $f(m,r,t)$ is a consequence of a result from van der Waerden [11] that uses arithmetic progressions. In Chapter 2, we use an alternative method to prove the existence of $f(m,r,t)$.

We compute $f(m,r,t)$ when any one parameter is equal to 1 in Chapter 2; they are routine proofs and as such, all published work studying this function involve all the parameters being at least 2. Below, we give a comprehensive list of known values for $f(m,r,t)$ beginning with those established by Bialostocki et al. [2].

Theorem 1.6 (Bialostocki et al. [2])**.** *Let m, r, and t be integers at least 2.*

(a) $f(m,2,2) = 5m-3$.

(b) $f(m,3,2) = 9m-7$.

(c) *If $r \geq 4$, then* $3r(m-1)+3 \leq f(m,r,2) < ((2m-2)-r+1)\cdot(2+\log_2 r)-1$.

(d) $f(m,2,t) \leq cmt^2$ *for some constant c.*

(e) $8m-4 \leq f(m,2,3) \leq 10m-6$.

(f) $f(2,r,t) \leq (r(t-1)+1)(r+1)$.

In 2005 Grynkiewicz [9] showed the lower bound of the inequality in Theorem 1.6(c) is sharp when $r = 4$.

Theorem 1.7 (Grynkiewicz [9]). *Let $m \geq 2$ be an integer. Then $f(m, 4, 2) = 12m - 9$.*

In 2015 Bernstein et al. [1] concluded the value of $f(m, 2, 3)$, which was initially bounded by Theorem 1.6(e).

Theorem 1.8 (Bernstein et al. [1]). *Let $m \geq 2$ be an integer. Then*

$$f(m, 2, 3) = 8m - 5 + \left\lfloor \frac{2m-2}{3} \right\rfloor + \delta,$$

where $\delta = 1$ if $m \in \{2, 5\}$ and $\delta = 0$ otherwise.

Recent work has been done on generalizations of our question, as well as investigations into its relationship to a theorem by Erdös, Ginzburg, and Ziv (see [3, 4, 5, 7, 8, 10] for examples). With possibly the only exception being the bounds given in Theorem 1.6(d) and (f), all work in this area focuses on parameter families with $t = 2$ and, more recently, $t = 3$ [1]. In our work we compute values for $f(2, 2, t)$ for arbitrarily large t which, after significant literature review, is the first set of computed values of $f(m, r, t)$ with $m, r \geq 2$ and arbitrarily large t.

In this thesis, we prove the following:

Theorem 1.9. *Let $t \geq 4$ be an integer. Then $f(2, 2, t) = 5t - 4$.*

Note $f(2, 2, t)$ has been computed when $t \leq 3$; $f(2, 2, 1) = 3$ which is a special case of Theorem 2.4, $f(2, 2, 2) = 7$ by Theorem 1.6(a), and $f(2, 2, 3) = 12$ by Theorem 1.8. Additionally, $f(2, 2, t) \leq 6t - 3$ by Theorem 1.6(f).

In Chapter 2, we discuss preliminary results as well as present a construction necessary for the proof of Theorem 1.9. We proceed in Chapter 3 by showing bounds for $s(m, r, t)$ and $f(m, r, t)$ as well as prove a series of lemmas which culminate in a proof of Theorem 1.9. We end the thesis in Chapter 4 with a conclusion and discussion of future work.

CHAPTER 2

PRELIMINARY RESULTS AND THE AST PARTITION

We begin this chapter in Section 2.1 by proving some preliminary results, such as the Pigeonhole Principle, and showing that $s(m, r, t)$ and $f(m, r, t)$ are both well-defined for all positive parameters. We proceed in Section 2.2 by looking at some additional results involving colorings of the positive integers. In Section 2.3, we present a construction necessary for proving Theorem 1.9 and close this chapter in Section 2.4 with a look at properties of alternating 2-colorings.

2.1 Preliminary Results

In this section, we begin by acheiving values for $s(m, r, t)$ and $f(m, r, t)$ when any single parameter is set to 1 and conclude by showing that both functions are well-defined for all positive parameters.

Observation 2.1. Let m, r, and t be positive integers. Clearly, $s(m, r, t) \geq mt$ since the union of t permissible m-sets that demonstrate $(m, r, t; d)$-permissibility must have cardinality no less than $s(m, r, t)$. Similarly, $f(m, r, t) \geq mt$.

We begin by showing that if $m = 1$ or $r = 1$, the above inequality is sharp.

Lemma 2.2. *Let m, r, and t be positive integers. Then $s(1, r, t) = f(1, r, t) = t$ and $s(m, 1, t) = f(m, 1, t) = mt$.*

Proof. We first show that $s(1, r, t) = t$. Let Δ_1 be an r-coloring of $[t]$. For $i \in [t]$, let $B_i = \{i\}$. Then $\{B_i \mid i \in [t]\}$ demonstrates that Δ_1 is $(1, r, t; 1)$-permissible. By definition, since Δ_1 is $(1, r, t; 1)$-permissible, it is also $(1, r, t)$-permissible. So it follows from Observation 2.1 that $s(1, r, t) = f(1, r, t) = t$.

Now, let $\Delta_2 : [mt] \to \{a\}$. For $i \in [t]$, let $B_i = \{(i-1)m + k \mid k \in [m]\}$. This takes consecutive sets of m integers. So we have that $\{B_i \mid i \in [t]\}$ demonstrates that Δ_2 is $(m, 1, t; m-1)$-permissible. By definition, since Δ_2 is $(1, r, t; m-1)$-permissible, it is also $(m, 1, t)$-permissible. It again follows from Observation 2.1 that $s(m, 1, t) = f(m, 1, t) = mt$. □

To handle the case where $t = 1$, we need the Pigeonhole Principle.

Theorem 2.3. ***The Pigeonhole Principle.*** *Let a and b be positive integers and let both X and Y be sets with $|X| = a$ and $|Y| = b$. Let $f : X \to Y$ and, for each $y \in Y$, let $f^{-1}(y) = \{x \in X \mid f(x) = y\}$. Then there exists $y \in Y$ such that $\left|f^{-1}(y)\right| \geq \left\lceil \frac{a}{b} \right\rceil$.*

Proof. Observe that $\left\lceil \frac{a}{b} \right\rceil - 1 < \frac{a}{b} \leq \left\lceil \frac{a}{b} \right\rceil$. Now suppose that for each $y \in Y$, $\left|f^{-1}(y)\right| \leq \left\lceil \frac{a}{b} \right\rceil - 1$. Then

$$a = \sum_{y \in Y} \left|f^{-1}(y)\right| \leq b\left(\left\lceil \frac{a}{b} \right\rceil - 1\right) < b\left(\frac{a}{b}\right) = a,$$

which produces a contradiction. Therefore, it must be that there exists $y \in Y$ such that $\left|f^{-1}(y)\right| \geq \left\lceil \frac{a}{b} \right\rceil$. □

Theorem 2.4. *Let m and r be positive integers. Then $s(m, r, 1) = f(m, r, 1) = (m - 1)r + 1$.*

Proof. We first show that $s(m, r, 1) > (m - 1)r$. Let $R = \{a_1, a_2, \ldots, a_r\}$ and consider $\Delta_1 : [(m - 1)r] \to R$ as $a_1^{m-1} a_2^{m-1} \ldots a_r^{m-1}$. Observe that Δ_1 is not $(m, r, 1; d)$-permissible for any d since each color appears only $m - 1$ times. Thus no m-set is monochromatic with respect to Δ_1.

We now show that $s(m, r, 1) \leq (m - 1)r + 1$. Let Δ_2 be an r-coloring of $[(m - 1)r + 1]$. Then by Theorem 2.3, there exists an $a_i \in R$ such that $\left|\Delta_2^{-1}(a_i)\right| \geq \left\lceil \frac{(m-1)r+1}{r} \right\rceil = m$. So Δ_2 is $(m, r, 1; d)$-permissible where $d = \text{diam}\left(\Delta_2^{-1}(a_i)\right)$. Therefore, $s(m, r, 1) \leq (m - 1)r + 1$.

Thus, since $s(m, r, 1) > (m - 1)r$ and $s(m, r, 1) \leq (m - 1)r + 1$, we have that $s(m, r, 1) = (m - 1)r + 1$. Hence, $s(m, r, 1) = f(m, r, 1) = (m - 1)r + 1$. □

By bootstrapping the previous result, we show that $s(m, r, t)$ and $f(m, r, t)$ are well-defined for all positive parameters.

Theorem 2.5. *Let m, r, and t be positive integers. Then $s(m, r, t)$ and $f(m, r, t)$ are both well-defined for all positive parameters.*

Proof. Let $q = (m - 1)r + 1$, $p = ((t - 1)(q - m + 1) + 1)$, and $M = qp$. Note that $s(m, r, 1) = f(m, r, 1) = q$. Let Δ be an r-coloring of $[M]$. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $[M]$, where $S_i = \{(i - 1)q + 1, \ldots, iq\}$ for each $i \in [p]$.

Let Δ_i be the restriction of Δ to S_i for each $i \in [p]$. So $\Delta_i(x) = \Delta(x)$ for all $x \in S_i$ and $i \in [p]$. By Theorem 2.4, for each $i \in [p]$, there exists $B_i \subseteq S_i$ such that $|B_i| = m$ and $|\Delta_i(B_i)| = |\Delta(B_i)| = 1$.

Let $g : [p] \to [m-1, q-1]$ with $g(i) = \text{diam}(B_i)$. Then by Theorem 2.3, there exists $d \in [m-1, q-1]$ such that

$$|g^{-1}(d)| \geq \left\lceil \frac{p}{q-1-(m-1)+1} \right\rceil = \left\lceil \frac{(t-1)(q-m+1)+1}{q-m+1} \right\rceil = \left\lceil (t-1) + \frac{1}{q-m+1} \right\rceil = t.$$

Then $\{B_i \mid i \in g^{-1}(d)\}$ is a set of monochromatic non-overlapping m-sets with respect to Δ of diameter d. Thus Δ is $(m, r, t; d)$-permissible and hence, any string of length M is $(m, r, t; d)$-permissible. It follows that $s(m, r, t)$ and $f(m, r, t)$ are both well-defined for all positive parameters. □

2.2 Colorings of the Positive Integers

In this section we show that, for positive integers m and r, in an r-coloring Δ of $\mathbb{N} = \{1, 2, \ldots\}$, there exists an infinite set of monochromatic m-sets with respect to Δ, ordered by precedence, of the same diameter. To do so, we first establish a derivative of the Pigeonhole Principle.

Theorem 2.6. *Let $h : \mathbb{N} \to Y$ be a function where Y is nonempty and finite. Then there exists $y \in Y$ such that $\left|h^{-1}(y)\right|$ is infinite.*

Proof. Assume otherwise, that for each $y \in Y$, $\left|h^{-1}(y)\right|$ is finite. Then $|\mathbb{N}| = \sum_{y \in Y} \left|h^{-1}(y)\right|$ is finite. Thus we have arrived at a contradiction. So there must be some $y \in Y$ for which $\left|h^{-1}(y)\right|$ is infinite. □

Theorem 2.7. *Let m and r be positive integers and Δ an r-coloring of $\mathbb{N}$. Then there exists an infinite set of monochromatic non-overlapping m-sets with respect to Δ of the same diameter.*

Proof. We proceed in a manner similar to Theorem 2.5. Again, let $q = (m-1)r + 1$. Let $\{S_1, S_2, \ldots\}$ be a partition of $\mathbb{N}$ where $S_i = \{(i-1)q + 1, \ldots iq\}$ for each $i \in \mathbb{N}$.

Let Δ_i be the restriction of Δ to S_i for each $i \in \mathbb{N}$. That is, $\Delta_i(x) = \Delta(x)$ for all $x \in S_i$. By Theorem 2.4, there exists $B_i \in S_i$ such that $|B_i| = m$ and $|\Delta_i(B_i)| = |\Delta(B_i)| = 1$.

Let $g : \mathbb{N} \to [m-1, q-1]$ with $g(i) = \text{diam}(B_i)$. Then by Theorem 2.6, there exists $d \in [m-1, q-1]$ such that $\left|g^{-1}(d)\right|$ is infinite. Thus there exists an infinite set of monochromatic m-sets all of diameter d, namely $\{B_i \mid i \in g^{-1}(d)\}$. $\square$

2.3 Alternating Substring-Triples Partition

To discuss the Alternating Substring-Triples Partition, or AST Partition, we first give some definitions relating to subsets of a coloring.

Definition 2.8. Let i, k, r, and n be positive integers and Δ be an r-coloring of $[n]$. For $i \in [n]$ and $k \in [n-i+1]$, we say that a set of consecutive elements $[i, i+(k-1)]$ is a ***k*-tuple** if Δ, when restricted to $[i, i+(k-1)]$, is monochromatic; that is, $|\Delta([i, i+(k-1)])| = 1$. We call a 2-tuple a **double** and a 3-tuple a **triple**. If $[i, i+(k-1)]$ is a k-tuple for some $i \in [n]$ and $k \in [n-i+1]$, we call the k-tuple **isolated** if

1. either $i = 1$ or $\Delta(i-1) \neq \Delta(i)$ and
2. either $n = i+(k-1)$ or $\Delta(i+(k-1)) \neq \Delta(i+k)$.

We say Δ is *alternating* if $\Delta(i) \neq \Delta(i+1)$ for each $i \in [n-1]$.

Having those definitions, we now give the AST Partition.

Construction 2.9. Alternating Substring-Triples Partition Let n be a positive integer and $\Delta : [n] \to \{a, b\}$. In what follows, we partition $[n]$ into $\mathcal{S}$, which contains *alternating substrings*, and $\mathcal{T}$, containing *triples*.

1. Find the maximum number of pairwise disjoint triples with respect to Δ. Suppose that there are w of them; call them $T_1, T_2, \ldots, T_w$, and suppose the minimal element in each is τ_i. We assume that either $\tau_i = 1$ or whenever $\tau_i > 1$, $\Delta(\tau_i - 1) \neq \Delta(\tau_i)$ or $\tau_i - 1 \in T_{i-1}$. This ensures that we "frontload" the triples. In other words, we take the first possible triple whenever possible. We define the collection of all triples as $\mathcal{T} := \{T_1, T_2, \ldots, T_w\}$.

2. Find all remaining doubles in $S = [n] \setminus (T_1 \cup T_2 \cup \cdots \cup \mathcal{T}_w)$ (we think of S as $w+1$ possibly empty subsets that are separated by the triples; if a string begins or ends with a triple, or if we have consecutive triples in a string, this corresponds to an empty subset of S). Say that there are v of them. Observe that these doubles occur precisely at the beginning and end of consecutive, maximal, alternating substrings in S. Now partition S into $v + w + 1$ (possibly empty) parts corresponding to maximal, alternating substrings, each called S_i for $i \in [v + w + 1]$. Define $\mathcal{S} := \{S_1, S_2, \ldots, S_{v+w+1}\}$. For $i \in [v + w + 1]$, let $k_i = |S_i|$, which we call this the **length** of S_i.

The partition of $[n]$ given by $\mathcal{S} \cup \mathcal{T}$, where elements of $\mathcal{S}$ and $\mathcal{T}$ correspond to alternating substrings and triples, respectively, is the Alternating Substring-Triple Partition, or AST Partition, of $[n]$ with respect to Δ. We denote the AST Partition of $[n]$ with respect to Δ with parameters v and w as $(\mathcal{S}, \mathcal{T}; v, w)$.

Example 2.10. Let $\Delta : [25] \to \{a, b\}$ be defined as $\Delta = ababababababbbbabaabbabab$. Then the AST Partition of [25] with respect to Δ is $(\{S_1, S_2, S_3, S_4\}, \{T_1\}; 2, 1)$ where $S_1 = [11]$, $S_2 = [15, 18]$, $S_3 = [19, 20]$, $S_4 = [21, 25]$, and $T_1 = [12, 14]$. This is illustrated below in (2.1).

$$\Delta : \underbrace{ababababababa}_{S_1}\,\underbrace{bbb}_{T_1}\,\underbrace{baba}_{S_2}\,\underbrace{ab}_{S_3}\,\underbrace{babab}_{S_4} \tag{2.1}$$

2.4 Properties of Alternating 2-colorings

Alternating 2-colorings have some characteristics which will be useful to us in the proof of Theorem 1.9. First, each alternating 2-coloring contains a particular number of what we call MCD2s, defined below in generality.

Definition 2.11. Let r and n be positive integers and Δ an r-coloring of $[n]$. A set $A \subseteq [n]$ is a *monochromatic set of diameter* d, or MCDd, if $|\Delta(A)| = 1$ and $\text{diam}(A) = d$.

Example 2.12. Let $\Delta = ababaab$. Then $A = \{1, 3, 6\}$ is an MCD5. Similarly, $B = \{2, 4\}$ is an MCD2.

Much of what we use in this thesis will refer to MCD2s. As such, we classify below how many MCD2s an alternating 2-coloring contains. Similarly, we describe a set of MCD2s any 2-coloring contains using the AST Partition.

Observation 2.13. Let k be a positive integer and Δ an alternating 2-coloring of $[k]$. Without loss of generality, we may assume that $\Delta = ab\cdots a$ if k is odd and $\Delta = ab\cdots ab$ if k is even. Then $\left\{\{3i-2, 3i\} \mid i \in \left[\left\lfloor \frac{k}{3}\right\rfloor\right]\right\}$ is a collection of $\left\lfloor \frac{k}{3}\right\rfloor$ MCD2s, none of which overlap. This means that any alternating string contains a permissible collection of $\left\lfloor \frac{k}{3}\right\rfloor$ MCD2s. Note that the MCD2s identified in a given substring are also "frontloaded". That is, we take the first possible MCD2 whenever possible.

Definition 2.14. Let n be a positive integer, Δ a 2-coloring of $[n]$, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of $[n]$ with respect to Δ. We know that $|\mathcal{S}| = v + w + 1$. For $i \in [v + w + 1]$, recall that $k_i = |S_i|$ and define $k_i' \in \{0, 1, 2\}$ as $k_i' = k_i \pmod 3$. Note that $n = 3w + \sum_{i=1}^{v+w+1} k_i$. By Observation 2.13, for each $i \in [v + w + 1]$, S_i contains $\left\lfloor \frac{k_i}{3} \right\rfloor = \frac{k_i - k_i'}{3}$ MCD2s. Furthermore, each of the w triples contains an MCD2. Then Δ contains $\sigma := w + \sum_{i=1}^{v+w+1} \left\lfloor \frac{k_i}{3} \right\rfloor = w + \sum_{i=1}^{v+w+1} \frac{k_i - k_i'}{3}$ permissible MCD2s. We call these σ MCD2s the *canonical MCD2s* of Δ. Then Δ is $(2, 2, \sigma; 2)$-permissible, or simply $(2, 2, \sigma)$-permissible.

Example 2.15. Let Δ be as defined in (2.1). Then Δ contains 6 canonical MCD2s which are $\{1, 3\}$, $\{4, 6\}$, $\{7, 9\}$, $\{12, 14\}$, $\{15, 17\}$, and $\{21, 23\}$, seen in (2.2). So Δ is $(2, 2, 6; 2)$-permissible.

$$\Delta : \underbrace{abababababa}_{S_1}\ \underbrace{bbb}_{T_1}\ \underbrace{baba}_{S_2}\ \underbrace{ab}_{S_3}\ \underbrace{babab}_{S_4} \tag{2.2}$$

We now discuss what it means for a coloring Δ to have a particular type.

Definition 2.16. Let n be a positive integer, Δ a 2-coloring of $[n]$, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST partition of $[n]$ with respect to Δ. Observe that $\mathcal{S} \cup \mathcal{T}$ is a partition of $[n]$ into $v + 2w + 1$ parts which have an implied order based on the way the construction parsed $[n]$ with respect to Δ. We often abbreviate Δ by its **type**. We use a $(v + 2w + 1)$-tuple to record the relative location of the alternating substrings and the triples associated with the AST Partition of Δ. For example, the coloring given in (2.2) is of type $(11, \tau, 4, 2, 5)$. Note that a coloring cannot be reconstructed given only its type.

Furthermore, the coloring in (2.2) *contains* $(\tau, 4, 2)$ and *ends* with $(2, 5)$. For ease, we use the notation $\bar{a}$, where a is a positive integer, to mean "is congruent to $a \pmod 3$ and at least a". So we may say that the coloring in (2.2) contains $(\tau, \bar{1}, 2)$ and ends with $(2, \bar{2})$.

Let ℓ be a positive integer and $a_1, a_2, \ldots, a_\ell \in \{a, b\}$. We say that Δ **contains** $a_1 a_2 \cdots a_\ell$ if there exists $k \in [n - \ell]$ such that Δ, when restricted to $[k + 1, k + \ell]$, is $a_1 a_2 \cdots a_\ell$. Further, we say that Δ **ends** with $a_1 a_2 \cdots a_\ell$ if $k = n - \ell$. So the coloring in (2.2) contains *ababbb* and ends with *babab*.

CHAPTER 3

PROOF OF THEOREM 1.9

In Section 3.1, we begin by establishing $s(2,2,t)$ for all positive integers t. We conclude the section by showing a lower bound for $f(m,r,t)$ for positive integers m, r, and t. We follow this in Section 3.2 by proving a series of lemmas necessary to show Theorem 1.9.

3.1 Bounds on $s(m,r,t)$ and $f(m,r,t)$

Lemma 3.1. *Let t be a positive integer. Then $s(2,2,t) > 5t-3$.*

Proof. Let $\Delta : [5t-3] \to \{a,b\}$ be represented by $(ababa)^{t-1}ab$. In what follows, we show that Δ is not $(2,2,t;d)$-permissible for any integer d. We proceed by induction.

Observe that when $t = 1$, we have that $\Delta = ab$ and clearly this is not $(2,2,1;d)$-permissible for any integer d as $[2]$ contains no monochromatic subsets of order 2 with respect to Δ.

Now assume that for some $t \geq 1$, $(ababa)^{t-2}ab$ is not $(2,2,t-1;d)$-permissible for any integer d and by way of contradiction assume that $\Delta = (ababa)^{t-1}ab$ is $(2,2,t;d)$-permissible for some integer d. Let $B_1, B_2, \ldots, B_t$ be the sets that realize Δ as $(2,2,t;d)$-permissible. Observe that $\Delta = (ababa)^{t-1}ab = ababa\left[(ababa)^{t-2}ab\right]$. So $\min(B_2) \leq 5$ because, otherwise, the sets $B_2, \ldots, B_t$ would realize $(ababa)^{t-2}ab$ to be $(2,2,t-1;d)$-permissible, contradicting the inductive hypothesis. So $B_1 \subseteq [4]$. Since we know that B_1 must be monochromatic with respect to Δ, $B_1 = \{1,3\}$ or $\{2,4\}$. This implies that $d = 2$. Since all of the sets have the same diameter, $\mathrm{diam}(B_2) = 2$. This means that $B_2 = \{4,6\}$ or $\{5,7\}$, neither of which are monochromatic with respect to Δ. Therefore, we have reached a contradiction and Δ is not $(2,2,t;d)$-permissible for any d. So $s(2,2,t) > 5t-3$. □

To show that $s(2,2,t) \leq 5t-2$, we first prove a lemma relating to the number of doubles and triples in a given 2-coloring.

Lemma 3.2. *Let t and n be a positive integers, Δ a 2-coloring of $[n]$, and $(\mathcal{S},\mathcal{T};v,w)$ the AST Partition of $[n]$ with respect to Δ. If $v+w \geq t$, then Δ is $(2,2,t;1)$-permissible.*

Proof. Note that each of the w triples contains a double. Hence, there are $v+w$ monochromatic, non-overlapping doubles in $[n]$ with respect to Δ. Since $v+w \geq t$, we have at least t permissible doubles. Thus Δ is $(2,2,t;1)$-permissible. □

Lemma 3.3. *Let t be a positive integer. Then $s(2,2,t) \leq 5t-2$.*

Proof. Let $\Delta : [5t-2] \to \{a,b\}$ and $(\mathcal{S},\mathcal{T};v,w)$ the AST Partition of $[5t-2]$ with respect to Δ. In this proof, we show that for any positive integer t, any 2-coloring of $[5t-2]$ is $(2,2,t;d)$-permissible for some integer $d \leq 2$. By Lemma 3.2, if $v+w \geq t$, then Δ is $(2,2,t;1)$-permissible. So in what follows, we assume that $v+w \leq t-1$ or $v+w+1 \leq t$. Let σ, k_i, k_i' be as defined in Definition 2.14. Then

$$\sigma = w + \sum_{i=1}^{v+w+1} \frac{k_i - k_i'}{3},$$

which implies

$$3\sigma = 3w + \sum_{i=1}^{v+w+1} k_i - \sum_{i=1}^{v+w+1} k_i'.$$

Since $k_i' \leq 2$ for all $i \in [v+w+1]$, we have that

$$3\sigma \geq 3w + \sum_{i=1}^{v+w+1} k_i - 2(v+w+1). \tag{3.1}$$

Also, from Definition 2.14, we have that

$$5t-2 = 3w + \sum_{i=1}^{v+w+1} k_i. \tag{3.2}$$

Additionally, $t \geq v+w+1$, so combining (3.1) and (3.2), we get

$$3\sigma \geq 3w + \sum_{i=1}^{v+w+1} k_i - 2(v+w+1) \geq 3w + (5t-2-3w) - 2t = 3t-2.$$

So $\sigma \geq t - \frac{2}{3}$, and since σ and t are integers, $\sigma \geq t$. Hence, by choosing t of the σ canonical MCD2s, we have a collection of t permissible sets realizing Δ to be $(2,2,t;2)$-permissible.

Hence, any coloring of length $5t-2$ is $(2,2,t;d)$-permissible with $d \leq 2$, so $s(2,2,t) \leq 5t-2$. □

Lemmas 3.1 and 3.3 then establish $s(2, 2, t)$ for all positive integers t.

Theorem 3.4. *Let t be a positive integer. Then $s(2, 2, t) = 5t - 2$.*

We conclude this section by showing a lower bound on $f(2, 2, t)$. Using a similar argument to Lemma 3.1, a lower bound for $f(m, r, t)$ is found.

Theorem 3.5. *Let m, r, and t be positive integers. Then $f(m, r, t) > (mr + 1)(t - 1)$.*

Proof. Let $R = \{a_1, a_2, \ldots, a_r\}$ and $\Delta : [(mr + 1)(t - 1)] \to R$ be an r-coloring of $[n]$. Let $\Delta = [(a_1 a_2 \cdots a_r)^m a_1]^{t-1}$. Similar to Lemma 3.1, we proceed by induction. When $t = 1$, Δ is empty and is therefore not (m, r, t)-permissible. Assume that for some $t \geq 2$ $\Delta = [(a_1 a_2 \cdots a_r)^m a_1]^{t-2}$ is not $(m, r, t-1)$-permissible. By way of contradiction, assume that $\Delta = [(a_1 a_2 \cdots a_r)^m a_1]^{t-1}$ is (m, r, t)-permissible. Let $B_1, B_2, \ldots, B_t$ be the t sets which realize Δ as (m, r, t)-permissible. Observe that $\min(B_2) \leq mr + 1$. Otherwise, $[(a_1 a_2 \cdots a_r)^m a_1]^{t-2}$ would be $(m, r, t-1)$-permissible with sets $B_2, B_3, \ldots, B_t$. So $B_1 \subseteq [mr]$. Since B_1 is monochromatic, B_1 is of the form $\{i, i + r, \ldots, i + r(m - 1)\}$ for some $i \in [r]$. Therefore $\text{diam}(B_1) = r(m - 1)$.

Similarly, $\max(B_{t-1}) \geq (mr + 1)(t - 1) - mr$. Otherwise, $[(a_1 a_2 \cdots a_r)^m a_1]^{t-2}$ would be $(2, r, t-1)$-permissible with sets $B_1, B_2, \ldots, B_{t-1}$. Then $B_t \subseteq \{(mr + 1)(t - 2) + i + 1 \mid i \in [mr]\}$. Since B_t is monochromatic, $B_t = \{(mr + 1)(t - 2) + i + 1 + (j - 1)r \mid j \in [m]\}$ for some $i \in [r]$. Therefore $\text{diam}(B_t) = r(m - 1)$. Since $\text{diam}(B_1) = \text{diam}(B_t) = r(m - 1)$ and Δ is (m, r, t)-permissible, it must be that Δ is actually $(m, r, t; r(m - 1))$ permissible. So we may assume $\text{diam}(B_2) = r(m - 1)$. If follows that if $\Delta(\min(B_2)) = a_1$, then $\Delta(\max(B_2)) = a_r$. Similarly, if $\Delta(\min(B_2)) = a_i$, for some $i \in [2, r]$, then $\Delta(\max(B_2)) = a_{i-1}$. In either case, B_2 is not monochromatic. Therefore Δ is not $(2, r, t)$-permissible and $f(m, r, t) > (mr + 1)(t - 1)$. □

The following corollary, a special case of Theorem 3.5, establishes a lower bound for $f(2, 2, t)$.

Corollary 3.6. *Let t be a positive integer. Then $f(2, 2, t) > 5t - 5$.*

3.2 Proof of Theorem 1.9

We now begin the proof of a series of lemmas which conclude in a proof of Theorem 1.9. These lemmas fit into two major categories. First, for a positive integer t, if a 2-coloring Δ of $[5t - 4]$

exhibits certain properties, then Δ is $(2,2,t)$-permissible. Second, for a positive integer t, if Δ is not $(2,2,t)$-permissible, then Δ must exhibit certain properties.

Lemma 3.7. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$, and $(\mathcal{S},\mathcal{T};v,w)$ the AST Partition of $[5t-4]$ with respect to Δ. If $v+w<t-1$, then Δ is $(2,2,t;2)$-permissible.*

Proof. If $v+w<t-1$, then $v+w+1\le t-1$. Let σ be as defined in Definition 2.14. Then

$$3\sigma = 3w+\sum_{i=1}^{v+w+1}\left(k_i-k_i'\right) = 5t-4-\sum_{i=1}^{v+w+1}k_i' \ge 5t-4-2(v+w+1)\ge 5t-4-2(t-1)=3t-2.$$

Since σ and t are both integers, $\sigma\ge t$. So if $v+w<t-1$, there are at least t canonical MCD2s. Therefore, Δ is $(2,2,t;2)$-permissible. □

By combining the previous result with Lemma 3.2, we have the following lemma.

Lemma 3.8. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$ which is not $(2,2,t)$-permissible, and $(\mathcal{S},\mathcal{T};v,w)$ the AST Partition of $[5t-4]$ with respect to Δ. Then $v+w=t-1$.*

Proof. By Lemma 3.2, since Δ is not $(2,2,t)$-permissible, $v+w\le t-1$ and, by Lemma 3.7, $v+w\ge t-1$. Therefore, if Δ is not $(2,2,t)$-permissible, $v+w=t-1$. □

Lemma 3.9. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$, and $(\mathcal{S},\mathcal{T};v,w)$ the AST Partition of $[5t-4]$ with respect to Δ. If a string contains an isolated 4-tuple or k-tuple with $k\ge 6$, then Δ is $(2,2,t)$-permissible.*

Proof. Assume by way of contradiction that Δ is not $(2,2,t)$-permissible. Then by Lemma 3.8, $v+w+1=t$. For each $i\in[w]$, let T_i' be the set containing the smaller two elements in T_i for each $T_i\in\mathcal{T}$.

Suppose $\{\ell,\ell+1,\ell+2,\ell+3\}$ is an isolated 4-tuple in Δ where $\ell\in[5t-7]$. We know from the AST Partition that $T_j=\{\ell,\ell+1,\ell+2\}$ for some $j\in[w]$ and $\ell+3$ does not belong to any double or triple identified by the AST Partition of $[5t-4]$ with respect to Δ. Let $\mathcal{D}$ be the set of all doubles in $[5t-4]$ with respect to Δ as identified using the AST Partition. Recall that $|\mathcal{D}|=v$. Then we have that $D\cup\{T_1',T_2',\dots,\mathcal{T}_w'\}\cup\{\ell+2,\ell+3\}$ is a permissible collection of $v+w+1=t$ doubles. So Δ is $(2,2,t;1)$-permissible hence Δ is $(2,2,t)$-permissible.

Now, let $\{\ell, \ell+1, \ldots, \ell+(k-1)\}$ be a k-tuple, $k \geq 6$, in Δ where $\ell \in [5t-3-k]$. Assume that $\ell = 1$ or $\Delta(\ell-1) \neq \Delta(\ell)$. Then $T_j = \{\ell, \ell+1, \ell+2\}$ and $T_{j+1} = \{\ell+3, \ell+4, \ell+5\}$ for some $j \in [w-1]$. Then $\mathcal{D} \cup \left(\{T_1', T_2', \ldots, \mathcal{T}_w'\} \setminus \{T_{j+1}'\}\right) \cup \{\ell+2, \ell+3\} \cup \{\ell+4, \ell+5\}$ is a permissible collection of $v+w+1=t$ doubles. So Δ is $(2,2,t;1)$-permissible. Hence Δ is $(2,2,t)$-permissible. □

Lemma 3.10. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$ which is not $(2,2,t)$-permissible, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of $[5t-4]$ with respect to Δ. Then Δ contains $t-1$ canonical MCD2s.*

Proof. The computation in this proof is similar to that in Lemma 3.3. Since Δ is not $(2,2,t)$-permissible, by Lemma 3.8, $v+w+1=t$. Let σ, k_i, and k_i' be as defined in Definition 2.14. Then

$$3\sigma = 3w + \sum_{i=1}^{v+w+1} \left(k_i - k_i'\right) = 3w + \sum_{i=1}^{v+w+1} k_i - \sum_{i=1}^{v+w+1} k_i' \geq 5t-4-2(v+w+1) = 5t-4-2t = 3t-4.$$

Thus $\sigma \geq t - \frac{4}{3}$. Since σ and t are both integers, $\sigma \geq t-1$. Further, since Δ is not $(2,2,t)$-permissible, $\sigma \leq t-1$. Therefore $\sigma = t-1$. □

Lemma 3.11. *Let t be an integer, Δ a 2-coloring of $[5t-4]$ which is not $(2,2,t)$-permissible, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of $[5t-4]$ with respect to Δ. Then each substring in $\mathcal{S}$ must have length congruent to $2 \pmod 3$ except for one which has length congruent to $1 \pmod 3$.*

Proof. Since Δ is not $(2,2,t)$-permissible, by Lemma 3.8, $v+w+1=t$. Let σ, k_i, and k_i' be as defined in Definition 2.14. Then

$$3\sigma = 3w + \sum_{i=1}^{v+w+1} k_i - \sum_{i=1}^{v+w+1} k_i' = 3w + \sum_{i=1}^{t} k_i - \sum_{i=1}^{t} k_i'. \tag{3.3}$$

Since Δ is not $(2,2,t)$-permissible, by Lemma 3.10, $\sigma = t-1$ or $3\sigma = 3t-3$. So

$$3t-3 = 3w + \sum_{i=1}^{t} k_i - \sum_{i=1}^{t} k_i',$$

$$3t-3 = 5t-4 - \sum_{i=1}^{t} k_i',$$

$$\sum_{i=1}^{t} k_i' = 2t-1.$$

Since $k_i' \le 2$ for all $i \in [t]$, the result follows. □

As a direct result of Lemma 3.11, we have the following corollary.

Corollary 3.12. *Let t be a positive integer and Δ a 2-coloring of $[5t-4]$. Then if Δ contains $(\bar{0})$ or at least two copies of $(\bar{1})$, then Δ is $(2,2,t)$-permissible.*

We next give a result relating to how a 2-coloring of $[5t-4]$ ends.

Lemma 3.13. *Let t be an integer and Δ a 2-coloring of $[5t-4]$ such that $\Delta(5t-6) = \Delta(5t-4)$. Then Δ is $(2,2,t)$-permissible.*

Proof. Let Δ_1 be the restriction of Δ to $[5t-7]$. That is $\Delta_1(x) = \Delta(x)$ for all $x \in [5t-7]$. Note that $5t-7 = 5(t-1)-2$. Then, by Theorem 3.4, Δ_1 is $(2,2,t-1;d)$-permissible with $d \le 2$. Let $B_1, B_2, \ldots B_{t-1}$ be the $t-1$ sets that realize Δ_1 as $(2,2,t-1;d)$-permissible. Define B_t as $\{5t-6, 5t-4\}$. Note that B_t is monochromatic with respect to Δ, $|B_t| = 2$ and $\mathrm{diam}(B_t) = 2$. Since $\mathrm{diam}(B_i) \le 2$ for all $i \in [t-1]$, $B_1, B_2, \ldots, B_t$ are t permissible 2-sets with respect to Δ. Therefore, $B_1, B_2, \ldots, B_t$ realize Δ as $(2,2,t)$-permissible. □

As a corollary to Lemma 3.13, for some positive integer t, a 2-coloring of $[5t-4]$ which is not $(2,2,t)$-permissible cannot end with a triple or an alternating substring with length 3 or larger.

Corollary 3.14. *Let t be an integer, Δ a 2-coloring of $[5t-4]$, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of $[5t-4]$ with respect to Δ. If Δ ends with (τ) or (k) where $k \ge 3$ is an integer, then Δ is $(2,2,t)$-permissible.*

Proof. If Δ ends with (τ), then we know that $\Delta(5t-6) = \Delta(5t-4)$. So by Lemma 3.13, Δ is $(2,2,t)$-permissible. Similarly, if Δ ends with (k) where $k \geq 3$ is an integer, then we know that $\Delta(5t-6) = \Delta(5t-4)$. Therefore, Δ is $(2,2,t)$-permissible. □

We now discuss the properties of triples contained in a coloring.

Lemma 3.15. *Let t be a positive integer and Δ a 2-coloring of $[5t-4]$. If Δ contains consecutive triples, then Δ is $(2,2,t)$-permissible.*

Proof. Either both triples have the same color or they have different colors. If the triples have the same color then, by Lemma 3.9, Δ is $(2,2,t)$-permissible. If the triples have different colors, then Δ contains $(\tau, \bar{0}, \tau)$. So, by Corollary 3.12, Δ is $(2,2,t)$-permissible. □

Lemma 3.16. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of $[5t-4]$ with respect to Δ. If Δ begins with a triple, then Δ is $(2,2,t)$-permissible.*

Proof. Let $S_1 \in \mathcal{S}$ be the first alternating substring in Δ. If Δ begins with a triple, then S_1 is empty and thus has length 0 so Δ contains (0). So by Corollary 3.12, Δ is $(2,2,t)$-permissible. □

Lemma 3.17. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of Δ. If $w > 0$ and Δ contains $(\bar{2}, \tau, \bar{2})$, then Δ is $(2,2,t)$-permissible. So if Δ is not $(2,2,t)$-permissible and contains any triples, then those triples are contained in copies of $(\bar{1}, \tau, \bar{2})$ or $(\bar{2}, \tau, \bar{1})$.*

Proof. Assume by way of contradiction that Δ is not $(2,2,t)$-permissible. Further, by Lemmas 3.14 and 3.16, Δ cannot begin or end with a triple. So each triple in Δ must be bounded on either side by an alternating substring.

Assume that Δ contains $(\bar{2}, \tau, \bar{2})$. Let $T = \{j, j+1, j+2\}$ for some $j \in [3, 5t-8]$ be the triple. Recall that for $i \in [v+w+1]$, $k_i = |S_i|$. Let S_i and S_{i+1} be the substrings immediately preceding and following T, respectively. By Observation 2.13, S_i contains $a_1 := \frac{k_i - 2}{3}$ MCD2s, S_{i+1} contains $a_2 := \frac{k_{i+1} - 2}{3}$ MCD2s, and the triple contains 1 MCD2. Say that there are p other canonical MCD2s in Δ. Since Δ is not $(2,2,t)$-permissible, by Lemma 3.10, there are $t-1$ canonical MCD2s in Δ so $a_1 + a_2 + p + 1 = t - 1$ or $a_1 + a_2 + p + 2 = t$.

Observe that, due to the construction of triples in the AST Partition of $[5t-4]$ with respect to Δ, we have that $\Delta(j) \neq \Delta(j-1)$. Note that $\Delta(j+3) \neq \Delta(j+4)$ since $k_{i+1} \geq 2$. Furthermore, $\Delta(j+2) \neq \Delta(j+3)$; otherwise, Δ would be $(2,2,t)$-permissible by Lemma 3.9. Observe that $S_i' = S_i \cup \{j\}$ is an alternating substring with length $3(a_1+1)$ so S_i' contains a_1+1 non-overlapping MCD2s. Similarly, $S_{i+1}' = S_{i+1} \cup \{j+2\}$ is an alternating substring with length $3(a_2+1)$ so S_{i+1}' contains a_2+1 non-overlapping MCD2s.

Using this regrouped partition of $[5t-4]$, Δ contains $(a_1+1)+(a_2+1)+p = a_1+a_2+p+2 = t$ non-overlapping MCD2s. Therefore, Δ is $(2,2,t)$-permissible. Note that a triple in Δ cannot be bounded on both sides by $(\bar{1})$ by Corollary 3.12. So if Δ contains any triples, they must be bounded on one side by $(\bar{1})$ and the other side by $(\bar{2})$ and the result follows. □

We now give an example to illustrate the previous result.

Example 3.18. Let $t=3$ and $\Delta = ababab bbab$, which is of type $(\bar{2}, \tau, \bar{2})$. By Definition 2.14, we have 2 canonical MCD2s; they are $\{1,3\}$ and $\{6,8\}$. Using the method described in the proof of Lemma 3.17, we find 3 MCD2s, which are $\{1,3\}$, $\{4,6\}$ and $\{8,10\}$, realizing Δ as $(2,2,3)$-permissible.

We now show that if, for some positive integer t, a 2-coloring of $[5t-4]$ contains more than 1 triple, then the 2-coloring is $(2,2,t)$-permissible.

Corollary 3.19. *Let t be a positive integer, Δ a 2-coloring of $[5t-4]$, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of Δ. If $w \geq 2$, then Δ is $(2,2,t)$-permissible.*

Proof. By Lemma 3.17, if Δ contains any triples, then those triples are contained in copies of $(\bar{1}, \tau, \bar{2})$ or $(\bar{2}, \tau, \bar{1})$. Suppose that Δ contains w_1 copies of $(\bar{1}, \tau, \bar{2})$ and w_2 copies of $(\bar{2}, \tau, \bar{1})$ such that $w_1 + w_2 = w \geq 2$. If $w_1 \geq 2$ or $w_2 \geq 2$, this implies that Δ contains at least 2 substrings whose length is congruent to $1 \pmod 3$. So Δ is $(2,2,t)$-permissible by Corollary 3.12. Therefore, $w_1 = w_2 = 1$ and thus Δ contains $(\bar{2}, \tau, \bar{1}, \tau, \bar{2})$.

If Δ contains $(\bar{2}, \tau, \bar{1}, \tau, \bar{2})$, then by way of contradiction assume that Δ is not $(2,2,t)$-permissible. Again recall that for $i \in [v+w+1]$, $k_i = |S_i|$. For $j \in [w-1]$, define $T_j = \{\alpha, \alpha+1, \alpha+2\}$ for $\alpha \in [3, n-5-k_{i+1}]$ and $T_{j+1} = \{\beta, \beta+1, \beta+2\}$ for $\beta \in [6, 5t-8]$. Let

S_i, S_{i+1}, and S_{i+2} for some $i \in [v+w-1]$ be the alternating substrings around T_j and T_{j+1} in order.

We first establish that T_j and T_{j+1} are isolated triples. Let the 2 colors used in this coloring be a and b. Without loss of generality, assume that $\Delta(T_j) = \{a\}$. Then $\Delta(\alpha - 1) = b$, otherwise the AST Partition is violated by having not taken the correct triple. Further, $\Delta(\alpha + 3) = b$. Otherwise, we would either have an isolated 4-tuple (if $k_{i+1} > 1$ or $k_{i+1} = 1$ and $\Delta(T_{j+1}) = \{b\}$) or isolated 7-tuple (if $k_{i+1} = 1$ and $\Delta(T_{j+1}) = \{a\}$) and Δ is $(2,2,t)$-permissible by Lemma 3.9. So T_j is an isolated triple.

Now, without loss of generality, assume that $\Delta(T_{j+1}) = \{a\}$. Then $\Delta(\beta - 1) \neq \Delta(\beta)$, because otherwise, the triples were not selected correctly in the AST Partition. Further, $\Delta(\beta + 3) = b$. Otherwise, we would either have an isolated 4-tuple (since $\Delta(\beta + 3) \neq \Delta(\beta + 4)$ because $\beta + 3, \beta + 4 \in S_{i+2}$ and S_{i+2} is alternating) so Δ is $(2,2,t)$-permissible by Lemma 3.9. So T_{j+1} is an isolated triple.

By Observation 2.13, S_i contains $a_1 := \frac{k_i - 2}{3}$ MCD2s, S_{i+1} contains $a_2 := \frac{k_{i+1} - 1}{3}$ MCD2s, S_{i+2} contains $a_3 := \frac{k_{i+2} - 2}{3}$ MCD2s, and both T_j and T_{j+1} contain 1 MCD2 each. Say that there are p other canonical MCD2s in Δ. Since Δ is not $(2,2,t)$-permissible, by Lemma 3.10 $a_1 + a_2 + a_3 + p + 2 = t - 1$ or $a_1 + a_2 + a_3 + p + 3 = t$. Observe that $S'_i = S_i \cup \{\alpha\}$ is an alternating substring with length $3(a_1 + 1)$ so S'_i contains $a_1 + 1$ non-overlapping MCD2s. Similarly, $S'_{i+1} = S_{i+1} \cup \{\alpha + 2, \beta\}$ is an alternating substring with length $3(a_2 + 1)$ so S'_{i+1} contains $a_2 + 1$ non-overlapping MCD2s. Likewise, $S'_{i+2} = S_{i+2} \cup \{\beta + 2\}$ is an alternating substring with length $3(a_3 + 1)$ so S'_{i+2} contains $a_3 + 1$ non-overlapping MCD2s. So Δ contains $(a_1 + 1) + (a_2 + 1) + (a_3 + 1) + p = a_1 + a_2 + a_3 + p + 3 = t$ non-overlapping MCD2s and we have reached a contradiction. Thus, Δ is $(2,2,t)$-permissible. □

We again give an example to illustrate the previous result.

Example 3.20. Let $t = 5$ and $\Delta = ababab\mathit{bb}ababaaaba$. Observe that Δ is of type $(\bar{2}, \tau, \bar{1}, \tau, \bar{2})$. By Definition 2.14, we have 4 canonical MCD2s; they are $\{1,3\}$, $\{6,8\}$, $\{9,11\}$ and $\{13,15\}$. Using the method described in the proof of Lemma 3.17, we find 5 MCD2s, which are $\{1,3\}$, $\{4,6\}$, $\{8,10\}$, $\{11,13\}$, and $\{15,17\}$ realizing Δ as $(2,2,5)$-permissible.

We first make some observations that will be useful in the final arguments.

Observation 3.21. Let $t \geq 4$ be an integer, Δ a 2-coloring of $[5t-4]$ which is not $(2,2,t)$-permissible, and $(\mathcal{S}, \mathcal{T}; v, w)$ the AST Partition of $[5t-4]$ with respect to Δ.

(a) By Lemma 3.8 and, since Δ is not $(2,2,t)$-permissible, $v+w+1=t$.

(b) Since $v+w+1=t$, there are t maximal, alternating substrings contained in Δ. Further, since $t \geq 4$, there are at least 4 alternating substrings of nonzero length in Δ.

(c) By Lemma 3.10 and, since Δ is not $(2,2,t)$-permissible, there are $t-1$ canonical MCD2s in Δ.

(d) $\Delta(5t-11) \neq \Delta(5t-9)$. Otherwise, by Lemma 3.13, Δ when restricted to $[5t-9]$ is $(2,2,t-1)$-permissible where the diameters of the t sets realizing Δ as $(2,2,t)$-permissible are at most 2. Then, by Theorem 2.3, either $\{5t-8, 5t-6\}$, $\{5t-8, 5t-4\}$, or $\{5t-6, 5t-4\}$ is a monochromatic set with respect to Δ. So by taking the $t-1$ sets realizing the restriction of Δ to $[5t-9]$ as $(2,2,t-1)$-permissible along with one of the other listed sets, we realize Δ as $(2,2,t)$-permissible.

We now show that $f(2,2,t) = 5t-4$.

Proof of Theorem 1.9. Recall that $t \geq 4$ be an integer and assume that Δ be a 2-coloring of $[5t-4]$ which is not $(2,2,t)$-permissible. Let $(\mathcal{S}, \mathcal{T}; v, w)$ be the AST Partition of $[5t-4]$ with respect to Δ. By Corollary 3.14, we know that Δ must end with (1) or (2). By Corollary 3.19, Δ contains at most 1 triple and by Lemma 3.17 if Δ contains a triple, then it either contains $(\bar{2}, \tau, \bar{1})$ or $(\bar{1}, \tau, \bar{2})$. With that in mind, Figure 3.1 illustrates the 12 types in which Δ may end. In each of these cases, we show that Δ is $(2,2,t)$-permissible and thus prove Theorem 1.9. Without loss of generality, assume that $\Delta(5t-4) = b$. Suppose that Δ ends with type:

(a) $(\tau, 2)$. By Lemma 3.13, Δ cannot end with *bbbab*. So Δ must end with *aaaab*. Since $t \geq 2$, there exists a nonempty alternating substring S_{t-1} which precedes the triple corresponding to (τ). Since the AST Partition frontloads triples, we know that the last character in S_{t-1} is b. So Δ ends with *baaaab* and thus contains an isolated quadruple. Therefore, by Lemma 3.9, Δ is $(2,2,t)$-permissible.

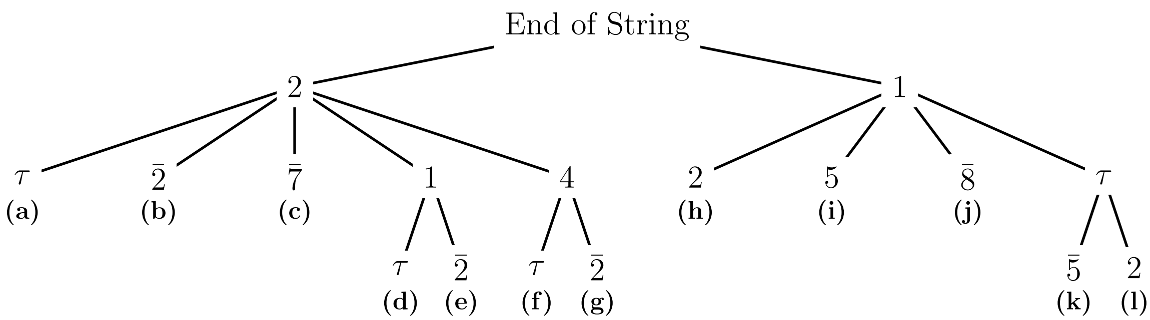

Figure 3.1: Tree of Types in Which a Coloring May End
The above outlines the types in which a coloring Δ, that is not $(2,2,t)$-permissible, may end. Moving down the tree starts at the end of the string and works backwards. The lettering corresponds to which part of the proof covers that particular ending.

(b) $(\bar{2}, 2)$. Then we know that Δ ends with *baab*. By Observation 3.21(c), we know that there are $t-1$ canonical MCD2s and all of them precede $[5t-7, 5t-4]$. Thus, by taking the $t-1$ canonical MCD2s along with $\{5t-7, 5t-4\}$, we realize Δ as $(2,2,t)$-permissible.

(c) $(\bar{7}, 2)$. Then $\Delta(5t-11) = \Delta(5t-9)$. So by Observation 3.21(d), Δ is $(2,2,t)$-permissible.

(d) $(\tau, 1, 2)$. By Lemma 3.11 and Observation 3.21(b), there exists a nonempty alternating substring S_{t-2} which precedes the triple corresponding to (τ) and has length congruent to 2 (mod 3). So Δ ends with type $(\bar{2}, \tau, 1, 2)$. So Δ ends with *abaaaaab*, *baaaaaab*, *babbbaab*, or *abbbbaab*. For the first and third case, $\Delta(5t-11) = \Delta(5t-9)$ so Δ is $(2,2,t)$-permissible by Observation 3.21(d). In the second case, Δ contains a 6-tuple and in the fourth case, Δ contains an isolated 4-tuple. So Δ is $(2,2,t)$-permissible by Lemma 3.9.

(e) $(\bar{2}, 1, 2)$. This implies that Δ ends with *abaab*. Observe that all of the $t-1$ canonical MCD2s precede $[5t-8, 5t-4]$. By taking the $t-1$ canonical MCD2s along with $\{5t-8, 5t-5\}$, we realize Δ as $(2,2,t)$-permissible.

(f) $(\tau, 4, 2)$. By Lemma 3.11 and Observation 3.21(b), there exists a nonempty alternating substring S_{t-2} which precedes the triple corresponding to (τ) and has length congruent to 2 (mod 3). So Δ ends with type $(\bar{2}, \tau, 4, 2)$. So Δ ends with *abaaababaab*, *abbbbbabaab*, *baaaababaab*, or *babbbbabaab*. In the first case, note that $\{5t-12, 5t-10\}$ and $\{5t-9, 5t-7\}$ are canonical MCD2s and all other canonical MCD2s precede $[5t-12, 5t-4]$. By grouping the other $t-3$ canonical MCD2s with the sets $\{5t-14, 5t-12\}$, $\{5t-10, 5t-8\}$, and $\{5t-7, 5t-4\}$, we realize Δ as $(2,2,t)$-permissible. In the second case, $\Delta(5t-11) = \Delta(5t-9)$, so Δ is $(2,2,t)$-permissible by Observation 3.21(d). In the third and fourth case, Δ contains an isolated 4-tuple so Δ is $(2,2,t)$-permissible by Lemma 3.9.

(g) $(\bar{2}, 4, 2)$. This implies that Δ ends with *abbabaab*. Note that $\{5t-9, 5t-7\}$ is a canonical MCD2 and all other canonical MCD2s precede $[5t-11, 5t-4]$. So by taking the other $t-2$ canonical MCD2s along with $\{5t-11, 5t-8\}$ and $\{5t-7, 5t-4\}$, we realize Δ as $(2,2,t)$-permissible.

(h) $(2,1)$. By Lemma 3.11 and Observation 3.21(b), there exists a nonempty substring S_{t-2} which precedes the substring corresponding to (2) and has length congruent to $2 \pmod 3$. So Δ ends with type $(\bar{2}, 2, 1)$ and Δ ends with *baabb*. Observe that all of the $t-1$ canonical MCD2s precede $[5t-8, 5t-4]$. By taking the $t-1$ canonical MCD2s along with $\{5t-8, 5t-5\}$, we realize Δ as $(2,2,t)$-permissible.

(i) $(5,1)$. By Lemma 3.11 and Observation 3.21(b), there exists a nonempty alternating substring S_{t-2} which precedes substring corresponding to (5) and has length congruent to 2 $\pmod 3$. So Δ ends with type $(\bar{2}, 5, 1)$ and Δ ends with *abbababb*. Note that $\{5t-9, 5t-7\}$ is a canonical MCD2 and all other canonical MCD2s precede $[5t-11, 5t-4]$. So by taking the other $t-2$ canonical MCD2s along with $\{5t-11, 5t-8\}$ and $\{5t-7, 5t-4\}$, we realize Δ as $(2,2,t)$-permissible.

(j) $(\bar{8}, 1)$. Then $\Delta(5t-11) = \Delta(5t-9)$. So by Observation 3.21(d), Δ is $(2,2,t)$-permissible.

(k) $(\bar{5}, \tau, 1)$. Then $\Delta(5t-11) = \Delta(5t-9)$. So by Observation 3.21(d), Δ is $(2,2,t)$-permissible.

(l) $(2, \tau, 1)$. By Lemma 3.11 and Observation 3.21(b), there exist two nonempty alternating substrings S_{t-2} and S_{t-3} which precede the substring corresponding to (2) and have length congruent to $2 \pmod 3$. So Δ ends with type $(\bar{2}, \bar{2}, 2, \tau, 1)$. So Δ ends with *baabaaab*, *baabbbbb*, *abbaaaab*, or *abbabbbb*. In the first case, Δ, when restricted to $[5t-9]$, ends with $(\bar{2}, \bar{2}, 1)$. Since $t-1 \geq 3$, Δ, when restricted to $[5t-9]$, is $(2,2,t-1)$-permissible where the $t-1$ sets have diameter at most 4 by (h), (i), and (j). By taking those $t-1$ sets along with

$\{5t-8, 5t-4\}$, we realize Δ as $(2,2,t)$-permissible. In the second case, the AST Partition is violated since the triples aren't frontloaded. Properly partitioning the ending yields that there are more than 1 substring with length congruent to $\pmod 3$ so by Corollary 3.12, Δ is $(2,2,t$-permissible. In the third and fourth cases, Δ contains an isolated 4-tuple so by Lemma 3.9, Δ is $(2,2,t)$-permissible.

In each case, Δ is $(2,2,t)$-permissible. Since those 12 cases are every possible way that Δ ends, it must be that any 2-coloring of $[5t-4]$ is $(2,2,t)$-permissible. Therefore, $f(2,2,t) = 5t-4$. $\square$

CHAPTER 4

CONCLUSION

Our further work on this topic has split into two directions: (1) increasing the set size (m) and number of colors used (r), or (2) focusing on colorings that are (m, r, t)-permissible for positive integers $t \geq 4$ and determining how many ways there are to realize that coloring as (m, r, t)-permissible.

4.1 Increasing Parameters

From Theorems 3.5 and 2.5, we have that a range in which value for $f(m, r, t)$ will live given by

$$(mr + 1)(t - 1) + 1 \leq f(m, r, t) \leq ((m - 1)r + 1)\left[(t - 1)(((m - 1)r + 1) - m + 1) + 1\right]. \quad (4.1)$$

Note that the bounds on $f(m, r, t)$ give us that f is $O(m^2r^2t)$ and f is $o(mrt)$; the upper bound is quadratic in both m and r but the lower bound is linear in both m and r (both are linear in t). Thus, for large values of m and r, the range in which $f(m, r, t)$ lives is enormous. Initial investigations into shrinking this range have not given much insight into where in these regions exact values for f lie. Our work particularly leverages the alternating structure of 2-colorings by exploiting their binary nature. When the set size or number of colors increases, the binary nature disappears. As such, other methods of proof will need to be developed in order to find more exact values for f.

4.2 Enumeration of Permissible Collections

The proofs presented in this paper simply show the existence of a permissible collection of sets that demonstrate (m, r, t)-permissibility for positive integers m, r, and t. It is a natural question to want to determine how many ways in which a coloring can be realized as (m, r, t)-permissible. For positive integers m, r, t, and n, we define the function $g(m, r, t; n)$ to be the number of ways a coloring of $[n]$ could be realized as (m, r, t)-permissible. Observe that $g(1, r, t; n) = \binom{n}{t}$. This means that there are $\binom{n}{t}$ ways to realize an r-coloring of $[n]$ as $(1, r, t)$-permissible. We now calculate $g(m, 1, t; n)$ for $m \geq 2$ and arbitrary positive integers t and n. Given a monochromatic coloring of $[n]$, we construct $B_1, B_2, \ldots, B_t$, the t sets which realize a 1-coloring as $(m, 1, t)$-permissible, in the following way:

- Select the diameter d_i for each B_i where $i \in [t]$. Observe that for $i \in [t]$, $d_i \geq m-1$ since B_i is an m-subset of $[n]$. Further, $d_1 + d_2 + \cdots + d_t \leq n - t$ because each B_i is contained in non-overlapping subsets of cardinality $d_i + 1$ for $i \in [t]$. Let the sum of the diameters be denoted as D.

- Choose the minimum element m_i (and therefore the maximum element $m_i + d_i$) of each B_i so that $B_i \subseteq [m_i, m_i + d_i]$ for $i \in [t]$. To achieve this we take one of the $\binom{n-D}{t}$ solutions to the equation

$$x_0 + x_1 + \cdots + x_t = n - D - t,$$

  where, for each $i \in [t]$, x_i is a nonnegative integer. Define $m_0 = 0$ and $d_0 = 0$. Then recursively define $m_i = m_{i-1} + x_{i-1} + d_{i-1} + 1$ for $i \in [t]$.

- Choose the remaining elements in each B_i for $i \in [t]$. By definition, $m_i, m_i + d_i \in B_i$, so there are $\binom{d_i-1}{m-2}$ ways to select the other elements of B_i for $i \in [t]$.

Therefore, we may enumerate $g(m, 1, t; n)$ as

$$g(m,1,t;n) = \sum_{\vec{d} \in I} \binom{n-D}{t} \binom{d_1 - 1}{m-2} \binom{d_2 - 1}{m-2} \cdots \binom{d_t - 1}{m-2}, \tag{4.2}$$

where $I = \{(d_1, d_2, \ldots, d_t) \mid m - 1 \leq d_1 \leq \cdots \leq d_t,\ d_1 + \cdots d_t = D \leq n - t\}$. Considering that the enumeration in (4.2) counts the number of realizations of $(m, 1, t)$-permissibility for a monochromatic coloring, extending this to r-colorings with $r \geq 2$ seems quite challenging.

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APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

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November 20, 2018

Adam O'Neal
1526 ½ Sixth Avenue
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Dear Mr. O'Neal:

This letter is in response to the submitted thesis abstract entitled *"On Monochromatic Pairs of Nondecreasing Diameter."* After assessing the abstract, it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction, it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

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