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Statistical Analysis of Tandem Queues With Markovian Passages in Porous Mediums

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STATISTICAL ANALYSIS OF TANDEM QUEUES WITH MARKOVIAN PASSAGES IN POROUS MEDIUMS

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APPROVAL OF THESIS/DISSERTATION

We, the faculty supervising the work of Gboyega David Adepoju, affirm that the thesis, STATISTICAL ANALYSIS OF TANDEM QUEUES WITH MARKOVIAN PASSAGES IN POROUS MEDIUMS, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the editorial standards of our discipline and the Graduate College of Marshall University. With our signatures, we approve the manuscript for publication.

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Now unto him who is able to do exceedingly and abundantly above all we can ask or think be glory, and honor, forever. Eph 3:20-21.

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ABSTRACT

Queuing theory is the mathematical study of queues or waiting lines. A queue is formed whenever the demand for service exceeds the capacity to provide service at that point in time. In this thesis, the birth-and-death process is used to model the movement of customers or units into and out of a network of queues in tandem. We start with the theoretical analysis of M/M/1 queues with Poisson arrival and exponential service time with first-come first-served (FCFS) discipline and one service station. We derive the global balance equation for each network. Using both the iterative and the probability generating function, we obtain the probabilities of the state for each service point in the network at equilibrium, and also discuss the statistical properties of the migration of customers from one service point to another. We generalize the probability generating function for the system with n states, and also the marginal for each of the queues in tandem. Specifically, two networks are considered, namely, one that allows customers into the system from the leading queue, and another with porous medium, which allows customers into the system of queues through any service stations. Finally, we simulate a queue network of 10,000 customers and generalize the traffic intensity, the proportion of customers moving from one station to another.

CHAPTER 1

INTRODUCTION

The beginning of queuing theory can be traced back to 1909, when Agner Krarup Erlang (1878-1929, now considered the father of the field) published his principal paper on congestion in telephone traffic [5]. Erlang established a strong foundation for queuing theory in terms of the nature, presumptions and procedures of analysis. His work with the Copenhagen Telephone Company is what prompted him to delve into the field. He pondered the problem of determining how many telephone circuits were necessary to provide phone service that would prevent customers from waiting too long for an available circuit. He discovered that the problem of minimizing waiting time was applicable to many fields [8]. The study of waiting line has made great contributions in modelling and designing communication systems since inception [8].

A queuing framework is composed of clients or units requiring a kind of service who arrive at a service station where they receive service. A queue is said to be formed when such clients or units are waiting to be served. Queuing theory is a field of operations research since the results are used for making choices around the assets required to supply service [11]. Applications of the queuing theory are traffic flow (vehicles, aircraft, people, communications), scheduling (patients in hospitals, jobs on machines, programs on computer), service design (banks, post offices, supermarkets), among many others [14].

The mechanism of tandem queue with $k \geq 1$ service stations is described as follows: Customers access the system through the first service station, after service completion at the station i, a customer has the option of either proceeding automatically and forming input into the jth station with the transition probability α_{ij} per unit time, $i = 1, 2, ...k$ and $j = 2, 3, ...k$, [1].

1.1 Queuing Network

A queuing system can be described by the flow of customers for service, forming or joining the queue. The term customer is used in a general sense and does not necessarily imply human customer [6]. Customers could be an airplane waiting in line to take off, a computer program waiting to be run, or items arranged together in a grocery store. A mechanism that performs the kind of service on customers or units that are fed into it is called a server or service channel. For example, jobs arriving at a component in a computer center are also regarded as customers and the component of the computing system (such as CPU, drum, disk, line printer, etc.) where such a facility is provided is considered the server.

There are cases where customers leave the system without joining the queue **(balked)** or leave without receiving service even after waiting for some time **(reneged)**. In an event that there are two or more parallel waiting lines, customers may switch from one line to another; that customer is said to have **(jockey)** for position. These are forms of queue with impatient customers. Queuing network can also be described as a group of nodes, where each of the nodes represents a service facility of some type. Customers can arrive from outside the system to any node and may leave the system from any node. Therefore, customers may enter the system at some node, transverse from node to node in the system and finally leave the system [15].

1.1.1 Queue Components

The arrival pattern of customers is the manner in which arrivals occur. It is specified by the inter-arrival time between any two consecutive arrivals. The inter-arrival may be deterministic, so it is the same between any two consecutive arrivals, or it may be stochastic. Usually in queuing situation, the inter-arrival time is stochastic, and it is necessary to know the probability distribution describing the times between successive customer arrivals (inter-arrival times). The arrival pattern also indicates whether arrivals occur singly or in groups or batches [10].

The service pattern is another queue component. It simply means the manner in which service is rendered. It is specified by the time taken to complete a service; the time may be constant (deterministic) or it may be stochastic. If it is stochastic, the pattern specification involves the distribution of service time of a unit. Sometimes service may be rendered in bulk or batches as in the case of an elevator, instead of personalised service of one at a time.

The manner in which customers are selected for service when they arrive is called queue discipline. The most common discipline is first-come, first-served **(FCFS)**. Some other common queuing disciplines are last-come, first-served **(LCFS)**, service in random order **(SIRO)**, etc.,

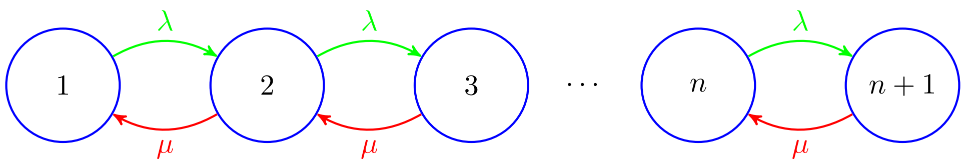

Figure 1: State Transition Rate Diagram.

The above state transition flow rate in Figure 1 shows that the system goes to state 1 if a service is completed or to state 3 if an arrival occurs.

which are independent of the time of arrival to the queue.

A system may have an infinite capacity; that is, the queue in front of the server(s) may grow to any length, this is called system capacity. However, there may be limitation to the waiting room, so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available as a result of a service completion. This type of queue is called gated queue or queue with finite capacity. This queue component may be viewed as one with forced balking where customer is forced to balk if it arrives when the queue size is at its limit.

A queuing system may have single service stage, as in a hair styling salon or several stages, like queuing system designed for physical examination procedure, where each patient must proceed through several stages, such as medical history, ear, nose, and throat examination, blood tests, electrocardiogram, eye examination and so on. Such systems are examples of tandem queue and the transition from one state to another is shown in figure 1.

The number of service channels is another major queuing component; this refers to the number of parallel service stations which can serve customers simultaneously. It is generally preferable to design multi-channel queuing systems to be fed by a single line [10]. A hair-styling salon with many chairs is an example of a multichannel system with one queue and parallel channels as shown in figure 2.

1.1.2 Queue Notations and Descriptions

Kendall (1953) [7] introduced a notation which is generally adopted to denote a queuing model. It consists of three basic characteristics: input, the service time, and the number of parallel servers. A queuing process is described by the notation **A/B/X/Y/Z**, where A indicates

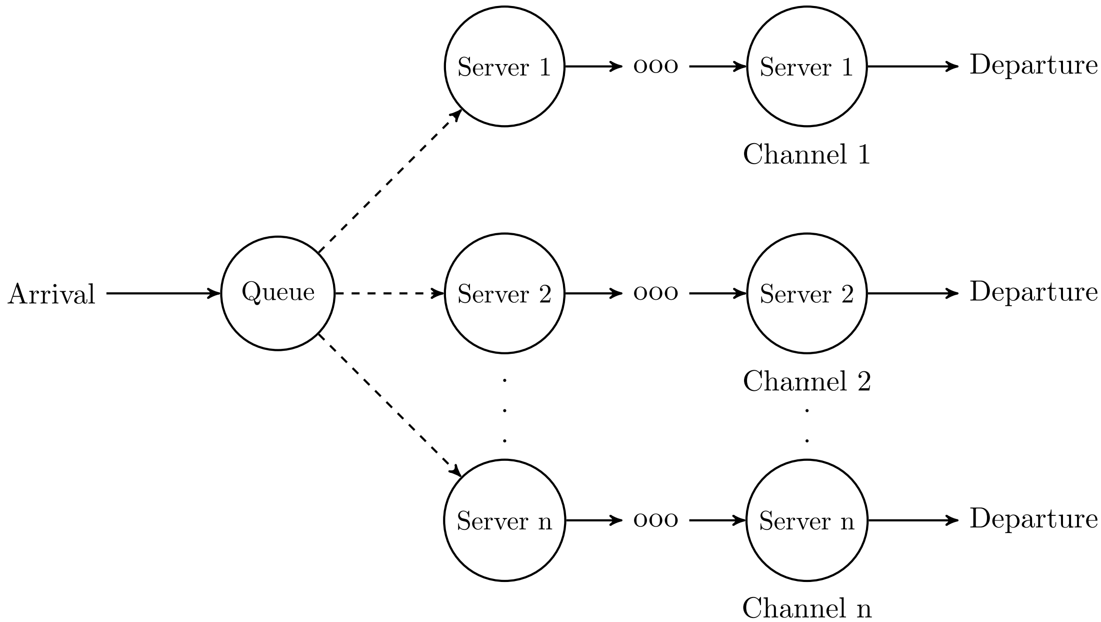

Figure 2: Multichannel Queuing Systems With Parallel Channels.

This queuing system contains three parallel channels.

inter-arrival time distribution, B the service pattern, X the number of parallel service channels, Y the restriction on system capacity, and Z the queuing discipline. For example, $D/M/2/\infty/FCFS$ denotes a queuing system with deterministic inter-arrival time, exponential service time, two parallel servers, infinite queue length and first-come, first-serve queue discipline. The Kendall notation is sometimes written in terms of only three items (For example $M/M/1$). In the case, there is no restriction on the system capacity and the queue discipline is assumed to be first-come first-served. A queuing system is said to have deterministic service time if the time is fixed. For example, jobs on machine can be scheduled for a fixed time. Other queuing notations used in this thesis and their meaning are listed below:

- λ = Average arrival rate.
- μ = Average service time.
- L_s = Average number of customers on the system.
- L_q = Average number of customers on the queue.

- W_s = Average waiting time in the system.
- α_{ij}= Transition probability.
- W_q = Average waiting time on the queue.
- c = Number of service channels.
- N(t)= Number of customers in the system at time t.
- $P_n(\mathbf{t})$ = Probability of exactly n customers are in the system at time t.
- $\rho = \frac{\lambda}{\mu}$=traffic intensity.

The traffic intensity is a measure of traffic congestion for one server system with the condition that $\frac{\lambda}{\mu} < 1$, that is– the mean arrival rate must be less than the mean maximum potential service rate of the system. If $\rho > 1$, then it means that the average arrival rate into the system exceeds the service rate of the system, and we would expect the queue to get bigger and bigger as time goes on unless customers are not allowed to enter again at some point.

In the case of a model with more than one service station like the $M/M/c/\infty$ model, where the first M is the interarrival time, the second M is service pattern, c is the number of parallel server and infinite queue length. The condition for traffic intensity of the system is given as $\frac{\lambda}{(c\mu)}$.

The Kendall's notation described in figure 1 is $M/M/1$. So the queue has M exponential inter-arrival time, deterministic service time D, servers arranged in series, infinite queue length and first come first serve discipline.

1.1.3 Types of Queuing Networks

- **M/M/n/** $\infty \rightarrow$ Queues with n parallel channels, Poisson inter-arrival time and exponential departure, ∞ possible length and FCFS discipline.
- **M/M/n/K** $\rightarrow$ Queues with n parallel channels, exponential arrival and departure, K possible length and FCFS discipline.
- $M^{[X]}$ **/ M / 1** $\rightarrow$ Queues with bulk arrivals or input i.e., exactly X at each time step.
- **M/**$M^{[Y]}$**/1** $\rightarrow$ Queues with bulk service.
- **M/**E_k**/1**, E_k/M/1, $E_j/E_k/1 \rightarrow$ Erlangian Models.
- **M/G/1** $\rightarrow$ Single server queue with exponential input and general service.
- **G/G/1** $\rightarrow$ General input, general service.
- **M/D/n** $\rightarrow$ Multichannel queue with exponential input and constant service.

1.2 Definitions

1.2.1 Little's Formula

One of the most relevant and useful relationship in queuing theory is the Little's formula, which was developed by John Little in the early 1960s [9]. The Little's formula is defined by

$$L = \lambda W, \tag{1.1}$$

where λ is the average arrival rate, L is the expected number of units in the system, and W is the expected waiting time in the system at steady state. Using similar notation, the expected number in the queue L_Q is defined by

$$L_Q = \lambda W_Q. \tag{1.2}$$

The Little's formula describes the long term average number of customers in a stable system as the product of the average arrival rate and the average time a customer spent in the system.

1.2.2 Poisson Process

The Poisson process is a special case of pure birth process with parameter λ. It is one of the most widely-used counting processes. Poisson process can be used as a model for a large class phenomena, like the number of car accidents at a site or in an area, the location of users in a wireless network and so on. The Poisson postulates are stated as follows:

1. The probability that an arrival occurs between time t and time $t + \triangle t$ is equal to $\lambda \triangle t + O(\triangle t)$, where λ is a constant independent of N(t), $\triangle t$ is an incremental element, and $O(\triangle t)$ denotes a quantity that becomes negligible when compared to $\triangle t$ as $\triangle t \rightarrow 0$.

2. The probability that there is more than one arrival between time t and time $t + \triangle t$ is equal to $O(\triangle t)$.

3. The numbers of arrivals in nonoverlapping intervals are statistically independent; i.e, the process has independent increments.

1.2.3 Properties of Poisson Process

- **Additive property:** The sum of n independent Poisson processes with parameter $\lambda_i, i = 1, 2, 3, ..., n$ is a Poisson process with parameter $\lambda_1 + \lambda_2 + ... + \lambda_n$.

- **Interarrival Time:** The interarrival times between two consecutive occurrences of a Poisson process with parameter λ are independently and identically distributed random variable having exponential distribution with parameter λ.

- **Memoryless Property of Exponential Distribution:** If the interval between two occurrences is exponentially distributed, then the memoryless property implies that the interval to the next occurrence is statistically independent of the time from the last occurrence.

Suppose X has the exponential distribution, then

$$Pr\{X \geq x + y | X \geq x\} = Pr\{X \geq y\} \tag{1.3}$$

is independent of x. That is,

$$
\begin{aligned}
Pr\{X \geq x+y | X \geq x\} &= \frac{Pr\{X \geq x+y | X \geq x\}}{Pr\{X \geq x\}} \\
&= \frac{Pr\{X \geq x+y\}}{Pr\{X \geq x\}} \\
&= \frac{\exp(-\lambda(x+y))}{\exp(-\lambda x)} \\
&= \exp(-\lambda y) = Pr\{X \geq y\}.
\end{aligned}
$$

1.2.4 The PASTA Property

Another very essential tool used in the analysis of many queuing systems is the PASTA (Poisson Arrivals See Time Average) property. This property proposed by Wolff (1982) [17], asserts that customers with poisson arrivals see the system as if they arrived at an arbitrary point in time despite the fact that they induce transitions in the system. This phenomenon arises from the lack of memory of an exponential interarrival time with the result that the arrival history just before a tagged arrival instant is stochastically identical to that of a random instant as well as that of the arrival instant. Markovian queuing systems with Poisson arrivals possess the PASTA property.

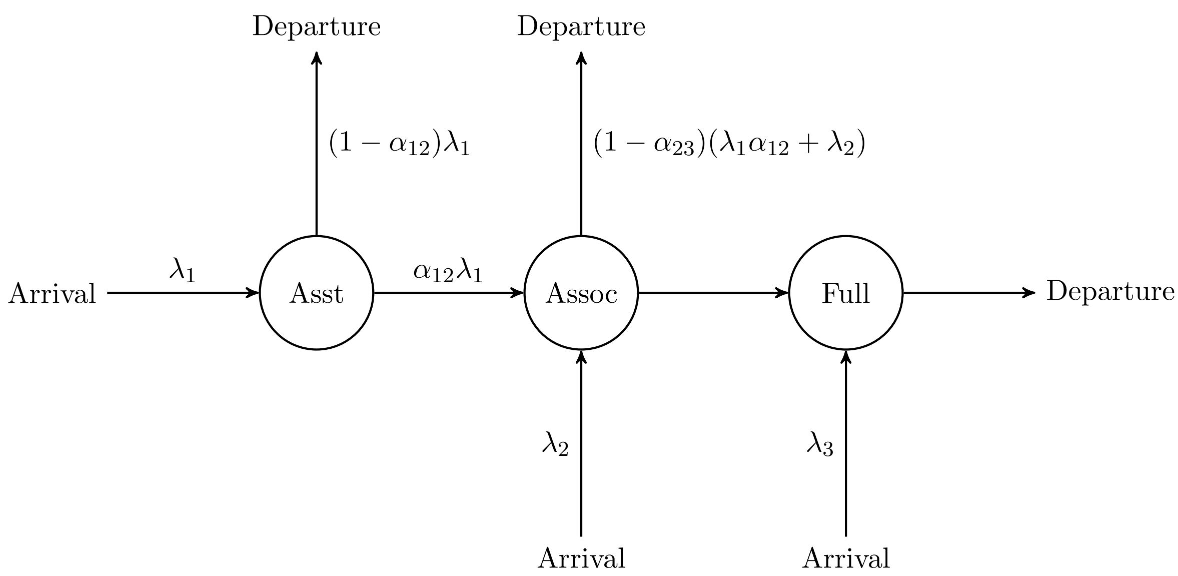

Figure 3: Tandem Queue and Faculty Mobility.

1.3 Description of Manpower System Using Tandem Queue

A Manpower system has been defined as any identifiable group of people working with a common end in view [3]. It will be essential to recognize inside such a framework a set of homogeneous classes. These classes may be based on any pertinent properties of their individuals such as age, grade, seniority, salary, job or location [3]. Each class contains a certain number of people, which may change based on the behaviour of individuals in the system. Hence, it has become of great interest to many operations researchers and statisticians to know how the manpower system changes with time. These changes are described by the flow of people which takes place between classes (e.g., promotions and transfers) and between the system and its environment (e.g., recruitment and retirement) [3].

In figure 3, we described the movement of faculty in a university system from Assistant professor to Associate professor and to Professor. These movements are described as recruitment (flow of faculty into the system), retention (faculty remaining at the same level), promotion (flow from one level to another) and retirement/resignation (flow of faculty out of the system) at any state in the system. Essentially, this movement can be modeled using stochastic processes, queuing theory, and compartmental models.

Compartmental models are a common technique for mathematical modeling of infectious diseases. It is used in epidemiology to model the transmission of infectious diseases from susceptible to infectious to recovered. The population is divided into homogeneous groups called compartments and the compartments of the model can either flow between each other or they can interact. As compartmental model, faculty splits into Assistant, Associate and full Professor as seen in figure 3.

In this thesis, birth-and-death processes are used to model the movement of customers or units into a queuing system. More specifically, we consider $M/M/1$ tandem queue with n queues. Customers enter the system with poisson inter-arrival rate λ_i and exponential departure time μ_i. Customers may exit the system after being served at station S_i or proceed to the next station S_{i+1} where $i \geq 1$.

We start with theoretical analysis of tandem queue with one service station. We compute some statistical properties of tandem queue, the global balance equation, probability of having n customers or units in the system at steady state using iterative method as well as probability generating function. We increased the service station by one (i.e., tandem queue with two service stations) and allow entrance to the second service station only from station one. Again we obtained the global balance equation for this model using probability generating function and also the probability of having n customers or units in the system at steady state. We further considered tandem queue with two service stations and allow arrival to and departure from both service stations. We generalized the traffic intensity, the proportion of customers moving from one service station to the other and also the marginal probability generating function for tandem queue with n service stations. Finally, we simulate a queue network of 10,000 customers.

1.4 Literature Review

In this section, related journals on manpower planning, queuing theory and compartmental models are reviewed. We start with an interesting work done by Ayyappan and Thamizhselvi [2] where a single server queuing system with two types of batches arrivals and services under non pre-emptive priority rule was considered. The server provides single service to high priority customers and the general bulk service rule for low priority customers on a FCFS discipline. The server starts service to the low priority customer only if the high priority queue is empty and the number of customers in the low priority queue is greater than or equal to a. If there are no customers in the high priority queue and the number of customers in the low priority queue are less than a, then the server becomes idle [2]. The average number of customers in the queues and the average waiting time are derived for this model.

In 2007, Yadavalli, Natarajan, and Udayabhaskaran [12] studied time dependent behaviour of stochastic models of manpower system considering the impact of pressure on promotion. They considered time dependent behaviour of three stochastic models of manpower system namely: 1) the pressure for promotion in a particular grade is contributed by the employees in the grade alone. 2) the pressure for promotion to a particular grade is proportional to the number of employees in the lower grades who have eligibility to get promoted to that grade. 3) the pressure for promotion is considered to be proportional to the number of employees in a particular grade as in (1) above. They considered length of service as the sole criterion for promotion [12].

Human behaviour is unpredictable and constitutes an essential aspect of manpower planning: hence, Ugwuowo and McClean reviewed methods of incorporating population heterogeneity into a manpower model [16]. The analysis of differentials in manpower system is stressed because they are a source of aggregation error in stochastic models. Ugwuowo and McClean [16] stressed two strategies: the use of observable sources of heterogeneity as they affect wastage and latent sources which cannot be identified precisely but are known to affect the key parameters of the models [16]. It has always been accepted that there are variations among individuals in life endeavours, thus the analysis of individual difference is clearly of central importance in the study of manpower systems. In survival analysis, it is well known that individuals differ substantially in their endowment for longevity [16].

Bleau [4] described academic flow model as one with three major stages and a minor stage, namely: fixed term, tenure track, tenured and part-time, respectively. Membership to a stage is determined by faculty member's current contract classification. The author considered a person changing from a part-time to a full-time appointment to have left the system and re-entered as a new hire. A person changing from full-time to part-time appointments are treated in the same manner. An assumption was made that a faculty member initially hired as a tenure-track appointment cannot change to a fixed-term appointment. It presented a better understanding of the complex phenomena of faculty movement through an institution and on its relationship to salary cost, composition of faculty and faculty turnover rate.

The model was implemented and tested at two different institutions. The findings suggested that the model is a viable means of gaining useful insights and quantitative data on the faculty profile, salary costs, expected departures and part-time trends and further, when used as a planning tool. The model apparently is comprehensive and flexible enough to analyze the probable effects, both in the short and long run, of alternative personnel policies on the faculty composition.

Knisley et al. [13] used linear compartmental model to model migration of salmon fishes. This model was introduced by dividing a river into a contiguous collection of habitat zones and assuming that the population is initially in zone 0, which corresponds to the river segment where the salmon hatched and developed into smolt. After living in zone 0 for almost a year, the smolt will actively swim down the river into the ocean estuaries through the sequence of zones (i.e, compartments). Thus, if $N(t)$ is the population of the habitat zone at time t in days, then the population one day later satisfies

$$N(t+1) = N(t) + Arrivals - Departures,$$

where arrivals and departures are those entering or leaving during that day. They in general wrote the model of the habitation zone as

$$N(t + \triangle t) = N(t) + I\triangle t - E\triangle t,$$

where I is the immigration rate, $I\triangle t =$ is the arrivals during a given time increment, $E\triangle t$ is the departures and E is the rate of emigration. Their approach was found to be of great value in introducing derivatives, integrals and fundamental theorem of calculus.

CHAPTER 2

INTRODUCTION

2.1 Model Description

The stochastic queuing models described in this section assume that customers' arrivals occur as a poisson process with parameter λ and the service times are independent and exponentially distributed with parameter μ and there is only one server. Customers exit the system after being served.

2.1.1 Poisson Process

Components of the model

- Probability of one arrival during $\Delta t = \lambda \Delta t$
- Probability of more than one arrival during $\Delta t = 0$
- Probability of no arrivals during $\Delta t = 1 - \lambda \Delta t$
- Probability of one service during $\Delta t = \mu \Delta t$
- Probability of no service during $\Delta t = 1 - \mu \Delta t$
- Probability of more than one departure during $\Delta t \; = 0$

2.2 Tandem Queue With One Service Station

We want to compute the probability of n arrivals in a time interval of length t, $P_n(t)$, n being an integer greater than or equal to zero. To do this, we first develop a differential-difference

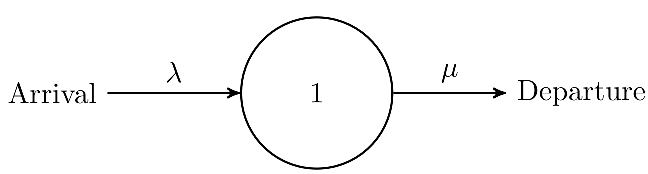

Figure 4: M/M/1 Queue With One Service Station.

equation for the arrival process which can be obtained by the birth and death process as

$$\begin{aligned}P_n(t+\Delta t) = &\text{ The probability of } n \text{ arrivals in time t and none in } \Delta t \\ &+ \text{probability of having } n-1 \text{ arrivals in time t and one arrival in } \Delta t \\ &+ \text{probability if having } n-2 \text{ arrivals in time t and two arrivals in } \Delta t \\ &+ \ldots \\ &+ \text{the probability of having no arrival in time t and } n \text{ arrivals in } \Delta t.\end{aligned}$$

Now for our model $M/M/1$ queue with one service station, poisson arrival rate and exponential service time figure 4, we obtain the differential-difference equation as follows.

$$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t)(1-\mu\Delta t) + P_{n-1}(t)\lambda\Delta t(1-\mu\Delta t) + P_{n+1}(t)(1-\lambda\Delta t)\mu\Delta t, \quad (2.1)$$

where $P_n(t)$ is the probability of having n units in the system at time t and $P_n(t+\Delta t)$ is the probability of having n units in the system at time $t+\Delta t$.

From equation (2.1) the probability of n units in the system at $t+\Delta t$ equals the sum of the probabilities of the three mutually exclusive events

(1) n units in the system at t, no arrival or service during Δt

(2) $n-1$ units in the system at t, one arrival and no service during Δt

(3) $n+1$ units in the system at t, no arrival and one service during Δt

Expanding the right-hand side of (2.1), moving $P_n(t)$ to the left-hand side, dividing by Δt and letting $\Delta t \to 0$ gives the following differential-difference equation:

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = P_n(t)(-\mu-\lambda) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu \quad (2.2)$$

$$\frac{dP_n(t)}{dt} = -P_n(t)(\mu+\lambda) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu. \quad (2.3)$$

At steady state (i.e, when $t\to\infty$), $\dfrac{dP_n(t)}{dt} = 0$. Here, $P_n(t)$ is no longer a function of t. Thus, t

can be excluded and (2.3) becomes the global balance equation

$$(\mu + \lambda)P_n = \lambda P_{n-1} + \mu P_{n+1} \tag{2.4}$$

If we consider state $n \geq 0$, the system can go to the next state $(n + 1)$ at rate λP_n, and it can come down from state $(n + 1)$ to the original state n at rate (μP_{n+1}). At equilibrium, these two rates (i.e, the rate up from a particular state n to the next state $(n + 1)$ and the rate down, that is, from state $(n + 1)$ to the original state n must be equal. This implies that

$$\lambda P_n = \mu P_{n+1}, (n \geq 0) \tag{2.5}$$

We obtain the probability that there are exactly n customers in the queue from the difference equation (2.4), first by using iterative method and then by probability generating function later.

For the iterative method, we start by setting $n = 0$ and find the boundary conditions. Clearly, there cannot be service when there is no one in the system; this makes $\mu P_0 = 0$ and also $P_{n-1}\lambda = 0$ because it is not possible to have -1 person on the queue, thus (2.4) becomes

$$\lambda P_0 = \mu P_1,$$

which implies that

$$P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0, \tag{2.6}$$

where $\rho = \dfrac{\lambda}{\mu}$ is a measure of traffic congestion for one server system with the condition that $\dfrac{\lambda}{\mu} < 1$. Now from the steady state equation (2.4), when $n = 1$;

$$(\mu + \lambda)P_1 = \lambda P_0 + \mu P_2 \tag{2.7}$$

Recall $P_1 = \frac{\lambda}{\mu} P_0$, so that

$$\frac{\lambda}{\mu}(\mu + \lambda)P_0 - \lambda P_0 = \mu P_2. \tag{2.8}$$

It implies that $\mu P_2 = \frac{\lambda^2}{\mu} P_0$. So that,

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0. \tag{2.9}$$

If we continue this process we will obtain $P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$, $P_4 = \left(\frac{\lambda}{\mu}\right)^4 P_0$ and so on. Thus, in general,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0. \tag{2.10}$$

Now to obtain the probability that there is no one in the system P_0, we rely on the fact that $\sum_{n=0}^{\infty} P_n = 1$ and sum both sides of (2.10). We obtain,

$$1 = P_0 \sum_{n=0}^{\infty} \rho^n. \tag{2.11}$$

Now we know that the sum of an infinite geometric series $\sum_{n=0}^{\infty} \rho^n = [1 + \rho + \rho^2 + ...]$ is given as $\dfrac{1}{1-\rho}$, where ρ is the common ratio. Hence, (2.11) becomes

$$P_0 \frac{1}{1-\rho} = 1,$$

which implies that the proportion of time that the system is empty is

$$P_0 = 1 - \rho. \tag{2.12}$$

2.2.1 Probability Generating Function

Now, solving the steady-state difference equation (2.4) using probability generating function. We recall from the literature that the probability generating function is given as

$$G(z) = \sum_{n=0}^{\infty} z^n P_n. \tag{2.13}$$

Now multiply equation (2.4) through by z^n and sum appropriately to have

$$\sum_{n=1}^{\infty} z^n P_n(\mu + \lambda) = \sum_{n=1}^{\infty} z^n P_{n-1}\lambda + \sum_{n=1}^{\infty} z^n P_{n+1}\mu. \tag{2.14}$$

But we cannot sum from $n = 0$, due to the P_{n-1} term. So we have

$$(\mu + \lambda)\sum_{n=1}^{\infty} z^n P_n = \lambda z \sum_{n=1}^{\infty} z^{n-1} P_{n-1} + \mu z^{-1} \sum_{n=1}^{\infty} z^{n+1} P_{n+1}. \tag{2.15}$$

We can write the above more appropriately as

$$(\mu + \lambda)[\sum_{n=0}^{\infty} z^n P_n - P_0] = \lambda z \sum_{n=0}^{\infty} z^n P_n + \mu z^{-1}[\sum_{n=0}^{\infty} z^n P_n - P_0 - zP_1]. \tag{2.16}$$

Thus,

$$(\mu + \lambda)[G(z) - P_0] = \lambda z G(z) + \mu z^{-1}[G(z) - P_0 - zP_1],$$

or

$$(\mu + \lambda - \lambda z - \mu z^{-1})G(z) = (\lambda + \mu)P_0 - \mu z^{-1} P_0 - \mu P_1. \tag{2.17}$$

Substituting $\frac{\lambda}{\mu}P_0$ for P_1 in (2.17) we have,

$$(\mu + \lambda - \lambda z - \mu z^{-1})G(z) = (\lambda + \mu)P_0 - \mu z^{-1} P_0 - \mu \frac{\lambda}{\mu} P_0, \tag{2.18}$$

which now gives,

$$\begin{aligned} G(z) &= \frac{(\mu z - \mu)\frac{P_0}{z}}{(\mu + \lambda - \lambda z - \mu z^{-1})}, \\ &= \frac{\mu(1 - z^{-1})P_0}{\lambda(1 - z) + \mu(1 - z^{-1})} \\ &= \frac{(\mu z - \mu)P_0}{(\mu z + \lambda z - \lambda z^2 - \mu)}. \end{aligned} \tag{2.19}$$

Now, $G(z)|_{z=0} = P_0$.

We obtain P_1 by finding the first derivative of $G(z)$ with respect to z and then setting

$z = 0$ to have,

$$G^{'}(z) = \frac{(\mu z + \lambda z - \lambda z^2 - \mu)\mu P_0 - (\mu z - \mu)P_0(\mu + \lambda - 2\lambda z)}{(\mu z + \lambda z - \lambda z^2 - \mu)^2},$$
$$= \frac{\lambda\mu z^2 P_0 + \lambda\mu P_0 - 2\lambda\mu z P_0}{(\lambda z + \mu z - \lambda z^2 - \mu)^2}.$$

$$G^{'}(z)|_{z=0} = \frac{\mu\lambda P_0}{\mu^2} = \frac{\lambda}{\mu}P_0. \tag{2.20}$$

Therefore $P_1 = \frac{\lambda}{\mu}P_0$, as obtained using the iterative method.

Again $P_2 = \frac{G^{''}(z)}{2!}|_{z=0}$.

$G^{''}(z) = \frac{(\lambda z+\mu z-\lambda z^2-\mu)^2(2\lambda\mu z P_0-2\lambda\mu P_0)-(\lambda\mu z^2 P_0+\lambda\mu P_0-2\lambda\mu z P_0)2[(\lambda z+\mu z-\lambda z^2-\mu)(\lambda+\mu-2\lambda z)]}{(\lambda z+\mu z-\lambda z^2-\mu)^4}$

$$\frac{G^{''}(z)|_{z=0}}{2!} = \frac{-2\lambda\mu^3 P_0 + 2\lambda^2\mu^2 P_0 + 2\lambda\mu^3 P_0}{2!\mu^4} = \frac{2\lambda^2\mu^2 P_0}{2\mu^4}$$

$$P_2 = \frac{2\lambda^2\mu^2 P_0}{2\mu^4} = \frac{\lambda^2}{\mu^2}P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \tag{2.21}$$

Thus, $P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$. So in general,

$$P_n = \frac{G^n(z)|_{z=0}}{n!} = \left(\frac{\lambda}{\mu}\right)^n P_0. \tag{2.22}$$

2.2.2 Expectation and Variance of the Model

To obtain the expected number of people in the system at steady state, let N represent the random variable, then

$$\begin{aligned} E(N) &= \sum_{n=0}^{\infty} nP_n = (0P_0 + P_1 + 2P_2 + ...) \\ &= \left(0(\rho)^0 P_0 + \rho P_0 + 2(\rho)^2 P_0 + ...\right) \end{aligned} \tag{2.23}$$

$$= P_0 \sum_{n=0}^{\infty} n\rho^n \tag{2.24}$$

Looking at the summation,

$$P_0 \sum_{n=0}^{\infty} n\rho^n = P_0(\rho + 2\rho^2 + 3\rho^3 + ...)$$
$$= P_0\rho(1 + 2\rho + 3\rho^2 + ...) \tag{2.25}$$

$$= P_0\rho \sum_{n=0}^{\infty} n\rho^{n-1} \tag{2.26}$$

Note that $\sum_{n=1}^{\infty} n\rho^{n-1}$ is the derivative of $\sum_{n=0}^{\infty} \rho^n$ with respect to ρ. Hence,

$$P_0\rho \sum_{n=0}^{\infty} n\rho^{n-1} = P_0\rho \sum_{n=0}^{\infty} \frac{d}{d\rho}\rho^n \tag{2.27}$$

$$= P_0\rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n. \tag{2.28}$$

We know that

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho},$$

thus,

$$P_0\rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n = P_0\rho \frac{d}{d\rho}(\frac{1}{1-\rho})$$

$$= P_0\rho \left(\frac{1}{(1-\rho)^2} \right)$$

$$= (1-\rho) \left(\frac{\rho}{(1-\rho)^2} \right)$$

So the expected number in the system at steady state is

$$\frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}. \tag{2.29}$$

We know from literature that the variance $Var(N) = E(N^2) - [E(N)]^2$.

$$E(N^2) = \sum_{n=0}^{\infty} n^2 P_n = \sum_{n=1}^{\infty} n^2 P_0 \rho^n = \sum_{n=1}^{\infty} n^2 (1-\rho)\rho^n \tag{2.30}$$

Note that the second derivative $\frac{d^2}{d\rho}(\sum_{n=0}^{\infty}\rho^n)$

$$= \sum_{n=0}^{\infty} n(n-1)\rho^{n-2} = \sum_{n=0}^{\infty}(n^2-n)\rho^{n-2}.$$

$E(N^2)$ in equation (2.30) can be written as

$$\begin{aligned}
E(N^2) &= (1-\rho)\sum_{n=1}^{\infty}[(n^2-n)+n]\rho^n \\
&= (1-\rho)\left[\sum_{n=1}^{\infty}(n^2-n)\rho^n + \sum_{n=1}^{\infty} n\rho^n\right] \\
&= (1-\rho)\left[\rho^2\sum_{n=1}^{\infty}\frac{d^2}{d\rho}\rho^n + \rho\sum_{n=1}^{\infty}\frac{d}{d\rho}\rho^n\right] \\
&= (1-\rho)\left[\rho^2\frac{d^2}{d\rho}\left(\frac{1}{1-\rho}\right) + \rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)\right] \\
&= (1-\rho)\left[\frac{2\rho^2}{(1-\rho)^3} + \frac{\rho}{(1-\rho)^2}\right] = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho} \\
&= \frac{\rho+\rho^2}{(1-\rho)^2}.
\end{aligned}$$

So that $Var(N) = E(N^2) - [E(N)]^2$

$$= \frac{\rho}{(1-\rho)^2} = \frac{\lambda}{\mu(1-\frac{\lambda}{\mu})^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}. \tag{2.31}$$

2.3 Tandem Queue With Two Service Stations

A queuing network in which a customer, served at point i, can immediately join the queue at point $i+l$, $l \geq 1$ is called tandem queues or queues in series. This type of queue is described in figure 5. Now consider a network of two queues in tandem. The global balance equation that an arriving customer finds n_1, customers in the queue 1 and n_2 customers in queue 2 may be written as,

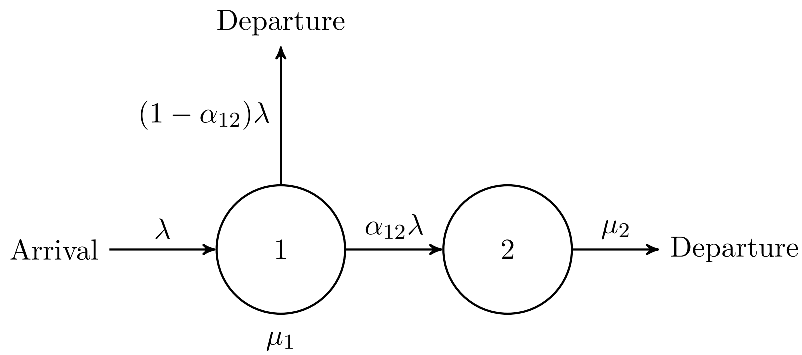

Figure 5: Tandem Queue With Two Service Stations and Intermediate Withdrawal.

$$\begin{aligned} P_{n_1,n_2}(t+\Delta t) = {} & P_{n_1,n_2}(t)(1-\lambda\Delta t)(1-\alpha_{12}\mu_1\Delta t)(1-\mu_1(1-\alpha_{12}))(1-\mu_2\Delta t) \\ & + P_{n_1-1,n_2}(t)\lambda\Delta t(1-\alpha_{12}\mu_1\Delta t)(1-\mu_1(1-\alpha_{12}))(1-\mu_2\Delta t) \\ & + \alpha_{12}\mu_1\Delta t P_{n_1+1,n_2-1}(t)(1-\lambda\Delta t)(1-\mu_2\Delta t) \\ & + (1-\alpha_{12})\mu_1\Delta t P_{n_1+1,n_2}(t)(1-\lambda\Delta t)(1-\mu_2\Delta t) \\ & + P_{n_1,n_2+1}(t)(1-\lambda\Delta t)(1-\alpha_{12}\mu_1\Delta t)(1-\mu_1(1-\alpha_{12}))\mu_2\Delta t. \end{aligned} \tag{2.32}$$

From the above equation, the probability that there are n_1, n_2 units in station 1 and 2 respectively at time $t + \triangle t$ is equal to the probabilities of the following mutually exclusive events, namely;

(1) n_1, n_2 units in the system at time t and no arrival or service during $\triangle t$ in both service stations.

(2) $n_1 - 1, n_2$ units in the system at time t, one arrival and no service at station 1, no arrival and no service at station 2 during $\triangle t$.

(3) $n_1 + 1, n_2 - 1$ units in the system at time t, no arrival and one service at station 1, one arrival and no service at station 2 during $\triangle t$.

(4) $n_1 + 1, n_2$ units in the system at time t, no arrival and one service at station 1, no arrival and no service at station 2 during $\triangle t$.

(5) $n_1, n_2 + 1$ units in the system at time t, no arrival and no service at station 1, no arrival and

one service at station 2 during $\triangle t$.

Simplifying equation (2.32) gives

$$P_{n_1,n_2}(t+\Delta t) = P_{n_1,n_2}(1-\mu_1-\mu_2-\lambda)\Delta t + \lambda \Delta t P_{n_1-1,n_2} + \alpha_{12}\mu_1 \Delta t P_{n_1+1,n_2-1} + (1-\alpha_{12})\mu_1 \Delta t P_{n_1+1,n_2} + \mu_2 \Delta t P_{n_1,n_2+1}. \tag{2.33}$$

$$lim_{\Delta t\to 0}\frac{P_{n_1,n_2}(t+\Delta t) - P_{n_1,n_2}(t)}{\Delta t} = -P_{n_1,n_2}(\mu_1+\mu_2+\lambda) + \lambda P_{n_1-1,n_2} + \alpha_{12}\mu_1 P_{n_1+1,n_2-1} + (1-\alpha_{12})\mu_1 P_{n_1+1,n_2} + \mu_2 P_{n_1,n_2+1}. \tag{2.34}$$

At steady state (i.e, when $t \to \infty$) $\dfrac{dP_{n_1,n_2(t)}}{dt} = 0$. Here, $P_n(t)$ is no longer a function of t. Equation (2.34) now becomes

$$P_{n_1,n_2}(\mu_1+\mu_2+\lambda) = \lambda P_{n_1-1,n_2} + \alpha_{12}\mu_1 P_{n_1+1,n_2-1} + (1-\alpha_{12})\mu_1 P_{n_1+1,n_2} + \mu_2 P_{n_1,n_2+1}. \tag{2.35}$$

Equation (2.35) is known as the steady-state difference equation or global balance equation.

Now, the boundary conditions for this model are as follows. When $n_1 = 0$, (2.35) becomes

$$(\mu_2+\lambda)P_{0,n_2} = \alpha_{12}\mu_1 P_{1,n_2-1} + (1-\alpha_{12})\mu_1 P_{1,n_2} + \mu_2 P_{0,n_2+1}. \tag{2.36}$$

The meaning of each of the quantities in (2.36) is as follows:

- $(\mu_2+\lambda)P_{0,n_2}$ is the probability of having no one at station 1 and n persons at station 2.
- $\alpha_{12}\mu_1 P_{1,n_2-1}$ is the probability that one person moved from station 1 to station 2 after been served.
- $(1-\alpha_{12})\mu_1 P_{1,n_2}$ is the probability that one person left the system from station 1.
- $\mu_2 P_{0,n_2+1}$ is the probability that one person left station two and there is no body in station 1.

When $n_2 = 0$, (2.35) becomes

$$(\mu_1+\lambda)P_{n_1,0} = \lambda P_{n_1-1,0} + (1-\alpha_{12})\mu_1 P_{n_1+1,0} + \mu_2 P_{n_1,1}. \tag{2.37}$$

The meaning of each of the quantities in (2.37) is as follows:

- $(\mu_1 + \lambda)P_{n_1,0}$ is the probability of having n persons at station one and none at station 2.
- $\lambda P_{n_1-1,0}$ is the probability that there is no one at station 2 and one person joined station 1.
- $(1 - \alpha_{12})\mu_1 P_{1,n_2}$ is the probability that one person left the system from station 1.
- $\mu_2 P_{n_1,1}$ is the probability that one person left station 2.

When $n_1 = 0, n_2 = 0$, (2.35) becomes

$$\lambda P_{0,0} = (1 - \alpha_{12})\mu_1 P_{1,0} + \mu_2 P_{0,1}. \tag{2.38}$$

Again, the meaning of each of the quantities in (2.38) is as follows:

- $\lambda P_{0,0}$ is the probability of having no customer at either of the two stations.
- $(1 - \alpha_{12})\mu_1 P_{1,0}$ is the probability that one person left the system from station 1.
- $\mu_2 P_{0,1}$ is the probability that one person left station 2.

Now, we further analyze the global balance equation for this model using probability generating function. We define the probability generating function as follows:

$$\begin{aligned}
G_{N_1,N_2}(z_1, z_2) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} \\
G_{N_1,0}(z_1) &= \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} \\
G_{0,N_2}(z_2) &= \sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} \\
G_{N_1,1}(z_1) &= \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,1} \\
G_{1,N_2}(z_2) &= \sum_{n_2=0}^{\infty} z_2^{n_2} P_{1,n_2}.
\end{aligned} \tag{2.39}$$

Now we simplify the left hand side of (2.35) in the generating function

$$(\mu_1+\mu_2+\lambda)\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} = (\mu_1+\mu_2+\lambda)\sum_{n_1=1}^{\infty} z_1^{n_1}\Bigg[\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - P_{n_1,0}\Bigg]$$

Let $\gamma = (\mu_1+\mu_2+\lambda)$. Hence,

$$\begin{aligned}
\gamma\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} &= \gamma\Bigg[\sum_{n_1=1}^{\infty} z_1^{n_1}\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - \sum_{n_1=1}^{\infty} z_1^{n_1} P_{n_1,0}\Bigg] \\
&= \gamma\Bigg[\sum_{n_1=1}^{\infty} z_1^{n_1}\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0}\Bigg] \\
&= \gamma\Bigg[\sum_{n_2=0}^{\infty} z_2^{n_2}\Bigg(\sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,n_2} - P_{0,n_2}\Bigg) - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0}\Bigg] \\
&= \gamma\Bigg[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} - \sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0}\Bigg] \\
&= \gamma\Bigg[G_{N_1,N_2}(z_1,z_2) - G_{0,N_2}(z_2) - G_{N_1,0}(z_1) + P_{0,0}\Bigg].
\end{aligned} \tag{2.40}$$

The first item on the RHS of (2.35) is

$$\begin{aligned}
\lambda\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1-1,n_2} &= \lambda z_1\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1-1} z_2^{n_2} P_{n_1-1,n_2} \\
&= \lambda z_1\sum_{n_1=1}^{\infty} z_1^{n_1-1}\Bigg[\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1-1,n_2} - P_{n_1-1,0}\Bigg] \\
&= \lambda z_1\Bigg[\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1-1} z_2^{n_2} P_{n_1-1,n_2} - \sum_{n_1=1}^{\infty} z_1^{n_1-1} P_{n_1-1,0}\Bigg] \\
&= \lambda z_1\Bigg[\sum_{n_2=0}^{\infty} z_2^{n_2}\sum_{n_1=1}^{\infty} z_1^{n_1-1} P_{n_1-1,n_2} - \sum_{n_1=1}^{\infty} z_1^{n_1-1} P_{n_1-1,0}\Bigg] \\
&= \lambda z_1\Bigg[\sum_{n_2=0}^{\infty} z_2^{n_2}\sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,n_2} - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0}\Bigg] \\
&= \lambda z_1\sum_{n_1=0}^{\infty} z_1^{n_1}\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - \lambda z_1\sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} \\
&= \lambda z_1\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} - \lambda z_1\sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} \\
&= \lambda z_1 G_{N_1,N_2}(z_1,z_2) - \lambda z_1 G_{N_1,0}(z_1).
\end{aligned} \tag{2.41}$$

The second item on the RHS of (2.35) is

$$
\begin{aligned}
\alpha_{12}\mu_1 P_{n_1+1,n_2-1} &= \alpha_{12}\mu_1 \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1+1,n_2-1} \\
&= \alpha_{12}\mu_1 z_1^{-1} z_2 \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1+1} z_2^{n_2-1} P_{n_1+1,n_2-1} \\
&= \alpha_{12}\mu_1 z_1^{-1} z_2 \sum_{n_2=0}^{\infty} z_2^{n_2} \sum_{n_1=1}^{\infty} z_1^{n_1+1} P_{n_1+1,n_2} \\
&= \alpha_{12}\mu_1 z_1^{-1} z_2 \sum_{n_2=0}^{\infty} z_2^{n_2} \Big(\sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,n_2} - P_{0,n_2} - z_1 P_{1,n_2} \Big) \\
&= \alpha_{12}\mu_1 z_1^{-1} z_2 \Big(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} - \sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} - z_1 \sum_{n_2=0}^{\infty} z_2^{n_2} P_{1,n_2} \Big) \\
&= \alpha_{12}\mu_1 z_1^{-1} z_2 G_{N_1,N_2}(z_1,z_2) - \alpha_{12}\mu_1 z_1^{-1} z_2 G_{0,N_2}(z_2) - \alpha_{12}\mu_1 z_2 G_{1,N_2}(z_2)
\end{aligned}
\tag{2.42}
$$

The third item on the RHS of (2.35) is

$$
\begin{aligned}
(1-\alpha_{12})\mu_1 P_{n_1+1,n_2} &= (1-\alpha_{12})\mu_1 \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1+1,n_2} \\
&= (1-\alpha_{12})\mu_1 z_1^{-1} \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1+1} z_2^{n_2} P_{n_1+1,n_2}
\end{aligned}
\tag{2.43}
$$

Let $\beta = (1-\alpha_{12})\mu_1 z_1^{-1}$. Then (2.43) becomes

$$
\begin{aligned}
(1-\alpha_{12})\mu_1 P_{n_1+1,n_2} &= \beta \sum_{n_1=1}^{\infty} z_1^{n_1+1} \Big[\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1+1,n_2} - P_{n_1+1,0} \Big] \\
&= \beta \Big[\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1+1} z_2^{n_2} P_{n_1+1,n_2} - \sum_{n_1=1}^{\infty} z_1^{n_1+1} P_{n_1+1,0} \Big] \\
&= \beta \Big[\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1+1} z_2^{n_2} P_{n_1+1,n_2} - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0} + z_1 P_{1,0} \Big] \\
&= \beta \Big[\sum_{n_2=0}^{\infty} z_2^{n_2} \Big(\sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,n_2} - P_{0,n_2} - z_1 P_{1,n_2} \Big) - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0} + z_1 P_{1,0} \Big] \\
&= \beta \Big[\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} - \sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} - z_1 \sum_{n_2=0}^{\infty} z_2^{n_2} P_{1,n_2} \\
&\qquad - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0} + z_1 P_{1,0} \Big]
\end{aligned}
$$

$$= \beta\Big[G_{N_1,N_2}(z_1,z_2) - G_{0,N_2}(z_2) - z_1 G_{1,N_2}(z_2) - G_{N_1,0}(z_1) + P_{0,0} + z_1 P_{1,0}.\Big] \tag{2.44}$$

The fourth item on the RHS of (2.35) is

$$\begin{aligned}
\mu_2 P_{n_1,n_2+1} &= \mu_2 z_2^{-1} \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} z_1^{n_1} z_2^{n_2+1} P_{n_1,n_2+1} \\
&= \mu_2 z_2^{-1}\Big[\sum_{n_1=1}^{\infty} z_1^{n_1}\Big(\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - P_{n_1,0} - z_2 P_{n_1,1}\Big)\Big] \\
&= \mu_2 z_2^{-1}\Big[\sum_{n_1=1}^{\infty} z_1^{n_1}\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - \sum_{n_1=1}^{\infty} z_1^{n_1} P_{n_1,0} - z_2 \sum_{n_1=1}^{\infty} z_1^{n_1} P_{n_1,1}\Big)\Big] \\
&= \mu_2 z_2^{-1}\Big[\sum_{n_1=1}^{\infty} z_1^{n_1}\sum_{n_2=0}^{\infty} z_2^{n_2} P_{n_1,n_2} - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} + P_{0,0} - z_2 \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,1} + z_2 P_{0,1}\Big] \\
&= \mu_2 z_2^{-1}\Big[\sum_{n_2=0}^{\infty} z_2^{n_2}\Big(\sum_{n_1=1}^{\infty} z_1^{n_1} P_{n_1,n_2} - P_{0,n_2}\Big) - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} \\
&\qquad + P_{0,0} - z_2 \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,1} + z_2 P_{0,1}\Big] \\
&= \mu_2 z_2^{-1}\Big[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} P_{n_1,n_2} - \sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} - \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,0} \\
&+ P_{0,0} - z_2 \sum_{n_1=0}^{\infty} z_1^{n_1} P_{n_1,1} + z_2 P_{0,1}\Big]
\end{aligned}$$

Hence $\mu_2 P_{n_1,n_2+1}$ equals

$$\mu_2 z_2^{-1}\Big[G_{N_1,N_2}(z_1,z_2) - G_{0,N_2}(z_2) - G_{N_1,0}(z_1) + P_{0,0} - z_2 G_{1,N_2}(z_2) + z_2 P_{0,1}\Big]. \tag{2.45}$$

From the boundary condition for the steady state equation (2.35) when $n_1 = 0$,

$$(\lambda+\mu_2)P_{0,n_2} = \alpha_{12}\mu_1 P_{1,n_2-1} + (1-\alpha_{12})\mu_1 P_{1,n_2} + \mu_2 P_{0,n_2+1}. \tag{2.46}$$

The left hand side of (2.46) can be expressed in terms of probability generating function as follows

$$\begin{aligned}
\sum_{n_2=1}^{\infty}(\lambda+\mu_2) z_2^{n_2} P_0, n_2 &= (\lambda+\mu_2)\sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} - (\lambda+\mu_2)P_{0,0} \\
&= (\lambda+\mu_2)G_{0,N_2}(z_2) - (\lambda+\mu_2 P_{0,0}).
\end{aligned} \tag{2.47}$$

The first item on the RHS of (2.46) is

$$\begin{aligned} \alpha_{12}\mu_1 P_1, n_{2-1} &= \alpha_{12}\mu_1 \sum_{n_2=1}^{\infty} z_2^{n_2} P_{1,n_2-1} \\ &= \alpha_{12}\mu_1 z_2 \sum_{n_2=1}^{\infty} z_2^{n_2-1} P_{1,n_2-1} \\ &= \alpha_{12}\mu_1 z_2 G_{1,N_2}(z_2). \end{aligned} \tag{2.48}$$

The second quantity on the RHS of (2.46) is

$$\begin{aligned} (1-\alpha_{12}\mu_1)P_1, n_2 &= (1-\alpha_{12})\mu_1 \sum_{n_2=1}^{\infty} z_2^{n_2} P_{1,n_2} \\ &= (1-\alpha_{12})\mu_1 \sum_{n_2=0}^{\infty} z_2^{n_2} P_{1,n_2} - (1-\alpha_{12}\mu_1 P_{1,0}) \\ &= (1-\alpha_{12})\mu_1 G_{1,N_2}(z_2) - (1-\alpha_{12}\mu_1 P_{1,0}). \end{aligned} \tag{2.49}$$

The third quantity on the RHS of (2.46) is

$$\begin{aligned} \mu_2 z_2^{n_2} P_{0,n_2+1} &= \mu_2 z_2^{-1} \sum_{n_2=1}^{\infty} z_2^{n_2+1} P_{0,n_2+1} \\ &= \mu_2 z_2^{-1} \sum_{n_2=0}^{\infty} z_2^{n_2} P_{0,n_2} - \mu_2 z_2^{-1} P_{0,0} - \mu_2 P_{0,1} \\ &= \mu_2 z_2^{-1} G_{0,N_2}(z_2) - \mu_2 z_2^{-1} P_{0,0} - \mu_2 P_{0,1}. \end{aligned} \tag{2.50}$$

Combining (2.47), (2.48) , (2.49), (2.50) we have the marginal probability

$$\begin{aligned} (\lambda+\mu_2)G_{0,N_2}(z_2) = (\lambda+\mu_2)P_{0,0} + \alpha_{12}\mu_1 z_2 G_{1,N_2}(z_2) + (1-\alpha_{12})\mu_1 G_1 1, N_2(z_2) \\ -(1-\alpha_{12})\mu_1 P_{1,0} + \mu_2 z^{-1} G_{0,N_2}(z_2) - \mu_2 z^{-1} P_{0,0} - \mu_2 P_{0,1}. \end{aligned} \tag{2.51}$$

We solve the boundary condition when $n_2 = 0$ of the steady state equation (2.37) using probability generating function method.

The left hand side of (2.37) becomes

$$\sum_{n=1}^{\infty}(\mu_1+\lambda)z_1^{n_1}P_{n_1,0} = (\mu_1+\lambda)\sum_{n_1=0}^{\infty} z_1^{n_1}P_{n_1,0} - (\mu_1+\lambda)P_{0,0}$$
$$= (\mu_1+\lambda)G_{N_1,0}(z_1) - (\mu_1+\lambda)P_{0,0}. \tag{2.52}$$

The first quantity on the right hand side of (2.37)

$$\sum_{n=1}^{\infty}\lambda z_1^{n_1}P_{n_1-1,0} = \lambda z_1\sum_{n_1=0}^{\infty} z_1^{n_1-1}P_{n_1-1,0} = \lambda z_1\sum_{n_1=0}^{\infty} z_1^{n_1}P_{n_1,0}$$
$$= \lambda z_1 G_{N_1,0}(z_1). \tag{2.53}$$

The second quantity on the right hand side of (2.37)

$$(1-\alpha_{12})\mu_1 P_{1,0} = (1-\alpha_{12})\mu_1 z_1^{-1}\sum_{n_1=1}^{\infty} z_1^{n_1+1}P_{n_1+1,0}$$
$$= (1-\alpha_{12})\mu_1 z_1^{-1}\sum_{n_1=0}^{\infty} z_1^{n_1}P_{n_1,0} - (1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0} - (1-\alpha_{12}\mu_1)P_{1,0} \tag{2.54}$$
$$= (1-\alpha_{12})\mu_1 z_1^{-1}G_{n_1,0}(z_1) - (1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0} - (1-\alpha_{12}\mu_1)P_{1,0}.$$

The third quantity on the right hand side of (2.37) is

$$\mu_2 P_{n_1,1} = \mu_2\sum_{n_1=0}^{\infty} z_1^{n_1}P_{n_1,1} - \mu_2 P_{0,1}$$
$$= \mu_2 G_{N_1,1}(z_1) - \mu_2 P_{0,1}. \tag{2.55}$$

Now, combining equation (2.52), (2.53), (2.54), (2.55) we have the marginal generating function

$$(\lambda+\mu_1)G_{N_1,0}(z_1) = (\lambda+\mu_1)P_{0,0} + \lambda z_1 G_{N_1,0}(z_1) + (1-\alpha_{12})\mu_1 z_1^{-1}G_{N_1,0}(z_1)$$
$$- (1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0} - (1-\alpha_{12})\mu_1 P_{1,0} + \mu_2 G_{N_1,0}(z_1) - \mu_2 P_{0,1}. \tag{2.56}$$

Finally, we combine equation (2.40) through equation (2.45) to obtain the probability generating function for this model.

$$
\begin{aligned}
&(\mu_1+\mu_2+\lambda z_1-\alpha_{12}\mu_1 z_1^{-1}z_2-(1-\alpha_{12})\mu_1 z_1^{-1}-\mu_2 z_2^{-1})G_{N_1,N_2}(z_1,z_2)\\
&\qquad=(\lambda+\mu_2)G_{0,N_2}(z_2)+\mu_1 G_{0,N_2}(z_2)+(\lambda+\mu_1)G_{n_1,0}(z_1)+\mu_2 G_{N_1,0}(z_1)\\
&\qquad-\lambda P_{0,0}-\mu_1 P_{0,0}-\mu_2 P_{0,0}-\lambda z_1 G_{N_1,0}(z_1)-\alpha_{12}\mu_1 z_1^{-1}z_2 G_{0,N_2}(z_2)\\
&\qquad-\alpha_{12}\mu_1 z_2 G_{1,N_2}-(1-\alpha_{12})\mu_1 z_1^{-1}G_{0,N_2}(z_2)-(1-\alpha_{12})\mu_1 G_{1.N_2}(z_2)\\
&\qquad-(1-\alpha_{12})\mu_1 z_1^{-1}G_{N_1,0}(z_1)+(1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0}\\
&\qquad+(1-\alpha_{12}\mu_1 P_{1,0}-\mu_2 z_2^{-1}G_{0,N_2}(z_2)-\mu_2 z_2^{-1}G_{N_1,0}(z_1)+\mu_2 z_2^{-1}P_{0,0}\\
&\qquad+\mu_2 G_{N_1,1}(z_1)+\mu_2 P_{0,1}.
\end{aligned}
$$

Substituting equation (2.51) for $(\lambda+\mu_1)G_{N_1,0}(z_1)$ and (2.56) for $(\lambda+\mu_2)G_{0,N_2}(z_2)$ in the above equation, we have

$$
\begin{aligned}
&(\mu_1+\mu_2+\lambda z_1-\alpha_{12}\mu_1 z_1^{-1}z_2-(1-\alpha_{12})\mu_1 z_1^{-1}-\mu_2 z_2^{-1})G_{N_1,N_2}(z_1,z_2)\\
&\qquad=(\lambda+\mu_2)P_{0,0}+\alpha_{12}\mu_1 z_2 G_{1,N_2}(z_2)+(1-\alpha_{12})\mu_1 G_{1,N_2}(z_2)-(1-\alpha_{12})\mu_1 P_{1,0}\\
&\qquad+\mu_2 z^{-1}G_{0,N_2}(z_2)-\mu_2 z^{-1}P_{0,0}-\mu_2 P_{0,1}+\mu_2 G_{N_1,0}(z_1)+(\lambda+\mu_1)P_{0,0}\\
&\qquad+\lambda z_1 G_{N_1,0}(z_1)+(1-\alpha_{12})\mu_1 z_1^{-1}G_{N_1,0}(z_1)-(1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0}\\
&\qquad-(1-\alpha_{12})\mu_1 P_{1,0}+\mu_2 G_{N_1,0}(z_1)-\mu_2 P_{0,1}+(\lambda+\mu_1)P_{0,0}+\lambda z_1 G_{N_1,0}(z_1)\\
&\qquad+(1-\alpha_{12})\mu_1 z_1^{-1}G_{N_1,0}(z_1)-(1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0}-(1-\alpha_{12})\mu_1 P_{1,0}\\
&\qquad+\mu_2 G_{N_1,0}(z_1)-\mu_2 P_{0,1}+\mu_2 G_{N_1,0}(z_1)-\lambda P_{0,0}-\mu_1 P_{0,0}-\mu_2 P_{0,0}\\
&\qquad-\lambda z_1 G_{N_1,0}(z_1)-\alpha_{12}\mu_1 z_1^{-1}z_2 G_{0,N_2}(z_2)-\alpha_{12}\mu_1 z_2 G_{1,N_2}-(1-\alpha_{12})\mu_1 z_1^{-1}G_{0,N_2}(z_2)\\
&\qquad-(1-\alpha_{12})\mu_1 G_{1.N_2}(z_2)-(1-\alpha_{12})\mu_1 z_1^{-1}G_{N_1,0}(z_1)+(1-\alpha_{12})\mu_1 z_1^{-1}P_{0,0}\\
&\qquad+(1-\alpha_{12})\mu_1 P_{1,0}-\mu_2 z_2^{-1}G_{0,N_2}(z_2)-\mu_2 z_2^{-1}G_{N_1,0}(z_1)\\
&\qquad+\mu_2 z_2^{-1}P_{0,0}+\mu_2 G_{N_1,1}(z_1)+\mu_2 P_{0,1}.
\end{aligned}
$$

Further simplification of the above now yields

$$G_{N_1,N_2}(z_1,z_2) = \frac{\mu_1\Big[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_2z_1^{-1})\Big]G_{0,N_2}(z_2)+\mu_2(1-z_2^{-1})G_{N_1,0}(z_1)}{\lambda(1-z_1)+\mu_1\Big[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_2z_1^{-1})\Big]+\mu_2(1-z_2^{-1})}. \tag{2.57}$$

By definition, we require $G_{N_1,N_2}(z_1=1,z_2=1)=\sum_{n=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}=1$.

Marginal Probability Generating Function

We obtain the first marginal probability generating function for this model by setting $z_2 = 1$ in (2.57) to have

$$G_{N_1}(z_1) = \frac{\mu_1\big[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_1^{-1})\big]G_{0,N_2}(1)}{\lambda(1-z_1)+\mu_1\big[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_1^{-1})\big]}$$

$$= \frac{\mu_1\big[(1-\alpha_{12})(z_1-1)+\alpha_{12}(z_1-1)\big]G_{0,N_2}(1)}{z_1\lambda(1-z_1)+\mu_1\big[(1-\alpha_{12})(z_1-1)+\alpha_{12}(z_1-1)\big]}.$$

Therefore,

$$G_{N_1}(z_1) = \frac{\mu_1(z_1-1)\big[1-\alpha_{12}+\alpha_{12}\big]G_{0,N_2}(1)}{(z_1-1)\big[\mu_1\big[1-\alpha_{12}+\alpha_{12}\big]-\lambda z_1\big]}$$

$$= \frac{\mu_1 G_{0,N_2}(1)}{\mu_1-\lambda z_1}.$$

But we know that $G_{N_1}(z_1=1)=1$ by definition, which implies that

$$\frac{\mu_1}{\mu_1-\lambda}G_{0,N_2} = 1 \tag{2.58}$$

$$G_{0,N_2} = 1-\frac{\lambda}{\mu_1} = 1-\rho. \tag{2.59}$$

This is the probability that station one is empty and there are n_2 people at station two.

Now we obtain the marginal probability generating function of N_2 by setting $z_1 = 1$ in equation (2.57)

$$G_{N_2}(z_2) = \frac{\mu_1\alpha_{12}(1-z_2)G_{0,N_2}(z_2)+\mu_2(1-z_2^{-1})G_{N_1,0}(1)}{\mu_1\alpha_{12}(1-z_2)+\mu_2(1-z_2^{-1})}$$

$$G_{N_2}(z_2) = \frac{-\mu_1\alpha_{12}G_{0,N_2}(z_2) + \mu_2 G_{N_1,0}(1)}{\mu_2 - \mu_1\alpha_{12}z_2}$$

By setting $z_2 = 1$ in the above equation, we have

$$\frac{-\mu_1\alpha_{12}G_{0,N_2}(z_2) + \mu_2 G_{N_1,0}(1)}{\mu_2 - \mu_1\alpha_{12}z_2} = 1,$$

which is equivalent to

$$\mu_2 G_{N_1,0}(1) = \mu_2 - \mu_1\alpha_{12} + \mu_1\alpha_{12}G_{0,N_2}(1). \tag{2.60}$$

Substituting $1 - \frac{\lambda}{\mu_1}$ for $G_{0,N_2}(1)$ in (2.60) we have

$$\mu_2 G_{N_1,0}(1) = \mu_2 - \mu_1\alpha_{12} + \mu_1\alpha_{12}\left(\frac{\mu_1 - \lambda}{\mu_1}\right) \tag{2.61}$$

$$\mu_2 G_{N_1,0}(1) = \mu_2 - \alpha_{12}\lambda. \tag{2.62}$$

Thus

$$G_{N_1,0}(1) = 1 - \alpha_{12}\frac{\lambda}{\mu_2}. \tag{2.63}$$

Now, we obtain the probability generating function for this model by substituting (2.59) and (2.63) into (2.57), to have

$$G_{N_1,N_2}(z_1, z_2) = \frac{\mu_1\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2z_1^{-1})\right]\frac{\mu_1 - \lambda}{\mu_1} + \mu_2(1-z_2^{-1})\frac{\mu_2 - \alpha_{12}\lambda}{\mu_2}}{\lambda(1-z_1) + \mu_1\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2z_1^{-1})\right] + \mu_2(1-z_2^{-1})}, \tag{2.64}$$

which reduces to

$$G_{N_1,N_2}(z_1, z_2) = \frac{\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2z_1^{-1})\right](\mu_1 - \lambda) + (1-z_2^{-1})(\mu_2 - \alpha_{12}\lambda)}{\lambda(1-z_1) + \mu_1\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2z_1^{-1})\right] + \mu_2(1-z_2^{-1})}. \tag{2.65}$$

CHAPTER 3

MODEL II

3.1 Tandem Queue With Two Porous Service Stations

In this chapter, we model the probability of having n_1, n_2 units in station 1 and 2 respectively at time $t + \triangle t$. Allowing arrival to and departure from both stations as seen in figure 6.

$$
\begin{aligned}
P_{n_1,n_2}(t+\triangle t) &= P_{n_1,n_2}(t)(1-\lambda_1\triangle t)(1-\alpha_{12}\mu_1\triangle t)(1-\mu_1(1-\alpha_{12}))(1-\mu_2\triangle t)(1-\lambda_2)\\
&+ P_{n_1-1,n_2}(t)\lambda_1\triangle t(1-\alpha_{12}\mu_1\triangle t)(1-\mu_1(1-\alpha_{12}))(1-\lambda_2)(1-\mu_2\triangle t)\\
&+ \alpha_{12}\mu_1\triangle t P_{n_1+1,n_2-1}(t)(1-\lambda_1\triangle t)(1-\mu_2\triangle t)\\
&+ (1-\alpha_{12})\mu_1\triangle t P_{n_1+1,n_2}(t)(1-\lambda_1\triangle t)(1-\mu_2\triangle t)\\
&+ P_{n_1,n_2+1}(t)(1-\lambda_1\triangle t)(1-\alpha_{12}\mu_1\triangle t)(1-\mu_1(1-\alpha_{12}))(1-\lambda_2\triangle t)\mu_2\triangle t\\
&+ P_{n_1,n_2-1}(t)(1-\lambda_1\triangle t)(1-\alpha_{12}\mu_1\triangle t)(1-\mu_1(1-\alpha_{12}))\lambda_2\triangle t(1-\mu_2\triangle t).
\end{aligned}
$$

From the above equation, the probability that there are n_1, n_2 units in station 1 and 2 respectively at time $t + \triangle t$ is equal to the probabilities of the following mutually exclusive events, namely;

(1) n_1, n_2 units in the system at time t and no arrival or service during $\triangle t$ in both stations.

(2) $n_1 - 1, n_2$ units in the system at time t, one arrival and no service at station 1, no arrival and no service at station 2 during $\triangle t$.

(3) $n_1 + 1, n_2 - 1$ units in the system at time t, no arrival and one service at station 1, one arrival and no service at station 2 during $\triangle t$.

(4) $n_1 + 1, n_2$ units in the system at time t, no arrival and one service at station 1, no arrival and no service at station 2 during $\triangle t$.

(5) $n_1, n_2 + 1$ units in the system at time t, no arrival and no service at station 1, no arrival and one service at station 2 during $\triangle t$.

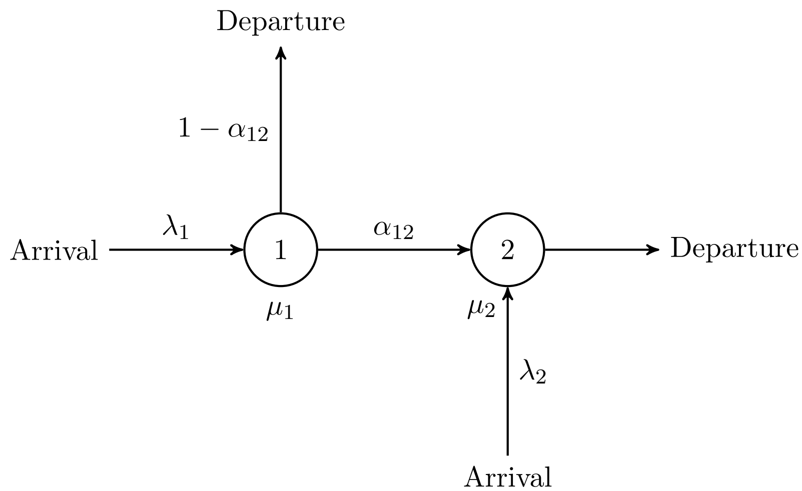

Figure 6: Tandem Queue with 2 Porous Stations.

(6) $n_1, n_2 - 1$ units in the system at time t, no arrival at station 1 and one arrival at station 2.

$$\begin{aligned} P_{n_1,n_2}(t+\triangle t) &= P_{n_1,n_2}(1-\mu_1-\mu_2-\lambda_1-\lambda_2)\triangle t + \lambda_1 \triangle t P_{n_1-1,n_2} + \alpha_{12}\mu_1 \triangle t P_{n_1+1,n_2-1} \\ &\quad + (1-\alpha_{12})\mu_1 \triangle t P_{n_1+1,n_2} + \mu_2 \triangle t P_{n_1,n_2+1} + \lambda_2 \triangle t P_{n_1,n_2-1}(t). \end{aligned} \tag{3.1}$$

$$\begin{aligned} \frac{P_{n_1,n_2}(t+\triangle t) - P_{n_1,n_2}(t)}{\triangle t} &= -P_{n_1,n_2}(\mu_1+\mu_2+\lambda_1+\lambda_2) + \lambda_1 P_{n_1-1,n_2} + \alpha_{12}\mu_1 P_{n_1+1,n_2-1} \\ &\quad + (1-\alpha_{12})\mu_1 P_{n_1+1,n_2} + \mu_2 P_{n_1,n_2+1} + \lambda_2 P_{n_1,n_2-1}. \end{aligned} \tag{3.2}$$

At steady state (i.e, when $t \to \infty$) $\frac{dP_{n_1,n_2}}{dt} = 0$. Here $P_{n_1,n_2}(t)$ is no longer a function of t. Equation (3.2) becomes

$$\begin{aligned} P_{n_1,n_2}(\mu_1+\mu_2+\lambda_1+\lambda_2) &= \lambda_1 P_{n_1-1,n_2} + \alpha_{12}\mu_1 P_{n_1+1,n_2-1} + (1-\alpha_{12})\mu_1 P_{n_1+1,n_2} \\ &\quad + \mu_2 P_{n_1,n_2+1} + \lambda_2 P_{n_1,n_2-1}. \end{aligned} \tag{3.3}$$

Equation (3.3) is the steady-state difference or global balance equation for this model.

Now, the boundary conditions for this model are as follows;

When $n_1 = 0$, (3.3) becomes

$$(\mu_2+\lambda_1+\lambda_2)P_{0,n_2} = \alpha_{12}\mu_1 P_{1,n_2-1} + (1-\alpha_{12})\mu_1 P_{1,n_2} + \mu_2 P_{0,n_2+1} + \lambda_2 P_{0,n_2-1}. \tag{3.4}$$

The meaning of each of the quantities in (3.4) is as follows:

- $\alpha_{12}\mu_1 P_{1,n_2-1}$ is the probability that one person moved from station 1 to station 2 after been served.
- $(1-\alpha_{12})\mu_1 P_{1,n_2}$ is the probability that one person left the system from station 1.
- $\mu_2 P_{0,n_2+1}$ is the probability that one person left station 2 and there is no body in station 1.
- $\lambda_2 P_{0,n_2-1}$ is the probability that no body is at station 1, and one person entered the system at station 2.

When $n_2 = 0$, equation (3.3) becomes

$$(\mu_1 + \lambda_1 + \lambda_2)P_{n_1,0} = \lambda_1 P_{n_1-1,0} + (1-\alpha_{12})\mu_1 P_{n_1+1,0} + \mu_2 P_{n_1,1}. \tag{3.5}$$

When $n_1 = 0, n_2 = 0$, (3.3) becomes

$$(\lambda_1 + \lambda_2)P_{0,0} = (1-\alpha_{12})\mu_1 P_{1,0} + \mu_2 P_{0,1}. \tag{3.6}$$

Now we analyze the steady state equation for this model (3.3) using probability generating function as we did in chapter two, we have

$$G_{N_1,N_2}(z_1,z_2) = \frac{\mu_1\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2 z_1^{-1})\right]G_{0,N_2}(z_2) + \mu_2(1-z_2^{-1})G_{N_1,0}(z_1)}{\lambda_1(1-z_1) + \lambda_2(1-z_2) + \mu_1\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2 z_1^{-1})\right] + \mu_2(1-z_2^{-1})}. \tag{3.7}$$

Marginal Probability Generating Function

We obtain the first marginal probability generating function by setting $z_2 = 1$ in (3.7),

$$
\begin{aligned}
G_{N_1}(z_1) &= \frac{\mu_1\big[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_1^{-1})\big]G_{0,N_2}(1)}{\lambda(1-z_1) + \mu_1\big[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_1^{-1})\big]} \\
&= \frac{\mu_1\big[(1-\alpha_{12})(z_1-1) + \alpha_{12}(z_1-1)\big]G_{0,N_2}(1)}{z_1\lambda_1(1-z_1) + \mu_1\big[(1-\alpha_{12})(z_1-1) + \alpha_{12}(z_1-1)\big]} \\
&= \frac{\mu_1(z_1-1)G_{0,N_2}(1)}{z_1\lambda_1(1-z_1) + \mu_1(z_1-1)} \\
&= \frac{\mu_1 G_{0,N_2}(1)}{\mu_1 - z_1\lambda_1}
\end{aligned}
$$

which implies that

$$\frac{\mu_1}{\mu_1-\lambda_1}G_{0,N_2}(1) = 1 \tag{3.8}$$

$$G_{0,N_2}(1) = 1 - \frac{\lambda_1}{\mu_1} = 1 - \rho_1. \tag{3.9}$$

This is the probability that station one is empty and there are n_2 people at station 2. Equation (3.9) can be written as $P(N_1 = 0, N_2 = n_2)$.

Now we obtain the second marginal probability generating function by setting $z_1 = 1$ in equation (3.7)

$$G_{N_2}(z_2) = \frac{\mu_1\alpha_{12}(1-z_2)G_{0,N_2}(z_2) + \mu_2(1-z_2^{-1})G_{N_1,0}(z_1)}{(1-z_2)\lambda_2 + \mu_1\alpha_{12}(1-z_2) + \mu_2(1-z_2^{-1})}$$

$$G_{N_2}(z_2) = \frac{-\mu_1 z_2\alpha_{12}(z_2-1)G_{0,N_2}(z_2) + \mu_2(z_2-1)G_{N_1,0}(z_1)}{\mu_2(z_2-1) - \mu_1\alpha_{12}z_2(z_2-1) - z_2\lambda_2(z_2-1)}$$

$$G_{N_2}(z_2 = 1) = \frac{\mu_2 G_{N_1,0}(1) - \mu_1\alpha_{12}G_{0,N_2}(1)}{\mu_2 - \mu_1\alpha_{12} - \lambda_2} = 1.$$

This implies that

$$\mu_2 G_{N_1,0}(1) = \mu_2 - \mu_1\alpha_{12} - \lambda_2 + \mu_1\alpha_{12}G_{0,N_2}(1). \tag{3.10}$$

Substituting $1 - \frac{\lambda_1}{\mu_1}$ for $G_{0,N_2}(1)$ in (3.10) we have

$$\mu_2 G_{N_1,0}(1) = \mu_2 - \mu_1\alpha_{12} - \lambda_2 + \mu_1\alpha_{12}\left(1 - \frac{\lambda_1}{\mu_1}\right) \tag{3.11}$$

$$\mu_2 G_{N_1,0}(1) = \mu_2 - \mu_1\alpha_{12} - \lambda_2 + \mu_1\alpha_{12} - \lambda_1\alpha_{12}. \tag{3.12}$$

Thus

$$G_{N_1,0}(1) = 1 - \frac{\lambda_1\alpha_{12} + \lambda_2}{\mu_2} = 1 - \rho_2. \tag{3.13}$$

Equation (3.13) is the probability of having n_1 in station 1 and station 2 is empty. It can also be written as $P(N_2 = 0, N_1 = n_1)$. Hence the probability generating function of this model becomes.

$$G_{N_1,N_2}(z_1,z_2) = \frac{\mu_1\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2z_1^{-1})\right]\frac{\mu_1-\lambda_1}{\mu_1} + \mu_2(1-z_2^{-1})\frac{\mu_2-\lambda_1\alpha_{12}+\lambda_2}{\mu_2}}{\lambda_1(1-z_1)+\lambda_2(1-z_2)+\mu_1\left[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_2z_1^{-1})\right]+\mu_2(1-z_2^{-1})}, \tag{3.14}$$

which finally becomes

$$G_{N_1,N_2}(z_1,z_2) = \frac{\left[(1-\alpha_{12})(1-z_1^{-1}) + \alpha_{12}(1-z_2z_1^{-1})(\mu_1-\lambda)\right] + (1-z_2^{-1})(\mu_2-\alpha_{12}\lambda_1+\lambda_2)}{\lambda_1(1-z_1)+\lambda_2(1-z_2)+\mu_1\left[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_2z_1^{-1})\right]+\mu_2(1-z_2^{-1})}. \tag{3.15}$$

We may now obtain the steady state probability $P(N_1 = n_1, N_2 = n_2)$ from equation (3.15) by the usual approach. That is,

$$P(N_1 = n_1, N_2 = n_2) = P_{n_1,n_2} = \frac{G^{(n_1+n_2)}}{n_1!n_2!}|_{z_1=0=z_2}.$$

3.1.1 Statistical Properties of A Tandem Queue With Two Service Stations

The expected number of people in the system at steady state for model with two service stations can be obtained as follows:

$$E(N_1N_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 n_2 P_{n_1,n_2}. \tag{3.16}$$

Because of the Markov property of the two queues, the joint probability

$$P(N_1 = n_1, N_2 = n_2) = P_{n_1,n_2} = P(N_1 = n_1)\cdot P(N_2 = n_2) = P_{n_1}\cdot P_{n_2}.$$

From equation (2.22), we have $P_{n_1} = \rho_1^{n_1}(1-\rho_1) = \rho_1^{n_1} G_{0,N_2}(1)$, and
$P_{n_2} = \rho_2^{n_2}(1-\rho_2) = \rho_2^{n_2} G_{N_1,0}(1)$,

Thus, equation (3.16) can be written as

$$E(N_1N_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 n_2 \rho_1^{n_1} G_{0,N_2}(1)\rho_2^{n_2} G_{N_1,0}(1). \tag{3.17}$$

From equation (3.9) and (3.13) $G_{0,N_2}(1) = 1-\rho_1$ and $G_{N_1,0}(1) = 1-\rho_2$. Where $G_{0,N_2}(1)$ is the marginal probability of having no customer in station 1 and n_2 customers in station 2 and $G_{N_1,0}(1)$ is the marginal probability of having n_1 customers in station 1 and none in station 2. Thus, equation (3.17) becomes

$$\begin{aligned} E(N_1N_2) &= (1-\rho_1)(1-\rho_2)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 n_2 \rho_1^{n_1}\rho_2^{n_2} \\ &= (1-\rho_1)(1-\rho_2)\sum_{n_1=0}^{\infty} n_1\rho_1^{n_1}\sum_{n_2=0}^{\infty} n_2\rho_2^{n_2} \\ &= (1-\rho_1)(1-\rho_2)\rho_1\rho_2\big[\sum_{n_1=1}^{\infty} n_1\rho_1^{n_1-1}\sum_{n_2=1}^{\infty} n_2\rho_2^{n_2-1}\big] \end{aligned} \tag{3.18}$$

Note that $\sum_{n_1=1}^{\infty} n_1\rho_1^{n_1-1}$ and $\sum_{n_2=1}^{\infty} n_1\rho_1^{n_2-1}$ are the derivatives of $\sum_{n_1=0}^{\infty}\rho_1^{n_1}$ and $\sum_{n_2=0}^{\infty}\rho_2^{n_2}$

with respect to ρ_1 and ρ_2 respectively. Thus, equation (3.18) becomes

$$E(N_1N_2) = (1-\rho_1)(1-\rho_2)\rho_1\rho_2\big[\frac{d}{d\rho_1}\sum_{n_1=0}^{\infty}\rho_1^{n_1}\frac{d}{d\rho_2}\sum_{n_2=0}^{\infty}\rho_2^{n_2}\big], \tag{3.19}$$

we know that $\sum_{n_1=0}^{\infty}\rho_2^{n_1} = \frac{1}{1-\rho_1}$ and $\sum_{n_2=0}^{\infty}\rho_2^{n_2} = \frac{1}{1-\rho_2}$. Hence, (3.19) becomes

$$\begin{aligned} E(N_1N_2) &= (1-\rho_1)(1-\rho_2)\rho_1\rho_2\big[\tfrac{d}{d\rho_1}(\tfrac{1}{1-\rho_1})\cdot\tfrac{d}{d\rho_2}(\tfrac{1}{1-\rho_2})\big] \\ &= (1-\rho_1)(1-\rho_2)\rho_1\rho_2\big[\tfrac{1}{(1-\rho_1)^2}\cdot\tfrac{1}{(1-\rho_2)^2}\big] \\ &= \frac{\rho_1\rho_2}{(1-\rho_1)(1-\rho_2)}. \end{aligned} \tag{3.20}$$

Covariance

Covariance is the measure of linear dependence between N_1 and N_2 (number of customers in station 1 and station 2), is defined

$$\begin{aligned} Cov(N_1, N_2) &= E\big[(N_1 - E(N_1))\cdot(N_2 - E(N_2))\big] \\ &= E(N_1N_2) - E(N_1)E(N_2). \end{aligned} \tag{3.21}$$

Recall from equation (2.29) that $E(N_1) = \dfrac{\rho_1}{(1-\rho_1)}$, similarly $E(N_2) = \dfrac{\rho_2}{(1-\rho_2)}$. Thus,

$$Cov(N_1, N_2) = \frac{\rho_1\rho_2}{(1-\rho_1)(1-\rho_2)} - \left(\frac{\rho_1}{1-\rho_1}\right)\cdot\left(\frac{\rho_2}{1-\rho_2}\right) = 0. \tag{3.22}$$

It implies that there is no linear dependence between the number of customers in station 1 and station 2.

3.1.2 Generalization

From figure 7, note that the proportion of customers or units that enters the system at a particular service station must be equal to the proportion that moved to the next station plus those that leave the system from that service station. For instance, for station 1, the rate of moving into the system is λ_1 which equals to the sum of the rate $(1-\alpha_{12})\lambda_1$ withdrawing from

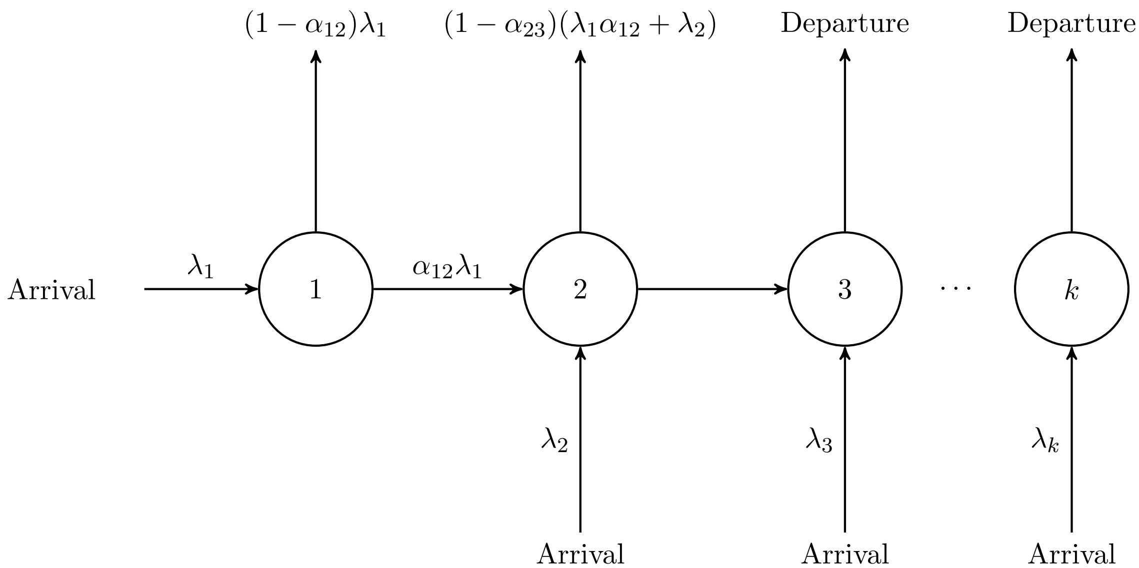

Figure 7: Tandem Queue With k Service Stations.

station 1, and proportion $\alpha_{12}\lambda_1$ that moves to service station 2 from station 1. Also, for station 2, the rate of moving into the system is $\alpha_{12}\lambda_1 + \lambda_2$ which equals to the proportion that moved out of the system at this station $(1-\alpha_{23})(\lambda_1\alpha_{23}+\lambda_2)$ and the proportion that moved to station 3 $\alpha_{23}(\alpha_{12}\lambda_1+\lambda_2)$. For M/M/1 with one service station, we obtained the probability generating function as

$$G(z) = \frac{\mu(1-z^{-1})P_0}{\lambda(1-z)+\mu(1-z^{-1})}.$$

Also for M/M/1 tandem queue with two service stations, we obtain the PGF as

$$G_{N_1,N_2}(z_1,z_2) = \frac{\mu_1\left[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_2z_1^{-1})\right]G_{0,N_2}(z_2)+\mu_2(1-z_2^{-1})G_{N_1,0}(z_1)}{\lambda_1(1-z_1)+\lambda_2(1-z_2)+\mu_1\left[(1-\alpha_{12})(1-z_1^{-1})+\alpha_{12}(1-z_2z_1^{-1})\right]+\mu_2(1-z_2^{-1})}.$$

For three service stations, the numerator of the probability generating function $G_{N_1,N_2,N_3}(z_1,z_2,z_3)$ is

$$\mu_1\left[(1-\alpha_{12})(1-z_1^{-1})+(1-\alpha_{23})(1-z_2^{-1})+\alpha_{12}(1-z_2z_1^{-1})+\alpha_{23}(1-z_3z_1^{-1})\right]G_{0,N_2,N_3}(z_2,z_3)$$
$$+\mu_2\left[(1-\alpha_{23})(1-z_2^{-1})+\alpha_{23}(1-z_3z_2^{-1})\right]G_{N_1,0,N_3}(z_1,z_3)+\mu_3(1-z_3^{-1})G_{N_1,N_2,0}(z_1,z_2),$$

Arrival Rate	State
$\alpha_{12}\lambda_1$	$Q_1 \rightarrow Q_2$
$\alpha_{23}(\alpha_{12}\lambda_1+\lambda_2)=\alpha_{12}\alpha_{23}\lambda_1+\alpha_{23}\lambda_2$	$Q_2 \rightarrow Q_3$
$\alpha_{12}\alpha_{23}\alpha_{34}\lambda_1+\alpha_{23}\alpha_{34}\lambda_2 + \alpha_{34}\lambda_3$	$Q_3 \rightarrow Q_4$
$\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45}\lambda_1+\alpha_{23}\alpha_{34}\alpha_{45}\lambda_2 + \alpha_{34}\alpha_{45}\lambda_3 + \alpha_{45}\lambda_4$	$Q_4 \rightarrow Q_5$
$\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45}\alpha_{56}\lambda_1+\alpha_{23}\alpha_{34}\alpha_{45}\alpha_{56}\lambda_2 + \alpha_{34}\alpha_{45}\alpha_{56}\lambda_3 + \alpha_{45}\alpha_{56}\lambda_4 + \alpha_{56}\lambda_5$	$Q_5 \rightarrow Q_6$
$\vdots$	$\vdots$
$\sum_{j=1}^{k-1} \lambda_j \prod_{i=j}^{k-1} \alpha_{i,i+1}$	$Q_{k-1} \rightarrow Q_k$

Table 1: Flow Rate Within Service Stations.

Queue	Traffic Intensity
Q_1	$\frac{\lambda_1}{\mu_1}$
Q_2	$\frac{\lambda_1\alpha_{12} + \lambda_2}{\mu_2}$
Q_3	$\frac{\lambda_1\alpha_{12}\alpha_{23} + \lambda_2\alpha_{23} + \lambda_3}{\mu_3}$
Q_4	$\frac{\lambda_1\alpha_{12}\alpha_{23}\alpha_{34} + \lambda_2\alpha_{23}\alpha_{34} + \lambda_3\alpha_{34} + \lambda_4}{\mu_4}$
$\vdots$	$\vdots$
Q_k	$\frac{\sum_{j=1}^{k-1} \lambda_j \prod_{i=j}^{k-1} \alpha_{i,i+1} + \lambda_k}{\mu_k}$

Table 2: Generalized Traffic Intensity.

and the denominator is

$$\begin{aligned}
&\lambda_1(1-z_1)+\lambda_2(1-z_2)+\lambda_3(1-z_3)\\
&+\mu_1\Big[(1-\alpha_{12})(1-z_1^{-1})+(1-\alpha_23)(1-z_2^{-1})+\alpha_{12}(1-z_2z_1^{-1})+\alpha_{23}(1-z_3z_1^{-1})\Big]\\
&+\mu_2\Big[(1-\alpha_{23})(1-z_2^{-1})+\alpha_{23}(1-z_3z_2^{-1})\Big]\\
&+\mu_3(1-z_3^{-1}).
\end{aligned}$$

The generalization of this PGF $G_{N_1,N_2,\dots,N_k}(z_1,z_2,...,z_k)$ is analytically demanding.

CHAPTER 4

THE FLOW CAPACITY, SAMPLE PATH AND RESIDENCE TIME IN QUEUE

The dynamics of the random arrivals and memoryless services in each of the queues in tandem giving rise into the flow capacity, the sample path and the residence time are discussed in this chapter.

We simulate a system of three queues in tandem at various time epochs. Figures 8 and figure 11 show the cumulative number of arrivals and departures in the network. The top (blue) line shows the cumulative number of arrivals and the bottom (red) line shows the cumulative number of departures from the system. The vertical distance between the two lines gives the number of customers present in the system at each time epoch, while the horizontal distance between the lines represents the waiting time in the system, which represents the waiting time in queue plus service time.

Figure 9 and figure 12 show the histograms of the average residence or waiting time in the system. The histograms are generated with 10,000 customers and 10,000 simulations. As expected, the histogram shows a normal distribution at steady state.

Figure 10 and figure 13 are the sample paths for queuing process with first-come first-served discipline. For figures 8, 10, 11 and 13, we have used the number of customers N=20, and service rate $\mu_j = \frac{4}{3}$, $(j = 1, 2, 3)$ in each of the queues. The values of the parameters $\alpha_{12}, \alpha_{23}, \lambda_1, \lambda_2$ and λ_3 in each of the queues are indicated in respective figures.

Figures 8A, 9A and 10A are for cases with only one queue. On the other hand, figures 8B, 9B and 10B describe systems with three queues with porous medium, where customers are allowed into the system through each of the service stations.

It is obvious from figure 8A how easy and fast a waiting line is formed in a system with one queue. The congestion is reduced by increasing the number of service points and allowing withdrawals within queues.

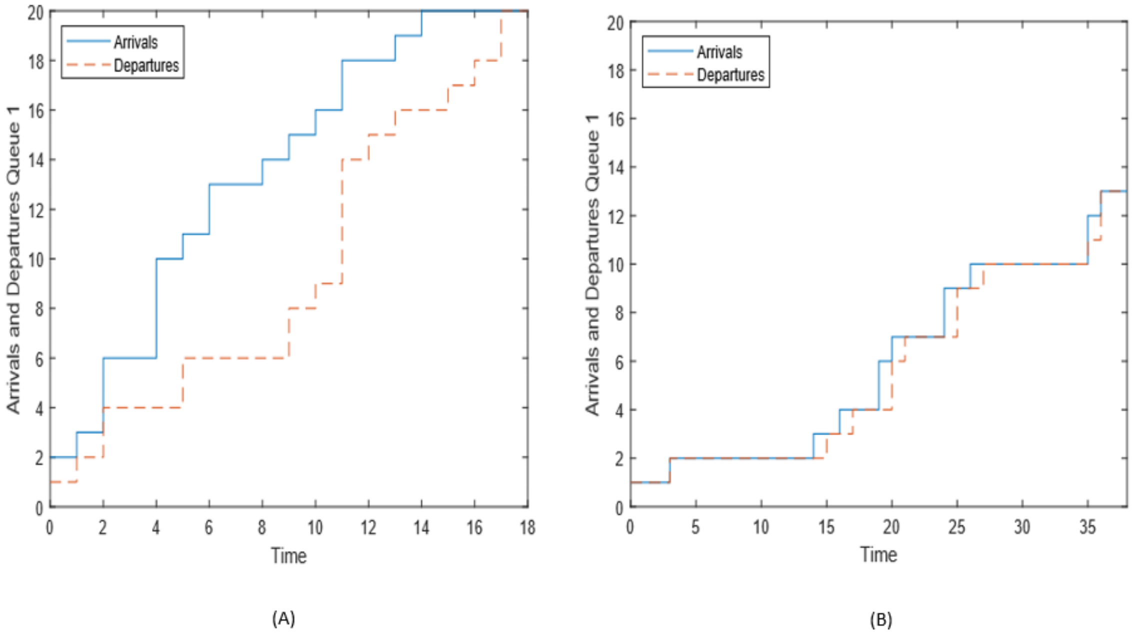

Figure 8: Cumulative Number of Arrivals and Departures from Queue 1.
For (A) $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$ and $\alpha_{12} = 0$. For (B): $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5$, and $\alpha_{12} = \frac{4}{3}$, $\alpha_{23} = \frac{4}{3}$.

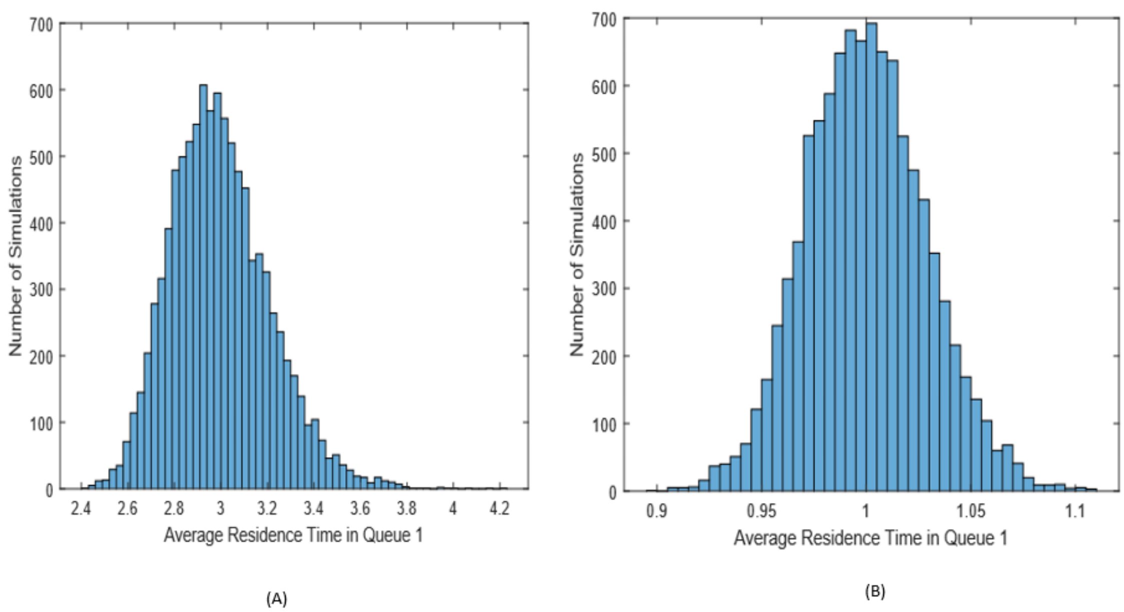

Figure 9: Histogram of Average Residence Time in Queue 1.
For (A) $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$ and $\alpha_{12} = 0$. For (B): $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5$, and $\alpha_{12} = \frac{4}{3}$, $\alpha_{23} = \frac{4}{3}$.

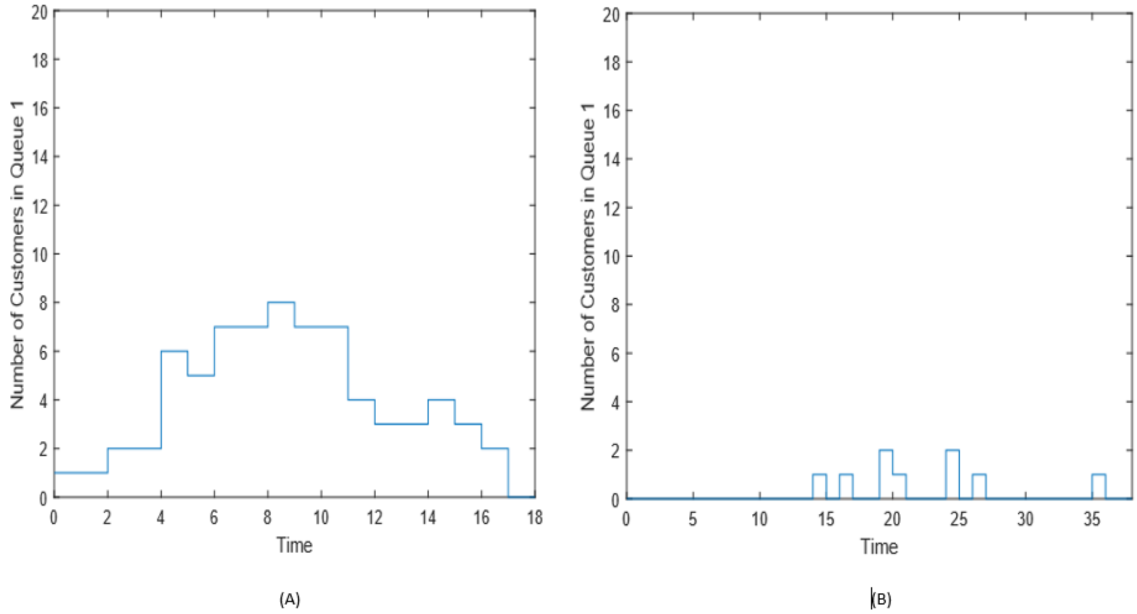

Figure 10: Sample Path for Queue 1.
For (A) $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$ and $\alpha_{12} = 0$. For (B): $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5$ and $\alpha_{12} = \frac{4}{3}$, $\alpha_{23} = \frac{4}{3}$.

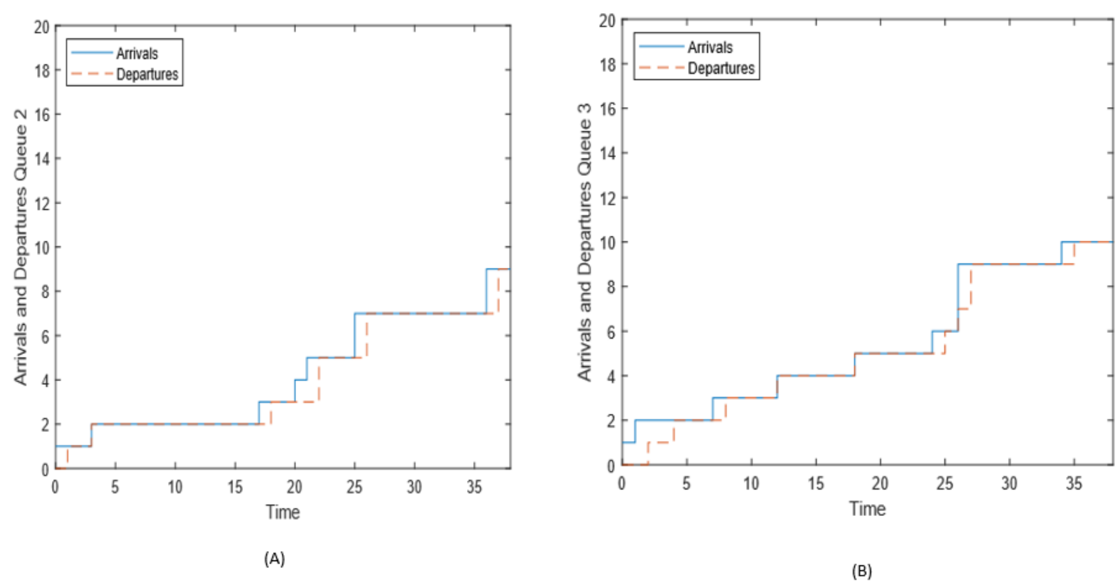

Figure 11: Cumulative Number of Arrivals and Departures.
For (A) $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5, \alpha_{12} = 0.75$ and $\alpha_{23} = 0.5$. For (B): $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5$.

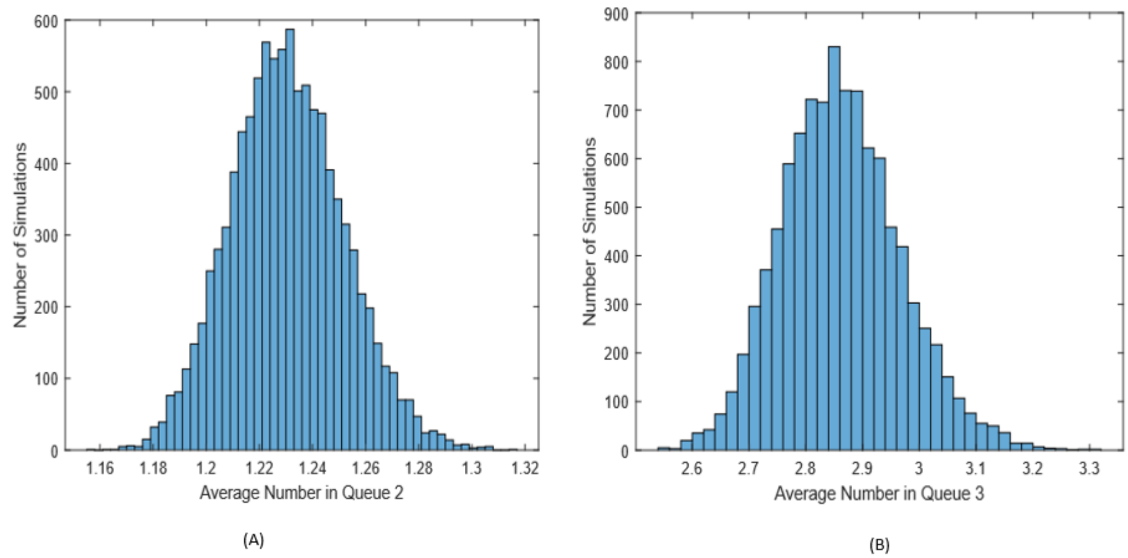

Figure 12: Histogram of Average Residence Time in Queue 2 and Queue 3.
For (A) $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5, \alpha_{12} = 0.75$ and $\alpha_{23} = 0.5$. For (B): $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5$.

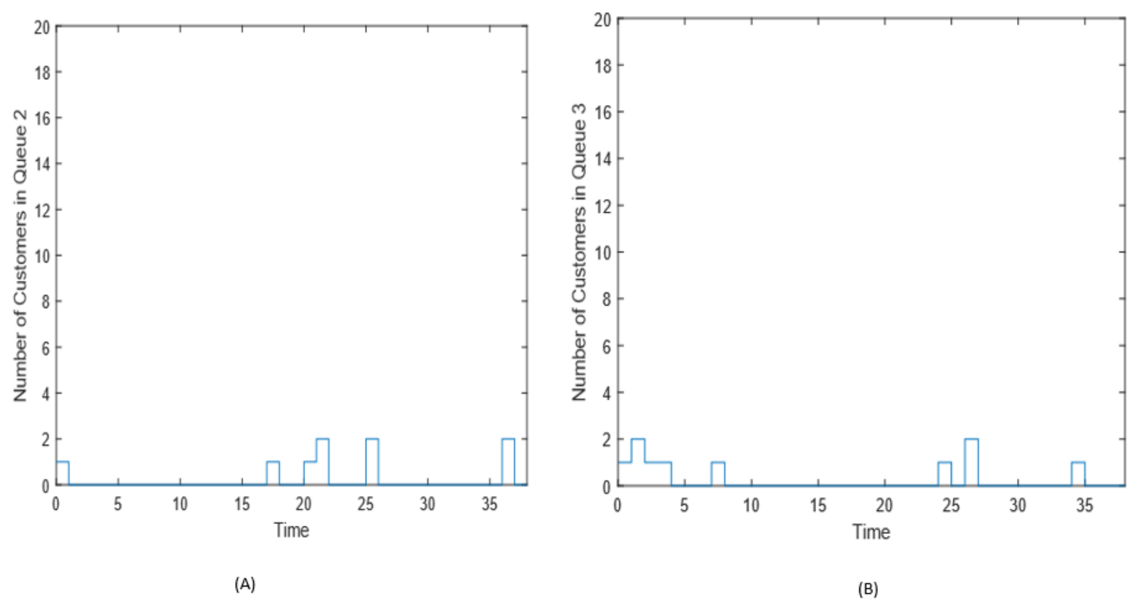

Figure 13: Sample Path for Queue 2 and 3.
For (A) $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5, \alpha_{12} = 0.75$ and $\alpha_{23} = 0.5$. For (B): $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5$.

CHAPTER 5

CONCLUSION

The mechanism of a system of tandem queue with $k \geq 1$ service stations was described in this thesis using some tools in stochastic process and queuing theory. For example, the birth-and-death process is used to model the movement of customers or units into and out of a queuing system. We start with the theoretical analysis of $M/M/1$ tandem queue with poisson arrival and exponential service time, first-come first-served $FCFS$ discipline and one service station. We compute some statistical properties of tandem queue and the global balance equation. From the global balance equation, we compute the probability of having n customers or units in the system at steady state using an iterative method as well as a probability generating function.

We consider a tandem queue with two service stations, and allow arrival to the second service station only from the first. The global balance equation for this model is analyzed using the generating function to obtain the probability of having given numbers of customers or units in the system at steady state. We further consider tandem queue with two service stations and allow arrivals and departure into and from the two service stations.

Finally, we simulate a queue network of 10,000 customers, we generalize the traffic intensity, the proportion of customers moving from one station to another and the marginal probability generating function for tandem queue with n service stations.

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APPENDIX A

APPROVAL LETTER

MARSHALL UNIVERSITY
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Office of Research Integrity

December 3, 2019

David Adepoju
622 Hal Greet Boulevard
Huntington, WV 25701

Dear Mr. Adepoju:

This letter is in response to the submitted thesis abstract entitled *"Statistical Analysis of Tandem Queues With Markovian Porous Medium."* After assessing the abstract, it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction, it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director

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