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Cauchy-Characteristic Patching with Improved Accuracy

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Introduction

- Cauchy-characteristic extractions (CCE) avoids the errors due to extraction at finite worldtube.
- The Cauchy and the characteristic approaches have complementary strengths and weaknesses.
- Unification of the two methods is a promising way of combining the strengths of both formalisms.
Advantages

• Avoids the errors due to gravitational waveform extraction at finite worldtube.
• The grid domain is exactly the region of the waves propagation (no artificial boundary).
• Gives the waveform and polarization state at infinity (no ongoing radiation).
• Offers flexibility and control in prescribing initial data (very little gauge freedom).
• Combine Cauchy & Characteristic methods.
The Characteristic Method

- Extract characteristic data at inner worldtube from Cauchy evolution,
- Propagate the characteristic solution along the outgoing light cones,
- Extracts the waveform at infinity for each retarded time.
The Stereographic Module

- Newman-Penrose eth-formalism on the sphere
  \[
  q_A q_B D^A U^B = \partial U
  \]
- Numerical noise introduced by inter-patch interpolation.
- Two improvements:
  - Circular boundary,
  - 4th order derivatives.
Higher Order Approximations

- 4\textsuperscript{th} order approximations in finite differences.
- 1\textsuperscript{st} and 2\textsuperscript{nd} derivatives of the 5\textsuperscript{th} order, 3\textsuperscript{rd} derivative of the 7\textsuperscript{th} order Lagrange polynomial.

\[ D_1 F(x_i) = \frac{F(x_{i-2}) - 8F(x_{i-1}) + 8F(x_{i+1}) - F(x_{i+2})}{12\Delta x} \]

\[ D_2 F(x_i) = -\frac{F(x_{i-2}) - 16F(x_{i-1}) + 30F(x_i) - 16F(x_{i+1}) + F(x_{i+2})}{12\Delta x^2} \]

\[ D_3 F(x_i) = \frac{F(x_{i-3}) - 8F(x_{i-2}) + 13F(x_{i-1}) - 13F(x_{i+1}) + 8F(x_{i+2}) - F(x_{i+3})}{8\Delta x^3} \]
Errors in Angular Derivatives

- 2D scalar wave propagation on the sphere:

\[-\partial_t^2 \Phi + \nabla^2 \Phi = 0, \Phi = \cos(\omega t)Y_{lm}, \omega = \sqrt{l(l+1)}\]

- Convergence rates for errors in \(\partial_1 \Phi, \partial_2 \Phi\) and \(\partial_3 \Phi\).

<table>
<thead>
<tr>
<th>Error</th>
<th>N=80</th>
<th>N=120</th>
<th>N=160</th>
<th>N=200</th>
<th>N=240</th>
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<tr>
<td>(\xi(\partial_1 \Phi))</td>
<td>3.99</td>
<td>4.04</td>
<td>4.11</td>
<td>4.35</td>
<td>4.85</td>
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<tr>
<td>(\xi(\partial_2 \Phi))</td>
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<td>4.07</td>
<td>4.24</td>
<td>4.80</td>
<td>3.95</td>
</tr>
<tr>
<td>(\xi(\partial_3 \Phi))</td>
<td>3.94</td>
<td>3.98</td>
<td>3.95</td>
<td>3.92</td>
<td>3.86</td>
</tr>
</tbody>
</table>
News for a Linearized Test

- Bondi News function: \[ N = N_+ + iN_\times = \partial_t h_+ + \partial_t h_\times \]
- Linearized vacuum Bondi-Sachs solutions to Einstein equations on Minkowski background
  \[ h = \sqrt{(l-1)l(l+1)(l+2)} Y_{lm} \text{Re}(h_l(r)e^{i\nu t}) \]
Alternate Method: \( \Psi_4 \)

- The Newman-Penrose Weyl component \( \psi_4 \)

\[
\Psi_4 = l \frac{\partial N}{\partial u} + O(l^2)
\]
Weyl tensor extraction $N_{\Psi}$ is slightly more accurate than the News function extraction $N$.

$$\Psi = N_{,u}, \quad N_{\Psi} = N\bigg|_{u=0} + \int_{0}^{u} \Psi du$$
The Patching Scheme

- Cauchy and characteristic evolution are patched in the vicinity of a worldtube,
- Characteristic data is provided by Cauchy evolution at worldtube B,
- Free initial characteristic data is given on the initial null hypersurface N,
- All is embedded in the Cauchy evolution.
Online Extraction

• Sets the Cartesian coordinates on the sphere on which to extract the Cauchy data,
• Interpolates the Cauchy metric, lapse, shift, and spatial derivatives of the metric on the sphere,
• Calculates the Jacobians from Cartesian to affine null metric, and from the affine to Bondi metric,
• Calculates the boundary data and puts it on the worldtube with a Taylor expansion to be evolved,
• Advances to the next time level and repeat.
Towards a Versatile Extraction

• CCE does not have to run simultaneously, if the data is given on the world tube, before extracting.

• Steps towards a new IO interface for Extraction:
  - Read from file Cauchy data in Cartesian coordinates between two determined radii, at a chosen resolution,
  - Convert data into a set of analytic functions using a Chebyshev and spherical harmonic decomposition,
  - Take the analytic functions and feed them to the extraction module (no interpolation necessary),
  - Evolve the data and compute the waveform at infinity.
Chebyshev Spherical Harmonic Decomposition

- Normalization for Chebyshev polynomials:

\[
\int_{-1}^{+1} w U_m U_n dt = \delta_{mn} \frac{\pi}{2} \implies f_n = \frac{2}{\pi} \int_{-1}^{+1} FwU_n dt, \quad w = \sqrt{1-t^2}
\]

- Change the integration limits, add \( Y_{lm} \) and do the coordinate transformation \((r, \theta, \phi)\rightarrow(x, y, z)\):

\[
f_{lmn} = \frac{2}{\pi} \int_{R_1}^{R_2} dt \frac{w}{dr} r^2 Y_{lm}^* U_n F dx dy dz
\]

- Recover the analytic function:

\[
F = \sum_{l} \sum_{m} \sum_{n} f_{lmn} Y_{lm} U_n
\]
Implementation

- Define a mask between two relevant radii ($R_1$, $R_2$)
- Read the functions in $(x,y,z)$ coordinates,
- Compute the expansion coefficients $f_{lmn}$ as a sum over the masked points,

$$f_{lmn} = \text{mask} \sum_{i,j,k} F_{lmn}^{i,j,k} \Delta x \Delta y \Delta z$$
- Reconstruct the function,
- Populate the worldtube.
Further Improvements

• Include higher order approximations in the post-processing of the news function

• Improve the characteristic boundary by changing the data on the inner worldtube from Dirichlet to Sommerfeld

• Produce a Cactus CCE module for wave extraction that will be freely available to the numerical relativity community