2006

The Dynamics of Newton's Method on Cubic Polynomials

Shannon N. Miller

THE DYNAMICS OF NEWTON'S METHOD ON CUBIC POLYNOMIALS

Thesis submitted to

The Graduate College of

Marshall University

In partial fulfillment of the

Requirements for the degree of

Master of Arts

Department of Mathematics

by

Shannon N. Miller

Dr. Ralph Oberste-Vorth, Committee Chairman

Dr. Ariyadasa Aluthge

Dr. Evelyn Pupplo-Cody

Marshall University

May 2006

ABSTRACT

THE DYNAMICS OF NEWTON'S METHOD ON CUBIC POLYNOMIALS

By Shannon N. Miller

The field of dynamics is itself a huge part of many branches of science, including the motion of the planets and galaxies, changing weather patterns, and the growth and decline of populations. Consider a function f and pick x_0 in the domain of f. If we iterate this function around the point x_0, then we will have the sequence x_0, $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, ..., which becomes our dynamical system. We are essentially interested in the end behavior of this system. Do we have convergence? Does it diverge? Could it do neither? We will focus on the functions obtained with Newton's method on polynomials and will apply our knowledge of dynamics. We also will be studying the types of graphs one would get if they looked at these same functions in the complex plane.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the contributions of the following people:

My committee:

Dr. Aluthge, who taught me in three semesters just how much I like complex variables and integral equations. He also took the time to sit down and really discuss the material in this work. Thank you.

Dr. Oberste-Vorth whose knowledge, time, and effort has been put forth to help me understand a subject in which I really enjoy begin involved. Even with the loss of a loved one, he managed to take the time out and help me finish this paper on time. Kudos.

Dr. Pupplo-Cody, who did not hesitate to talk to a crying freshman when she was crushed with an ever so slightly imperfect grade in her first college math course, has put forth so much effort into completely understanding my topic and giving me new ideas to make the presentation so much more inviting to watch. Thanks.

Also, Dr. Lawrence, my graduate advisor and mentor, was always there for me when I was in need. Whether it be a math problem or wanting to learn to ballroom dance, this is your woman!

To my fiance Rob-Roy, thank you for supporting me through stressful times. I love you.

Finally, to my grandmother, the person that wanted nothing more for me in life than to be a better person. Mamaw, I am.

Contents

1 Preliminaries **1**

1.1 Chaotic Dynamics 1

1.2 The Mandlebrot, Julia and Fatou Sets 7

2 Newton's Method **11**

2.1 An Interation 11

2.2 Conjugacy Classes of Quadratics 15

3 Newton's Method Extended **23**

3.1 Cubics 23

3.2 Conjugacy Classes of Cubics 24

4 Graphical Analysis **31**

4.1 The Symmetry of N_λ 31

4.2 Newton's Method for Cubic Polynomial N_λ 38

A MatLab Code **43**

A.1 Newton's Method on Quadratic Polynomials 43

A.2 Newton's Method on Cubic Polynomials 45

A.3 Newton's Method on Lambda Cubic 47

A.4 Parameterspace 49

List of Figures

2.1 Newton's method on the quadratic polynomial $p(z) = z^2 - 2z - 3$. 20

2.2 Newton's method on the quadratic polynomial $p(z) = z^2 - 2.5$. 20

3.1 Newton's method a generic polynomial with three roots for initial guess x_0. 23

4.1 Newton's method on the cubic polynomial $p(z) = z^3 + z^2 + z + 1$. Here, $\lambda =$ i. 33

4.2 The parameter space created with $z = \frac{1}{3}$. 39

4.3 Excerpt from parameter space figure. 40

4.4 The set of periodic points from our parameter space. 40

4.5 N_λ for $\lambda = .589 + 605i$ from -1.5 to 1.5. 41

4.6 N_λ for $\lambda = .589 + .605i$ from 0 to 1.5. 41

4.7 "The Rabbit": N_λ for $\lambda = .589 + .605i$ from 0.2 to .48. 42

Chapter 1

Preliminaries

1.1 Chaotic Dynamics

What exactly is a dynamical system? These systems involve iterating a mathematical function and then asking what happens. No one really knows the answer to this question...completely. Even with such simple functions as the quadratic or cubic, no one can say precisely what would happen.

We will be considering specific functions and their behaviors. For example, we will consider the following functions:

1. $f_1(x) = x^2$
2. $f_2(x) = x^3$
3. $f_3(x) = \sin x$
4. $f_4(x) = \frac{1}{x}$

Let us begin by looking at a few preliminaries. We can think about this iteration process as one you would take if you were finding the square root or the square, by repeating an action over and over again. This list of iterates around a point is called the orbit of that point. We can define the orbit in a recursive manner, *i.e.* $f^0(x) = x$ and for all $n \in \mathbb{N}, f^n(x) = f(f^{n-1}(x))$. However, we employ the following definition.

Definition 1 (Orbit) *Let $f(x)$ be a function defined on a domain. The orbit of the function, denoted $f^n(x)$, for a point x is created by iterating a function starting at that* *point to get a list of numbers. The forward orbit of x is the set of points in the sequence x, $f(x)$, $f(f(x))$, $f(f(f(x)))$, written more succinctly as x, $f(x)$, $f^2(x)$, $f^3(x)$,... and will be denoted $f^n(x)$ for $n \in \mathbb{N}$. Similarly, the backward orbit of x is the set of points in the sequence x, $f^{-1}(x)$, $f^{-1}(f^{-1}(x))$, $f^{-1}(f^{-1}(f^{-1}(x)))$, written more succinctly as x, $f^{-1}(x)$, $f^{-2}(x)$, $f^{-3}(x)$,... and will be denoted by $f^{-n}(x)$ for $n \in \mathbb{N}$ whenever the inverses exist.*

For the case when $f_1(x) = x^2$, depending on the position of the point that you pick, *i.e.* $0 < |x| < 1$, $|x| > 1$, $x = 1$, $x = -1$, or $x = 0$, there will be a possibility of three different types of behavior of the orbit for that specific point. The first type of behavior we will see is convergence. This is when the orbit of a particular point approaches a finite value when we iterate the function infinitely, or $f^n(x) \to 0$ as $n \to \infty$. The other two are divergence, $f^n(x) \to \infty$ as $n \to \infty$, and having fixed points, $f^n(x) = 1$ or $f^n(x) = 0$ for all n. In the latter case, we can say that the point does not "move" over the iterate. This gives us the following definition.

Definition 2 (Fixed Point) *The point x is a fixed point for f if $f(x) = x$. We will denote the set of all fixed points as $Fix(x)$.*

Suppose x_0 is a fixed point of an analytic function f, that is, $f(x_0) = x_0$. The number $\lambda = f^{'}(x_0)$, where $f^{'}$ is the derivative of f, is called the multiplier of f at x_0. A possible reason for being called a multiplier is because of its position is the equation used for the Mean Value Theorem $|f(x) - f(y)| = f^{'}(x)|x - y|$. We classify fixed points according to λ as follows:

Superattracting : $\lambda = 0$.

Attracting : $|\lambda| < 1$.

Repelling: $|\lambda| > 1$.

Rationally neutral: $|\lambda| = 1$ and $\lambda^n = 1$ for some integer n.

Irrationally neutral: $|\lambda| = 1$ but λ^n is never 1.

If we choose a point that may not be a fixed point, we find that other points in the plane exhibit these same characteristics. We will concentrate on the first three properties right now, because the last two properties on the list only occur when we have a complex function.

Now, let us take a look at $f_3(x) = \sin x$. Consider the point x_0=1.57. We see that the first few iterates are approximately

$$f_3(1.57) = .999$$

$$f_3^2(1.57) = .841$$

$$f_3^3(1.57) = .745$$

$$f_3^4(1.57) = .678$$

and the points are decreasing on the average by .107. If we jump a few iterations ahead,

$$f_3^{73}(1.57) = .197$$

$$f_3^{74}(1.57) = .196$$

$$f_3^{75}(1.57) = .194$$

the average decrease is .0015. Even further, we have

$$f_3^{298}(1.57) = .099527$$

$$f_3^{299}(1.57) = .099362$$

$$f_3^{300}(1.57) = .099199$$

As we can see, this is slowly approaching 0, which is a fixed point for this function. These iterations are getting closer and closer to 0, and thus we say this point, $x_0 = 1.57$ is attracted to zero. So, we would call $x_0 = 1.57$ a convergent point. Furthermore, our second function f_2 illustrates this property. We can see immediately that zero is a fixed point. Also, ± 1 are fixed points. So, let us pick some points between these with which we have discussed. If we choose $x_0 = -1.5, x_0 = -.5, x_0 = .5$, and $x_0 = 1.5$, then we will get the following chart of iterations.

Iteration	$x_0 = -1.5$	$x_0 = -.5$	$x_0 = .5$	$x_0 = 1.5$
$f_2(x)$	-3.375	-.125	.125	3.375
$f_2^2(x)$	-38.4434	-.0019	.0019	38.4434
$f_2^3(x)$	-56815.3088	-7.450×10^{-9}	7.450×10^{-9}	56815.3088
$f_2^4(x)$	-1.8340×10^{14}	-4.1359×10^{-25}	4.1359×10^{-25}	1.8340×10^{14}
$f_2^5(x)$	-6.1688×10^{42}	-7.0747×10^{-74}	7.0747×10^{-74}	6.1688×10^{42}

The points chosen in the interval from -1 to 0 and the interval 0 to 1 are attracted to zero very quickly. In this case, zero is a super-attractive fixed point, since $f'(0) = 0$. However, if we look at the point chosen to the left of the fixed point -1, as well at the point chosen to the right of 1, these two iterations seem to push away from each of their corresponding fixed points. This tells us that that if we keep iterating we will approach infinity. When the iterations tend to infinity, the points -1 and 1 are called repelling fixed points, since $|f'(-1)| = 3$ and $|f'(1)| = 3$.

However, let us choose a point for $f_4(x) = \frac{1}{x}$. We would like to find all the fixed points, i.e. all the points that satisfy the equation $\frac{1}{x} = x$. This is equivalent to solving the equation $1 = x^2$. We can see that the only two points that will satisfy this equation are $x = 1$ and $x = -1$. If we choose a nonzero number between -1 and 1, say .5, and iterate it, we see that the first few iterates are,

$$f_4(.5) = 2$$

$$f_4^2(.5) = .5$$

$$f_4^3(.5) = 2$$

$$f_4^4(.5) = .5$$

where the points seem to repeat a "cycle" of numbers that includes the original. We call this iterate a periodic orbit, or cycle. If we jump a few iterations ahead,

$$f_4^{73}(.5) = 2$$

$$f_4^{74}(.5) = .5$$

$$f_4^{75}(.5) = 2$$

the orbit is still the same. We say that .5 is a periodic point of period 2, since the cycle repeats the value of .5 every second iterate. We expect this behavior from the recursiveness of our iteration. This gives us another definition.

Definition 3 (Periodic Point) *The point x is a periodic point for f if $f^n(x) = x$, for some integer n. The least positive n for which $f^n(x) = x$ is called the prime period of x. We will denote the set of periodic points of period n by $Per_n(f)$*

In this case, every nonzero point in the interval that contains .5, i.e. $(-1, 1)$, will be a periodic point of period 2. There are two types of periodic points: hyperbolic and non-hyperbolic. We will be dealing with hyperbolic periodic points unless told otherwise, but what does it mean to be hyperbolic?

Definition 4 (Hyperbolic) *Let p be a periodic point of prime period n. The point p is hyperbolic if $|(f^n)'(p)| \neq 1$. The number $(f^n)^{'}(p)$ is called the multiplier of the periodic point.*

One important thing about these hyperbolic periodic points is the fact that if we pick some other point that is relatively close to them, the orbit of that particular point will converge to the hyperbolic periodic point. Note that a fixed point is a periodic point. The following proposition illustrates this idea.

Proposition 1 *Let f be C^1 and p be a hyperbolic fixed point with $|f^{'}(p)| < 1$. Then there is an open interval U about the hyperbolic fixed point $p \in U$ such that if $x \in U$, then*

$$\lim_{n \to \infty} f^n(x) = p.$$

Proof. Since f is C^1, $f^{'}(x)$ exists and is continuous at all x in some interval U about p. So, there is $\epsilon > 0$ small enough and $A \in \mathbb{R}_+$ so that $|f^{'}(x)| < A < 1$ when $x \in [p-\epsilon, p+\epsilon]$ and $0 < A < 1$. Since p is a fixed point of f, we know that $|f(x)-p| = |f(x) - f(p)|$, and from the definition of the derivative $f^{'}(x) = \lim_{n\to\infty} \frac{f(x)-f(p)}{x-p}$. From the Mean Value Theorem we have that $|f(x) - f(p)| < A|x - p|$. Using our previous statement, we have the following

$$|f(x) - p| = |f(x) - f(p)| < A|x - p| < |x - p| \leq \epsilon,$$

since x is in the ϵ-ball around p. Hence $f(x)$ is contained in $[p - \epsilon, p + \epsilon]$ and, in fact, is closer to p than x is. Via the same argument

$$|f^n(x) - p| < A^n|x - p|$$

so that $f^n(x) \to p$ as $n \to \infty$. □

The previous proposition leads us to the following definition about the points around a particular x_0.

Definition 5 (Basin of Attraction) *The set of all points whose orbits are attractive periodic orbits is the basin of attraction of that orbit.*

So far we have only been considering points in the the real line. We have seen all the different types of behavior that occur at these points. What about complex numbers in the complex plane? Suppose we are given a complex-valued function $g : \mathbb{C} \to \mathbb{C}$ defined as

$$g(x, y) = u(x, y) + iv(x, y)$$

where u and v are maps from $\mathbb{R}^2$ to $\mathbb{R}$. Again, we are interested in the behavior of orbit of g at a particular point.

One thing that we will discuss is the graphical nature of these complex iterates which create beautiful pictures called fractals. When we graph these, we simply graph the real part in the horizontal direction and the imaginary part in the vertical direction. We now dive into the realm of the complex chaotic dynamics.

1.2 The Mandlebrot, Julia and Fatou Sets

It is useful to think of the iteration of a function being defined on the Riemann sphere, $\hat{C}$, consisting of the complex plane with a point at infinity. We know that looking at the complex plane will open up solutions that we would not have in the real $x-y$ plane. Any complex differentiable map $f : \hat{C} \to \hat{C}$ on the Riemann sphere can be expressed as a rational function, that is as the quotient,

$$f(z) = \frac{p(z)}{q(z)},$$

where p and q are polynomials. In this case, we have to assume that p and q have no common roots. The degree, d, of f is equal to the largest the degrees of p and q.

The global study of Newton's method can now be analyzed using this theory of complex dynamics of rational maps on this Riemann sphere. Any complex analytical mapping will decompose the plane into two complementing sets, the stable set, where the dynamics are mostly tame, and the unstable set, where the dynamics become chaotic and unpredictable. The study of this idea was started by G. Julia and P. Fatou in the 1920's. Let us take a look at some of their findings.

We start out by examining some of the special properties of maps defined in the complex plane. In fact, it is these properties that give the dynamics of complex analytic mappings a much richer structure that we previously mentioned.

Definition 6 (Normal Family) *For a function F, the family F^n of iterates is a normal family on an open set U if every sequence of the F^n's has a subsequence which either*

1. *converges uniformly on compact subsets of U, or*
2. *converges uniformly to ∞ on U.*

If you let $R(z)$ be a rational function, where

$$R(z) = \frac{P(z)}{Q(z)}.$$

Here P and Q are polynomials that have no common factors and d =max(deg(P), deg(Q))≥ 2. The Fatou set F of R is defined to be the set of points $z_0 \in \mathbb{C}$ such that R^n is a normal family in some neighborhood of z_0. The Julia set J is the complement of the Fatou set. The Julia Set is the boundary of the sets that escape, i.e. those that do not converge, and those that never escape, i.e. those that do converge.

As an example, consider the squaring map $f : z \to z^2$. The entire open unit disk D is contained in the Fatou set of f, since successive iterates on any compact subset of D converge uniformly to zero. Similarly, the exterior, $\hat{C} \setminus \bar{D}$ is contained in the Fatou set, since the iterates of f converge to the constant function $z \to \infty$ outside of $\bar{D}$. On the other hand, if z_0 belongs to the unit circle, then in any neighborhood of z_0 any limit of iterates f^n would necessarily have a jump discontinuity as we cross the unit circle. We arrive to the very descriptive definition of the Julia set.

Definition 7 *Consider a complex mapping of the form $z_{n+1} = f(z_n)$. The points that lie of the boundary between points that are bounded and those that are unbounded are collectively referred to as the Julia set. The following properties of a Julia set, J, are well known:*

- *The set J is invariant.*
- *An orbit on J is either periodic or eventually periodic.*
- *All repelling periodic points are on J. So, set J is a repellor.*
- *The set J is either connected or totally disconnected.*
- *The set J nearly always has fractal structure.*

If we have a polynomial, the Julia set is the closure of the set of all repelling periodic points. Let us explore what Julia and Fatou discovered by considering once again the square function $f_1(x) = x^2$. In $\mathbb{R}$, we have observed that if $|x| < 1$, then $f^n(x) \to 0$, and $|x| > 1$, then $f^n(x) \to \infty$. The map has only two fixed points and all other points eventually approach one of them or go off to infinity. Thus, we can say that the dynamics are fairly tame. If we translate this to the complex plane, the set of all attracting convergent points is contained in the open unit disk, and the set of all repelling divergent points are on the outside of the closure of this same disk. This tells us that the Julia set is precisely equal to the unit circle.

To summarize, we have to following definition.

Definition 8 *Let S be a Riemann surface, let $f : S \to S$ be a non-constant holomorphic, or analytic, mapping, and let f^n be its nth-interate, or orbit. Fixing some point $z_0 \in S$, we have the dichotomy(a mutually exclusive bipartition of elements): If there exists some neighborhood U of z_0 so that the sequence of iterates f^n restricted to U forms a normal family, then we say that z_0 is a regular or normal point, or that z_0 belongs to the Fatou set of f. Otherwise, if no such neighborhood exists, we say that z_0 belongs to the Julia set.*

Now that we have a few definitions under our belts, let us look at a few facts about these two sets. Julia sets define the border between bounded and unbounded orbits. Now, the Julia set associated with the point $c = a + ib$ is denoted by J_c.

Theorem 1 *Every attracting periodic orbit is contained in the Fatou set. In fact the entire basin of attraction Ω for an attracting periodic orbit is contained in the Fatou set. However the boundary of the basin $\partial\Omega$ is contained in the Julia set, and every repelling periodic orbit is contained in the Julia set.*

Proof. We need only consider the case of a fixed point. If z_0 is attracting, the it follows from Taylor's Theorem that the successive iterates of f, restricted to a small neighborhood of z_0, converge uniformly to the constant function $z \to z_0$. The corresponding statement for any compact subset of the basin Ω then follow easily. On the other hand, around a boundary point of this basin, that is, a point which belongs to the closure $\overline{\Omega}$ but not to Ω itself, it is clear that no sequence of iterates can converge to a continuous limit. If z_0 is repelling, then no sequence of iterates can converge uniformly near z_0, since the derivative $\frac{df^n(z)}{dz}$ at z_0 takes the value λ^n, which diverges to infinity as $n \to \infty$. $\Box$

The previous theorem leads us to the following conclusion. If we define the basin of attraction of z_0 for a rational map $R(z)$, $A(z_0) = \{z \in \mathbb{C} : R^n(z) \to z_0\}$. This notation allows for another definition of the Julia set in the following theorem.

Theorem 2 *For $P(z)$ with deg$(P) \geq 2$, the boundary of $A(z)$ coincides with the Julia set of P.*

Proof. From Theorem 1, we see that $\partial A(z) \in J(P)$. Suppose w_0 is on the boundary of $\partial A(z)$ and V is any neighborhood of w_0. Then for $z_0 \in A(z) \bigcap V$, $P^n(z) \to \infty$, but the orbit of w_0, $P^n(w_0)$ remains bounded. Thus P^n is not normal in V so w_0 is in the Julia set.$\Box$

Looking ahead in our studies, it is with this theorem that we will see the complexity in the case of the cubic, which in turn adds character to the iteration of the complex function. This theorem simply says that all the basins have the same boundary. So, if we have the case of three attracting fixed points, then any disk that intersects two of the basins must intersect all the basins.

Only when we iterate these functions in the complex plane will we see the Mandelbrot set. It is the set of complex numbers c for which the iteration $z_{n+1} = z_n + c$ produces finite z_n for all n. The Mandelbrot set is seen when we choose to color the points of the plane according to their convergence to the root. We will see more of this in chapters to come. For different parameter values of the Mandelbrot set, we will show the many different forms of the Julia set.

Chapter 2

Newton's Method

2.1 An Interation

Newton's method is used to calculate approximately the roots of any real-valued differentiable function. Newton's method is the best known iteration method for finding these real and/or complex roots. We begin with an initial point, say x_0, on the real axis. Using the point $(x_0, f(x_0))$, we can obtain the equation for the tangent line at this point, and it is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

If we let the x-intercept of this line be the point x_1, we have the following equation

$$-f(x_0) = f'(x_0)(x_1 - x_0).$$

Solving this equation for x_1, we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now, repeat the same steps using our new point x_1 to get x_2. Continuing these same actions recursively, we will be able to represent our pattern as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f'(x_n) \neq 0$. What we want to do is observe the behavior after iteration. As $n \to \infty$, this system exhibits several possible behaviors. The first and most trivial is where x_0 is a root of the polynomial f, and here we see that $x_n = x_0$ for all n. The orbit in this case will look like $x_0, x_0, x_0, ...$ The second type of behavior occurs when x_0 is not a root of our polynomial and $f'(x_0) = 0$. This tells us that we cannot use Newton's Method, and our orbit will will go off, or converge, to infinity. In the third type, if in the orbit one of the previous points is repeated, we will have a cycling behavior in which none of the points are roots. In other words, $x_n = x_0$ for some n. Again, this sequence does not converge, nor does it diverge. So, we will refer to this as periodic, and the points in the orbit will be our periodic points. Other points will simply fail to converge.

We will be considering throughout the remainder of our discussion the following function defining Newton's Method on a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

Starting with some initial approximate value z_0, we define the $(n + 1)$ approximation by $z_{n+1} = N_f(z_n)$. If the function is a polynomial (or a rational function), then the iteration function N_f, after some manipulation, will be a rational map of the form

$$N_p(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials with real or complex coefficients.

Aside from the particular dynamics of Newton's Method, one goal is to discover the relationship between any particular polynomial and it's Newton function. This leads us to the definition of topological conjugacy.

Definition 9 *Let $f : A \to A$ and $g : B \to B$ be two functions, i.e. the original polynomial p and the function created after applying Newton's Method. These two functions are called topologically conjugate if there exists a homeomorphism m that* *maps $A \to B$ such that the following diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle m}\downarrow & & {\scriptstyle m}\downarrow \\ B & \xrightarrow{g} & B \end{array}$$

The basin of attraction for a root of a polynomial is the set of starting points which eventually converge to this root. Since every root is at least an attracting or even a super-attracting fixed point, this basin includes a neighborhood of the root and is non-empty.

So what is the relationship between fixed points of Newton's method and points of the original function? The roots of any polynomial equation are the fixed points of the Newton function. One would think that these points should be of some interest in the beginning function, i.e., possibly critical points of the original function? This is sometimes true, however, the following proposition describes the relationship when the roots are not critical points, the points where the function and/or its derivative is equal to zero.

Proposition 2 *Given an analytic function f whose roots are not critical points of f, then the roots of f are superattractive fixed points of N_f.*

Proof. Let r be any root of f such that $f^{'}(r) \neq 0$. Then,

$$N_f(r) = r - \frac{f(r)}{f'(r)} = r$$

since r is a root of f, or $f(r) = 0$, and $f^{'}$ does not vanish at r, or $f^{'}(r) \neq 0$. We conclude that r is a fixed point of N_f by the definition of fixed point. Now, if we look at the derivative,

$$\begin{aligned} N_f^{'}(r) &= 1 - \frac{f^{'}(r)f^{'}(r) - f(r)f^{''}(r)}{(f^{'}(r))^2} \\ &= \frac{(f^{'}(r))^2 - (f^{'}(r))^2 + f(r)f^{''}(r)}{(f^{'}(r))^2} \end{aligned}$$

$$= \frac{f(r)f''(r)}{(f'(r))^2}.$$

For the same reasons as given for $N_f(r) = r$, the multiplier, which we have discussed before, is $\lambda = N_f'(r) = 0$, we have that r is a superattracting fixed point of N_f. □

What about those functions whose roots are in fact critical points of the original equation? We should be able to say something stronger about these points. In fact, we have another proposition that has a particular claim about these particular points.

Proposition 3 *Given an analytic function f whose roots are critical points of f, then the roots of f are attractive fixed points of N_f such that the $N_f'(r) = \frac{n-1}{n}$ where r is the root and n is the order of that root.*

Proof. Let r be a zero of order $n \geq 2$ for f. We then take the Taylor series representation of f about r. The first $n-1$ terms of f vanish at r, so

$$f(z) = \frac{f^{(n)}(r)}{n!}(z-r)^n + \frac{f^{(n)}(r)}{(n+1)!}(z-r)^{n+1} + \ldots$$

$$\begin{aligned} N_f'(r) &= \lim_{z \to r} \frac{f(z)f''(z)}{(f'(z))^2} \\ &= \lim_{z \to r} \frac{\left(\frac{f^{(n)}(r)}{n!}(z-r)^n + \ldots\right)\left(\frac{f^{(n)}(r)}{(n-2)!}(z-r)^{n-2} + \ldots\right)}{\left(\frac{f^{(n)}(r)}{(n-1)!}(z-r)^{n-1} + \ldots\right)^2} \\ &= \lim_{z \to r} \frac{\frac{(f^n(r))^2}{n!(n-2)!}(z-r)^{2n-2} + \ldots}{\frac{(f^n(r))^2}{((n-1)!)^2}(z-r)^{2n-2} + \ldots} \end{aligned}$$

Dividing the numerator and denominator by the greatest common factor $(z-r)^{2n-2}$ and applying the limit yields the following leading coefficients of $N_f'(r)$.

$$\frac{\frac{(f^n(r))^2}{n!(n-2)!}}{\frac{(f^n(r))^2}{((n-1)!)^2}} = \frac{n-1}{n}.$$

Thus, r is an attractive fixed point for N_f. $\square$

2.2 Conjugacy Classes of Quadratics

Let us consider once again the square function, but this time as a complex function. Here, $f(z) = z^2$, where $z = x + iy$ is a complex number. In terms of its real and imaginary parts, $f(x+iy) = x^2 - y^2 + i(2xy)$, so $f(z)$ is also a complex number. From this, we can say that the orbit of any complex number with this function will be a complex number. If we iterate f, will we see many of the same types of behavior?

We begin by working with functions of the type $z_{n+1} = z_n^2 + c$. Thus we can say that

$$x_{n+1} = x_n^2 - y_n^2 + a,$$

and

$$y_{n+1} = 2x_n y_n + b,$$

are the real and imaginary parts, where $z_n = x_n + iy_n$ and $c = a + ib$. We need to look at the inverse mapping of this function in order to utilize the properties. To do this one must find expressions for x_n and y_n in terms of x_{n+1} and y_{n+1}. Now,

$$x_n^2 - y_n^2 = x_{n+1} - a.$$

Also, note that

$$\begin{aligned}(x_n^2 + y_n^2)^2 &= (x_n^2 - y_n^2)^2 + 4x_n^2 y_n^2 \\ &= (x_n^2 - y_n^2 + a - a)^2 + (2x_n y_n + b - b)^2 \\ &= (x_{n+1} - a)^2 + (y_{n+1} - b)^2,\end{aligned}$$

and hence,

$$x_n^2 + y_n^2 = +\sqrt{(x_{n+1} - a)^2 + (y_{n+1} - b)^2},$$

since $x_n^2 + y_n^2 > 0$. Let

$$u = \sqrt{(x_n^2 - y_n^2)^2 + (2x_n y_n)^2} \text{ and } v = x_n^2 - y_n^2.$$

Then,

$$x_n = \pm\sqrt{\frac{u+v}{2}} \text{ and } y_n = \frac{y_{n+1} - b}{2x_n}.$$

In terms of the computation, there will be a problem if $x_n = 0$. To overcome this difficulty, the following simple algorithm is applied. Suppose that the two roots of equation are given by $x_1 + iy_1$ and $x_2 + iy_2$. If $x_1 = \sqrt{\frac{u+v}{2}}$, then $y_1 = \sqrt{u-v}$ if $y > b$, or $y_1 = -\sqrt{u-v}$ if $y < b$. The other root is then given by $x_2 = -\sqrt{\frac{u+v}{2}}$ and $y_2 = -y_1$.

It is not hard to determine the fixed points for mapping $z_{n+1} = f_c(z_n) = z_n^2 + c$. Suppose that z is a fixed point of period one. Then $z_{n+1} = z_n = z$, and

$$z^2 - z + c = 0,$$

which gives us two solutions,

$$z_1 = \frac{1 + \sqrt{1-4c}}{2}$$

or

$$z_2 = \frac{1 - \sqrt{1-4c}}{2}.$$

The stability of these fixed points can be determined in the usual way. Hence the fixed point is stable if for $f_c = z^2 - z + c$

$$\left|\frac{df_c}{dz}\right| < 1,$$

and it is unstable if

$$\left|\frac{df_c}{dz}\right| > 1.$$

As a simple example, let us consider

$$z_{n+1} = z_n^2.$$

This is exactly the same as before, except this time we are in the complex plane. One of two fixed points of the equation lies at the origin. Call it z^*, and there is also a fixed point at $z = 1$. Initial points that start inside the unit circle are attracted to z^*, i.e. an initial point starting on $|z| = 1$ will generate points that again lie on $|z| = 1$. The initial points starting outside the $|z| = 1$ will be repelled to infinity, since $|z| > 1$. Therefore, the circle defines the Julia set J that is a repellor, invariant, and connected. The interior of $|z| = 1$ defines the basin of attraction (or domain of stability) for the fixed point at z^*.

Since we know so much about this simple complex quadratic, it would be nice if we could compare any general complex quadratic. We can in fact do this and show that these two polynomials behave the same dynamically.

Theorem 3 *Any complex quadratic polynomial f is conjugate to exactly one polynomial in the one parameter family $g_c : z \to z^2 + c$.*

Proof. Let $f(z) = az^2 + bz + d$ represent a general quadratic. Let $g_c(z) = z^2 + c$ and define $h(z) = Az + B$, $A, B \neq 0$. From the definition of topological conjugacy, we want to find A and B such that $h \circ f = g \circ h$. The commutativity of this expression depends of the equality of the two equations

$$(h \circ f)(z) = Aaz^2 + Abz + Ad + B,$$

$$(g_c \circ h)(z) = A^2z^2 + 2ABz + B^2 + c.$$

If we set each of the corresponding coefficients of both equations equal to one another, it yields the following three equations

$$Aa = A^2$$

$$Ab = 2AB$$

$$Ad + B = B^2 + c.$$

Solving the first two equations, we see that $A = a$ and $B = b/2$, whence we obtain that f is topologically conjugate to the family g_c, denoted by $f \sim g_c$, via the affine transformation

$$h(z) = az + \frac{b}{2}$$

and on the condition that the third equation be satisfied, i.e.

$$c = ad + \frac{b}{2} - \frac{b^2}{4}.$$

□

In Propositions 2 and 3, we saw that the behavior of Newton's method is different when considering two different types of roots, ones that are critical points and those that are not. What are some other traits that distinguish roots of polynomials? Well, some quadratics have roots that are the same. For example, consider the quadratic $f(z) = z^2 - 6zi - 9$. Here, $f(z)$ can be factored such that we can see its roots $f(z) = (z - 3i)(z - 3i)$. In this case, $f(z)$ does not have two distinct roots, but one root of multiplicity two. Is the behavior of Newton's method different for the roots that are distinct as opposed to the ones that are not distinct? The subsequent lemma illustrates part of the answer.

Lemma 1 *The Newton function of a general quadratic, f, with two distinct roots is conjugate to the Newton function of $g_c : z \to z^2 + c$ for some nonzero c.*

Proof. Denoting Newton's method on $f(z) = az^2 + bz + d$ and Newton's method on $g_c(z) = z^2 + c$ for $c \neq 0$ as the functions

$$N_f(z) = \frac{az^2 - d}{2az + b} \text{ and } N_{g_c}(z) = \frac{z^2 - c}{2z},$$

respectively, let's determine a conjugating map, $k(z) = Az + B$, such that

$$(k \circ N_f)(z) = A\left(\frac{az^2 - d}{2az + b}\right) + B = \frac{Aaz^2 - Ad + 2Bza + Bb}{2az + b}$$

and

$$(N_{g_c} \circ k)(z) = \frac{(Az+B)^2 - c}{2(Az+B)} = \frac{A^2z^2 + 2ABz + B^2 - c}{2(Az+B)}$$

Again, we want to set these two expression equal to one another and solve. This means cross multiplying the two rational functions to get these two expressions:

$$2aA^2z^3 + 6aABz^2 + (4aB^2 + 2bAB - 2dA^2)z + 2bB^2 - 2dAB$$

and

$$2aA^2z^3 + (bA^2 + 4aAB)z^2 + (2aB^2 - 2ac + 2bAB)z + bB^2 - bc.$$

For the z^3 terms we get the tautology $2aA^2 = 2aA^2$. The z^2 term yields

$$\begin{aligned} 6AaB &= 4AaB + A^2b \\ 2AaB &= A^2b \\ B &= \frac{A^2b}{2Aa} = \frac{Ab}{2a}. \end{aligned}$$

We find $c = \frac{A^2d - aB^2}{a}$ by comparing the coefficients of the z terms. Since A is any arbitrary non-zero value, we can choose $A = 1$ to get the general solution $k(z) = z + \frac{b}{2a}$ with $c = -\frac{b^2-4ad}{4a^2}$. □

Let us look at an example of this behavior. First, consider the quadratic $f(z) = z^2 - 2z - 3$ and its conjugate function $g(z) = z^2 - 2.5$. We can see, in Figures 2.1 and 2.2, the rate of convergence for each point around the roots of the polynomial. The picture generated with f and that of g show exactly the same behavior.

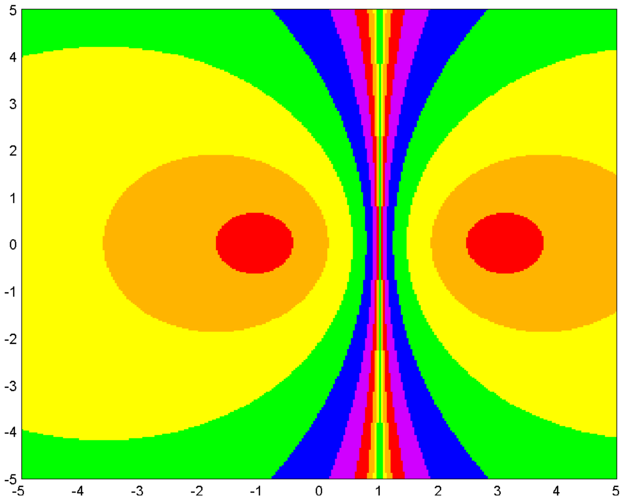

Figure 2.1: Newton's method on the quadratic polynomial $p(z) = z^2 - 2z - 3$.

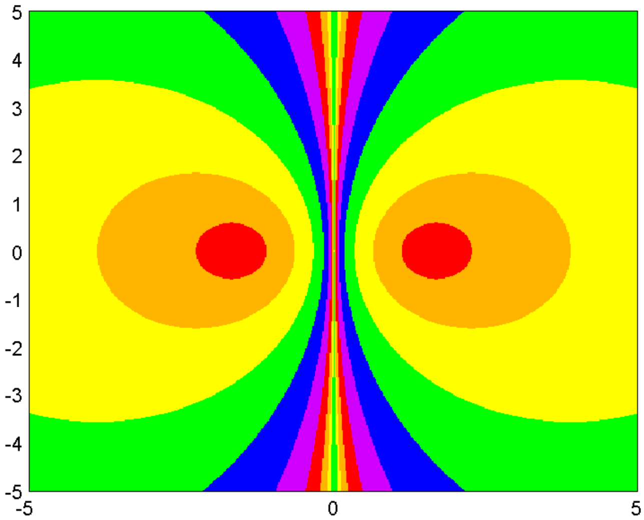

Figure 2.2: Newton's method on the quadratic polynomial $p(z) = z^2 - 2.5$.

We have seen that Newton's method of any general quadratic is conjugate to Newton's method of some polynomial in the one parameter family $z^2 + c$. Can we make this any easier? As a matter of fact, the next lemma takes what we have shown and makes it even simpler.

Lemma 2 *Newton's method on* $g_c : z \to z^2 + c$ *for some nonzero* c *is conjugate to the simple quadratic* $g_0 = z^2 + 0$.

Proof. Recall that $N_{g_c}(z) = \frac{z^2-c}{2z}$. We will express our conjugating map as a Möbius transformation rather than as an affine map. Given $l(z) = \frac{Az+B}{Cz+D}$, we want to find A, B, C, and D such that $l \circ g_0 = N_{g_c} \circ l$. Well,

$$(l \circ N_{g_c})(z) = \frac{Az^2 - Ac + 2Bz}{Cz^2 - Cc + 2Dz} \text{ and } (g_0 \circ l)(z) = \frac{A^2z^2 - 2ABz + B^2}{Cz + D^2}.$$

Cross multiplying these rational functions gives the following

$$AC^2z^4 + (2BC^2 + 2ADC)z^3 + (4BDC - AcC^2 - AD^2)z^2 + 2BD^2 - 2AcDC)z - AcD^2$$

and

$$CA^2z^4 + (2DA + 2CAB)z^3 + (4DAB - CcA^2 + CB^2)z^2 + (2DB^2 - 2CcAB)zCcB^2.$$

One can verify that we get $A = C \neq 0$, $B = \pm D \neq 0$ as general solutions where $c = \frac{-D^2}{C^2}$; however, we exclude the solution $B = D$ since that makes the conjugating map, l, a constant function. Since, $c = \frac{-D^2}{C^2}$, so $\frac{D}{C} = \pm i\sqrt{c}$. For $\frac{D}{C} = \pm i\sqrt{c}$, we get $h_c(z) = \frac{z-i\sqrt{c}}{z+i\sqrt{c}}$ conjugating n_{g_c} to g_0. If $\frac{D}{C} = -i\sqrt{c}$, then our conjugating map is of the form $h_c(z) = \frac{z+i\sqrt{c}}{z-i\sqrt{c}}$. □

In this case, the point we start with in our iteration always converges to the nearest root. Also, the Julia set is the line created with the perpendicular bisector of the two roots. If we tried to "measure" this line, we would see a measure of zero. In other words, there is no girth to the Julia set. This line divides the plane into two symmetric parts. In Chapters 3 and 4, we will see that this is not the case.

We have really simplified the way we look at Newton's method on a general quadratic. The only way that this would be any easier for us is if these functions were conjugate to some linear function. The next theorem exemplifies this.

Theorem 4 *Suppose f has an attracting fixed point at z_0, with multiplier λ satisfying $0 < |\lambda| < 1$. Then there is a conformal map $\zeta = \varphi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates $f(z)$ to the linear function $g(\zeta) = \lambda\zeta$. The* *conjugating function is unique, up to multiplication by a nonzero scaler factor.*

Proof. Suppose $z_0 = 0$. We need to define $\varphi_n(z) = \lambda^{-n} f^n(z) = z + \cdots$. Then φ_n satisfies

$$\varphi_n \circ f = \lambda^{-n} f^{n+1} = \lambda \varphi_{n+1}.$$

Thus if $\varphi_n \to \varphi$, then $\varphi \circ f = \lambda\varphi$, so $\varphi \circ f \circ \varphi^{-1} = \lambda\zeta$, and φ is a conjugation.

To show convergence note that for $\delta > 0$ small

$$|f(z) - \lambda z| \leq C|z|^2, \qquad |z| \leq \delta.$$

Thus, if $|f(z)| \leq |\lambda||z| + C|z|^2 \leq (|\lambda| + C\delta)|z|$, and with $|\lambda| + C\delta < 1$,

$$|f^n(z)| \leq (|\lambda| + C\delta)|z|, \qquad |z| \leq \delta.$$

We choose $\delta > 0$ so small that $\rho = \frac{(|\lambda|+C\delta)^2}{|\lambda|} \leq 1$, and we obtain

$$|\varphi_{n+1}(z) - \varphi_n(z)| = \left| \frac{f^n(f(z)) - \lambda f^n(z)}{\lambda^{n+1}} \right| \leq \frac{c|f^n(z)|^2}{|\lambda|^{n+1}} \leq \frac{\rho^n C|z|^2}{|\lambda|}$$

for $|z| \leq \delta$. Hence, $\varphi_n(z)$ converges uniformly on a neighborhood of z_0 to 0 for $|z| \leq \delta$, and the conjugation exists. □

Now, we take our explorations one step further and examine cubic polynomials, where the behavior is much more complicated.

Chapter 3

Newton's Method Extended

3.1 Cubics

Consider the generic polnomial in Figure 3.1. Here we can see that a particular initial guess need not converge to the nearest root. By Theorem 2 of Chapter 1, the Julia set must be a complicated fractal set when there are more than two basins of attraction.

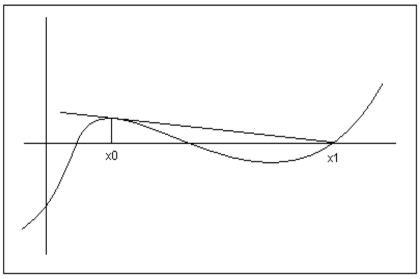

Figure 3.1: Newton's method a generic polynomial with three roots for initial guess x_0.

We know that a polynomial is most generally represented as

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0,$$

which is the exact way we have been viewing the quadratics in the previous chapter.

It is also known that we can view this same polynomial in terms of its roots

$$f(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n)$$

where $r_1, r_2, \ldots, r_n$ are the roots of the polynomial. We can take the same approach when we look at general cubic functions $a_3z^3 + a_2z^2 + a_1z + a_0$ and represent them in the same manner. So, if we first factor out a_3, we now have

$$p(z) = a_3(z - a)(z - b)(z - c)$$

where a, b, c are the roots of p.

We have discovered in previous sections the assumptions needed to say that any quadratic is conjugate to $z^2 + c$. In this section, we will extend this idea to cubic polynomials. In fact, Newton's method on any general cubic polynomial $p : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that has at least two distinct roots will look dramatically the same for quadratics,

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

To simplify the understanding of the dynamical properties of Newton's method on cubic polynomials, we will utilize the one-parameter family $p_\lambda(z) = (z-\lambda)(z+\lambda)(z-1)$ rather than p itself. This still contains cubics with at least two distinct roots, and we will refer to the function created after applying Newton's method as N_λ. This is defined by $N_\lambda(z) = \frac{2z^3-z^2-\lambda^2}{3z^2-2z-\lambda^2}$.

3.2 Conjugacy Classes of Cubics

Recall from Chapter 1, the properties of the Fatou and Julia sets. We saw that they were compliments of one another, and the Julia set was the boundary for the points that converge that those that do not. Since the dynamics of cubic polynomials are more complicated than that of quadratics, we shall be relying more on computer graphics to illustrate their behavior and help give us a better understanding of what is actually happening. The conclusions for quadratics were somewhat easily obtained, and it would be ideal if the same approach could be taken for cubics, along with getting the same sort of results. However, the global conditions of the dynamics of N_λ seem to be much more complex. We will assign a coloring scheme to the points on the dynamical plane and use it for part of our investigation. We do not want to get ahead of ourselves, thus we are going to develop ideas like those in Chapter 2 for a cubic polynomial.

In previous sections, we discussed the idea of being topologically conjugate. Here we must introduce the notion of a cross ratio to find a conjugacy between N_λ and N_p. We are allowed to use this category of maps, since they are the class that is analytical and differentiable on the Riemann sphere. We define the cross-ratio of four distinct points z_0, z_1, z_2, z_3 as the Möbius transformation

$$(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_2 - z_1)},$$

which brings us to the following proposition.

Proposition 4 *For any complex cubic polynomial p with at least two distinct roots, N_p is topologically conjugate to N_λ for some $\lambda \in \mathbb{C}$.*

Proof. Let us assume we have a polynomial p with at least two distinct roots, so we will have to consider two cases for this. The first case is when a, b, c are all distinct. The second, with two distinct roots, follows from the first case. First, let's choose a complex cubic polynomial $p(z) = a_3(z-a)(z-b)(z-c)$. Considering the cross-ratio of the roots of p and the roots of $p_\lambda = (z-\lambda)(z+\lambda)(z-1)$, we can derive a Möbius transformation, call it m, which conjugates N_p to N_λ. We find m by setting $(z, a, b, c) = (w, \lambda, -\lambda, 1)$,

$$\begin{aligned}
\frac{(z-a)(b-c)}{(z-c)(b-a)} &= \frac{(w-\lambda)(-\lambda-1)}{(w-1)(-\lambda-\lambda)} \\
\frac{bz - cz - ab + ac}{bz - az - bc + ac} &= \frac{\lambda w - w + \lambda^2 - \lambda}{-\lambda w - \lambda w + \lambda + \lambda}
\end{aligned}$$

$$\frac{(b-c)z+(ac-ab)}{(b-a)z+(ac-bc)} \quad = \quad \frac{(\lambda-1)w+(\lambda^2-\lambda)}{(-2\lambda)w+(2\lambda)}.$$

Cross multiplying and solving for w yields

$$w = m(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$$

where

$$a_1 = \lambda(b - 2c + a\lambda + a - b\lambda)$$

$$b_1 = \lambda(-2ab + ac - ac\lambda + bc\lambda + bc)$$

$$c_1 = b\lambda - 2c\lambda + a\lambda + a - b$$

$$d_1 = -2ab\lambda + ac\lambda - ac + bc\lambda + bc.$$

This is somewhat tedious to work with, so let's simplify by finding a value of λ for which the map m is an affine map. That is, make the function a linear mapping. We set $c_1 = 0$ we get the following transformation

$$m(z) = \frac{a_1}{d_1}z + \frac{b_1}{d_1},$$

and solve for λ to give us

$$\lambda = \frac{a-b}{2c-(a+b)}.$$

Now, if we substitute this value of λ, we get the affine transformation $m(z) = Az + B$ where

$$A = \frac{2}{2c-(a+b)} \text{ and } B = -\frac{a+b}{2c-(a+b)}$$

In this case, we need $2c \neq a+b$, then m sends $a \to \lambda$, $b \to -\lambda$, and $c \to 1$. It immediately follows that $N_p \sim N_\lambda$ where

$$p(z) = k(p_\lambda \circ m)(z) = 8k\frac{(z-a)(z-b)(z-c)}{(2c-(a+b))^3}$$

and k is the constant $a_3\frac{(2c-(a+b))^3}{8}$. This is topological conjugacy and is easily verified by calculating

$$N_p(z) = m^{-1} \circ N_\lambda \circ m(z) = \frac{2z^3 - cz^2 - bz^2 - az^2 + abc}{3z^2 - 2az - 2bz - 2cz + ab + bc + ca}. \tag{3.1}$$

Here we find $m^{-1}(z)$ by simply using $m(z)$ and obtaining $m^{-1}(z) = (c - \frac{a-b}{2})z + \frac{a+b}{2}$. In the case where $2c = a + b$, we have $c = \frac{a+b+c}{3}$ as a critical point of N_p. The critical point for N_λ is $\frac{1}{3}$, the average of the roots of p_λ. Since $c \to 1$, our transformation m cannot be a conjugacy between N_p and N_λ. Our conjugacy must send the critical point of N_p to the critical point of N_λ. Now, if we solve either the transformations $(z, \frac{a+b}{2}, a, b) = (w, \frac{1}{3}, -\frac{1}{3}, 1)$ or $(z, \frac{a+b}{2}, b, a) = (w, \frac{1}{3}, -\frac{1}{3}, 1)$ for w, we will have the map we so desire. In fact, $w = m(z) = \frac{3a+b-4z}{3(a-b)}$ where equation (3.1) holds for $\lambda = \frac{1}{3}$.

Now, for the second case, let p have only two distinct roots, a and b. By letting $c = a$ in our original equation for c_1, our map reduces to $m(z) = \frac{2z-(a+b)}{a-b}$. This sends the point a to 1 and b to -1. If we choose $\lambda = \pm 1$, m will conjugate N_p to $N_{\pm 1}$ where $p = a_3(z-a)^2(z-b)$. $\square$

The cross ratio produces a transformation that depends on the order of the points z_1, z_2, z_3. So, we can find all the possible conjugacies between N_p and N_λ simply by considering the set of all permutations of the set $A = a, b, c$ which contains the roots of N_λ. We would solve $(z, z_i, z_j, z_k) = (w, \lambda, -\lambda, 1)$ for w which will represent a Möbius transformation such that $w : z_i \to a, z_j \to b, z_k \to c$. We will do this for each element z_i, z_j, z_k in the set of permutations of A.

As for the case where p has only two distinct roots, we have

$$m_a(z) = \frac{z-a}{b-a} \text{ and } m_b(z) = \frac{z-b}{a-b}$$

for $p_a(z) = a_3(z-a)^2(z-b)$ and $p_b(z) = a_3(z-a)(z-b)^2$, respectively. These conjugate

N_p to N_λ for $\lambda = 1$. Also,

$$m_1(z) = \frac{2z}{a+b} - \frac{a+b}{a-b} \text{ and } m_2(z) = \frac{-2z}{a+b} + \frac{a+b}{a-b}$$

conjugate N_p to N_1. These m's corresponding to $p_1(z) = (z-a)^2(z-b)$ and $p_2(z) = (z-a)(z-b)^2$, respectively.

If we have a cubic polynomial with three distinct roots, we will have to consider two cases. The first is where any one root is the average of the other two. In this case, we have the following conjugating maps

$$m_1(z) = \frac{4z}{3(b-c)} - \frac{b+3c}{3(b-c)} \text{ and } m_2(z) = \frac{-4z}{3(b-c)} + \frac{3b+c}{3(b-c)} \text{ for } a = \tfrac{b+c}{2},$$

$$m_3(z) = \frac{4z}{3(a-c)} - \frac{3a+c}{3(a-c)} \text{ and } m_4(z) = \frac{-4z}{3(a-c)} + \frac{3a+c}{3(a-c)} \text{ for } b = \tfrac{a+c}{2},$$

$$m_5(z) = \frac{4z}{3(a+b)} - \frac{a+3b}{3(a-b)} \text{ and } m_6(z) = \frac{-4z}{3(a-b)} + \frac{3a+b}{3(a-b)} \text{ for } c = \tfrac{a+b}{2}.$$

These conjugate N_p to N_λ where $\lambda = \pm\frac{1}{3}$ since $p_{\frac{1}{3}} = p_{-\frac{1}{3}}$.

On the other hand, for any cubic polynomial p having three distinct roots in which no one root is the average of the other two, there are three conjugating maps with six distinct values of λ that conjugate N_p to N_λ. These conjugacies are

$$m_1(z) = \frac{2z}{2c-(a+b)} - \frac{a+b}{2c-(a+b)} \text{ where } 2c \neq a+b$$

$$m_2(z) = \frac{2z}{2c-(b+c)} - \frac{a+b}{2c-(b+c)} \text{ where } 2c \neq b+c$$

$$m_3(z) = \frac{2z}{2c-(a+c)} - \frac{a+c}{2c-(a+c)} \text{ where } 2c \neq a+c$$

with corresponding λ values

$$\lambda_1 = \frac{a-b}{2c-(a+b)}, \lambda_2 = \frac{b-c}{2a-(b+c)}, \lambda_3 = \frac{a-c}{2b-(a+c)}.$$

At this point, let us recall the simple roots of the quadratic $f(z) = az^2 + bz + d$

are in fact the two superattractive fixed points of the rational function of degree two

$$N_f = \frac{az^2 - d}{2az + b}.$$

Also, we have seen that the simple roots of the cubic polynomial

$$p(z) = 8k\frac{(z-a)(z-b)(z-c)}{(2c-(a+b))^3}$$

are indeed the three superattractive fixed points of the rational function of degree three

$$N_p(z) = \frac{2z^3 - cz^2 - bz^2 - az^2 + abc}{3z^2 - 2az - 2bz - 2cz + ab + bc + ca}.$$

We now come to the following proposition.

Proposition 5 *If R is a rational function of degree n with exactly n distinct superattractive fixed points, then $R \sim N_p$ for an nth-degree polynomial p.*

Proof. First, let us define $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, where $R(z) = z - \frac{p(z)}{q(z)}$ and p and q share no common factors. Also, let $\rho_i \in \mathbb{C}$ for $i = 1, 2, ..., n$ be the n distinct superattractive fixed points of R. If ρ_i is fixed for R then, $\frac{p(\rho_i)}{q(\rho_i} = 0$. We know, since ρ_i is a root of our polynomial p, that $(z-\rho_i)$ divides p for $i = 1, 2, ..., n$. Also, since ρ_i is a superattractive fixed point for R, then $R^{'}(\rho_i) = 0$. Given that $R^{'} = 1 - \frac{qp^{'} - pq^{'}}{q^2} = \frac{q^2 - qp^{'} + pq^{'}}{q^2}$, this tells us that $(z - \rho_i)$ divides $q^2 - qp^{'} + pq^{'}$. Furthermore, since we know that $(z-\rho_i)$ divides p, and thus divides $pq^{'}$, we also know that it divides $q^2 - qp^{'} + pq^{'} - pq^{'} = q(q - p')$. Moreover, p and q have no common factors and $(z - \rho_i)$ divides p, so $(z - \rho_i)$ divides $(q - p^{'})$ which implies that $p^{'}(\rho_i) = q(\rho_i)$ for $i = 1, 2, ..., n$.

We know from assumption that the $deg(R) = n$. Starting with this fact we get the following implications

$$deg(R) = n \Longrightarrow$$

$$deg(zq - p) = n \Longrightarrow$$

$$deg(q) \leq n \Longrightarrow$$

$$deg(zq) \leq n + 1 \Longrightarrow$$

$$deg(p) \leq n + 1.$$

This gives us two possibilities for the powers of p and q. If $deg(p) = n + 1$, then $deg(zq) = n + 1$ and $deg(q) = n$. In this case, let us assume that R is Newton's method for p and let the degree of p be $n + 1$ where

$$p = a_{n+1}z^{n+1} + a_n z^n + ... + a_1 z + a_0.$$

Thus,

$$R(z) = z - \frac{a_{n+1}z^{n+1} + a_n z^n + ... + a_1 z + a_0}{(n+1)a_{n+1}z^n + (n)a_n z^{n-1} + ... + a_1}$$

$$= \frac{(n+1)a_{n+1}z^{n+1} + (n)a_n z^n + ... + a_1 z}{(n+1)a_{n+1}z^n + (n)a_n z^{n-1} + ... + a_1} - \frac{a_{n+1}z^{n+1} + a_n z^n + ... + a_1 z + a_0}{(n+1)a_{n+1}z^n + (n)a_n z^{n-1} + ... + a_1}$$

$$= \frac{na_{n+1}z^{n+1} + (n-1)a_n z^n + ... + a_2 z^2 - a_0}{(n+1)a_{n+1}z^n + (n)a_n z^{n-1} + ... + a_1}.$$

However, $deg(R) = n$ implies that the first term must be disappear, i.e. $na_{n+1} = 0$, but in either case, $n = 0$ or $a_{n+1} = 0$, we get a contradiction to the fact that $deg(p) = n+1$.

On the other hand, if $deg(p) = n$, then $deg(zq) = n$ and $deg(q) < n$. Recall that if we have two distinct points, then we are able to determine a unique line in the plane. Furthermore, if we have three distinct points, we can determine a unique quadratic. This is true for any number n. So, we make the generalization that if we have n distinct points that we will determine a unique polynomial of degree $n - 1$. Since $p^{'}$ and q have n distinct point in common, with $deg(q) < n$ and $deg(p^{'}) = n - 1$, we see that $p^{'} = q$. Thus, $R(z) = z = \frac{p}{q}$ is Newton's method for p. $\square$

At last, we are ready to apply our knowledge to a specific function, examine it closely, and reveal our findings.

Chapter 4

Graphical Analysis

4.1 The Symmetry of N_λ

Before we dive into our discussion of N_λ, let us first define a triangle created by the roots of N_λ when $\lambda \in \mathbb{C} - \mathbb{R}$. We will denote this triangle T_λ, where each side of T_λ is determined by any pair of distinct roots. Note that if $\lambda \in \mathbb{R}$, then all the roots will lie on the real axis and thus will be collinear. We will be able to infer some things about N_λ just by looking at this triangle. We can even generalize the following property to generic cubic polynomials.

Proposition 6 *If p and q are cubic polynomials whose roots form similar triangles T_p and T_q, respectively, then N_p is conjugate to N_q via some affine map.*

Proof. For this proof, we will need to recall a theorem from complex analysis which states any Möbius transformation can be represented as the composition of a finite number of inversions, created by the function $v(z) = \frac{1}{z}$, magnifications, created by the function $m(z) = Az$ where $A \in \mathbb{R}^+$, rotations, created by the function $r(z) = e^{i\theta} z$ where $\theta \in \mathbb{R}$, and translations, created by the function $t(z) = z + B$ where $B \in \mathbb{C}$. Consider the geometric representation of the two triangles T_p and T_q. We only need to make three manipulations to achieve our goal. Recall a linear transformation is of the form

$$g(z) = az + b, \text{ where } a, b \in \mathbb{C} \text{ and } a \neq 0.$$

Now, let us consider

$$\begin{aligned} g(z) &= (m \circ r \circ t)(z) \\ &= m(r(z + c) \\ &= m(e^{i\theta}(z + c)) \\ &= A(e^{i\theta}(z + c)) \\ &= Ae^{i\theta}z + A(e^{i\theta}c) \\ &= A_1 z + B_1 \end{aligned}$$

where $A_1 = Ae^{i\theta}$ and $B_1 = Ae^{i\theta}c$. Each of these mappings are non-fractional, linear mappings, thus $g(z)$ is conformal for all points in the complex plane. With a few calculations, one can see that the angles of T_p are preserved under each mapping. This is independent from our choice of c, θ, and A. Therefore, we can choose these such that g will conjugate T_p to T_q. Thus implying that p conjugates to q, and from our previous knowledge, making N_p conjugate to N_q through this affine map. □

The following pictures illustrates the rate of convergence for each point in the plane with a different color. We can see that the points form a right triangle in the plane. Thus, any roots from another polynomial that form a right triangle will be conjugate to this polynomial.

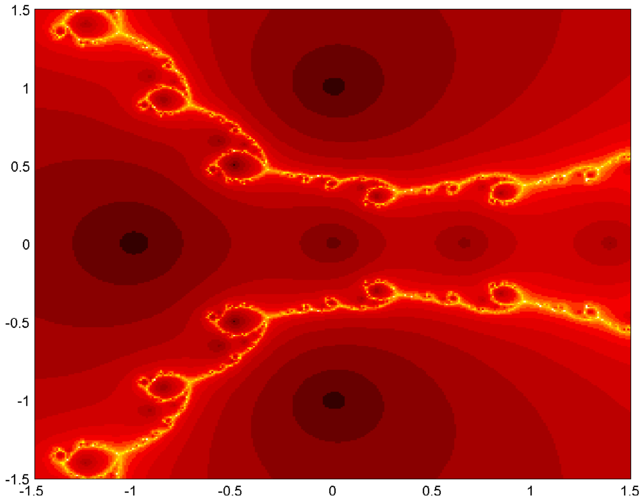

Figure 4.1: Newton's method on the cubic polynomial $p(z) = z^3 + z^2 + z + 1$. Here, λ = i.

Not only can the function created with the Newton's method of our particular family of cubic polynomials be examined by the triangles created with the roots, even when we do not see a triangle, N_λ exhibits some very nice properties. In fact, what we realize is that the roots of the triangle have become collinear and form a line the plane.

This exemplifies the following proposition.

Proposition 7 *If $\lambda \in \mathbb{R}$, then N_λ is symmetric with respect to the real axis.*

Proof. For this proof, all we need to show is that for the two complex roots of the function

$$Re(N_a(z)) = Re(N_a(\overline{z})) \text{ and } Im(N_a(z)) = Im(N_a(\overline{z}))$$

$$Re(N_a(a + bi) = Re(N_a(a - bi)) \text{ and } Im(N_a(a + bi)) = -Im(N_a(a - bi)).$$

In this case, we call our function N_a since we are assuming that we have $\lambda \in \mathbb{R}$. Refer back to our original Newton function,

$$N_\lambda = \frac{2z^3 - z^2 - \lambda^2}{3z^2 - 2z - \lambda^2},$$

Now, if we substitute $z = x + yi$ and $\lambda = a$ and simplify, we get

$$\begin{aligned} N_a(z) &= \frac{2(x+yi)^3 - (x+yi)^2 - a^2}{3(x+yi)^2 - 2(x+yi) - a^2} \\ &= \frac{2(x^3 + 3x^2yi + 3x(yi)^2 + (yi)^3) - (x^2 + 2xyi + (yi)^2) - a^2}{3(x^2 + 2xyi + (yi)^2 - 2(x+yi) - a^2} \\ &= \frac{2x^3 + 6x^2yi - 6xy^2 - 2y^3i - x^2 - 2xyi - y^2 - a^2}{3x^2 + 6xyi - 3y^2 - 2x - 2yi - a^2} \\ &= \frac{(2x^3 - 6xy^2 - x^2 - y^2 - a^2) + (6x^2y - 2y^3 - 2xy)i}{(3x^2 - 3y^2 - 2x - a^2) + (6xy - 2y)i} \end{aligned}$$

With the same approach, let us examine $\overline{z} = a - bi$

$$\begin{aligned} N_a(\overline{z}) &= \frac{2(x-yi)^3 - (x-yi)^2 - a^2}{3(x-yi)^2 - 2(x-yi) - a^2} \\ &= \frac{2(x^3 - 3x^2yi + 3x(yi)^2 - (yi)^3) - (x^2 - 2xyi + (yi)^2) - a^2}{3(x^2 - 2xyi + (yi)^2 - 2(x-yi) - a^2} \\ &= \frac{2x^3 - 6x^2yi - 6xy^2 + 2y^3i - x^2 + 2xyi - y^2 - a^2}{3x^2 - 6xyi - 3y^2 - 2x + 2yi - a^2} \\ &= \frac{(2x^3 - 6xy^2 - x^2 - y^2 - a^2) + (-6x^2y + 2y^3 + 2xy)i}{(3x^2 - 3y^2 - 2x - a^2) + (-6xy + 2y)i} \end{aligned}$$

This is very cumbersome to work with so let us make a substitution to make our equations much easier to manipulate. Consider the following substitutions

$$r_1 = 2x^3 - 6xy^2 - x^2 - y^2 - a^2$$

$$s_1 = -6x^2y + 2y^3 + 2xy$$

$$r_2 = 3x^2 - 3y^2 - 2x - a^2$$

$$s_2 = -6xy + 2y.$$

So, we have that

$$N_a(z) = \frac{r_1 - s_1 i}{r_2 - s_2 i} \text{ and } N_a(\overline{z}) = \frac{r_1 + s_1 i}{r_2 + s_2 i}.$$

Multiplying the top and bottom by the complex conjugate of each denominator, we achieve

$$N_a(z) = \frac{r_1r_2 + s_1s_2}{r_2^2 + s_2^2} + \frac{(-r_2s_1 + r_1s_2)}{r_2^2 + s_2^2}i$$

$$N_a(\overline{z}) = \frac{r_1r_2 + s_1s_2}{r_2^2 + s_2^2} + \frac{(r_2s_1 - r_1s_2)}{r_2^2 + s_2^2}i$$

With this representation it is easy to see that the real parts of $N_a(z)$ and $N_a(\overline{z})$ are in fact equal, and the corresponding imaginary parts are complex conjugates. □

Back to the feature of N_λ that we have discussed about the appearance of a triangle created from the roots of p_λ when $\lambda \in \mathbb{C} - \mathbb{R}$. We viewed this triangle before in Figure 3.2. We have the following proposition in which T_λ tells us something about N_λ.

Proposition 8 *If T_λ is an isosceles triangle, then N_λ is symmetric with respect to the line determined by the median of T_λ which divides T_λ into two equal parts.*

Proof. If we have an isosceles triangle, then we can say that any point in the plane z_0 is of equal distance from exactly two roots of the polynomial. For this, we must consider the following three cases of an isosceles triangle T_λ, 1) $|\lambda - z_0| = |-\lambda - z_0|$, 2) $|1 - z_0| = |\lambda - z_0|$, and 3) $|1 - z_0| = |-\lambda - z_0|$.

Let us begin with case one. Well, we start with $\lambda = a + bi$ and see that

$$\begin{aligned} |\lambda - z_0| &= |-\lambda - z_0| \\ \sqrt{(a - \frac{1}{3})^2 + b^2} &= \sqrt{(-a - \frac{1}{3})^2 + b^2} \\ (a - \frac{1}{3})^2 + b^2 &= (-a - \frac{1}{3})^2 + b^2 \\ a - \frac{1}{3} &= \pm(a + \frac{1}{3}) \end{aligned}$$

This gives us the two equations $a - \frac{1}{3} = a + \frac{1}{3}$ and $a - \frac{1}{3} = -a - \frac{1}{3}$. Thus, $a = 0$ and $b \in \mathbb{R}$. Here, we find that $\lambda = bi$ lies along the imaginary axis, and the real axis contains the median that divides the two equal sides of T_λ. So, this comes down to showing in the same manner as in Proposition 7 that

$$Re(N_{bi}) = Re(N_{bi}(\overline{z})) \text{ and } Im(N_{bi}(z)) = Im(N_{bi}(\overline{z})).$$

In fact, we have

$$N_{bi}(z) = \frac{(2x^3 - 6xy^2 - x^2 + y^2 + b^2) + (6x^2y + 2y^3 + 2xy)i}{(3x^2 - 3y^2 - 2x + b^2) + (6xy - 2y)i}$$

and

$$N_{bi}(\overline{z}) = \frac{(2x^3 - 6xy^2 - x^2 + y^2 + b^2) + (-6x^2y + 2y^3 + 2xy)i}{(3x^2 - 3y^2 - 2x + b^2) + (-6xy + 2y)i}$$

which satisfy the given equations, because they are complex conjugates.

Now, look at case two. Again, let us start with

$$\begin{aligned} |1 - z_0| &= |\lambda - z_0| \\ \sqrt{(1 - \frac{1}{3})^2 + 0^2} &= \sqrt{(a - \frac{1}{3})^2 + b^2} \\ (\frac{2}{3})^2 &= (a - \frac{1}{3})^2 + b^2 \end{aligned}$$

What we have now is the equation for a circle centered at the point $(\frac{1}{3}, 0)$ with radius $\frac{2}{3}$. For each λ not on the real or imaginary axis, there exist two distinct values of γ such that $Re(\gamma) = 0$ and T_λ is similar to T_γ. This follows from the fact that we have six distinct values of λ which are conjugate to each other and three distinct sets whose members represent λ values for which one root is equidistant to any other two. These three sets consist of two circles centered at $\frac{1}{3}$ and $-\frac{1}{3}$ both with radius $\frac{2}{3}$. Thus, by proposition 6, N_λ conjugate to N_γ. Since γ lies on the imaginary axis, we have by proposition 7 that N_γ is symmetric with respect to the real axis. Therefore, N_λ is symmetric with respect to the median of T_λ which divides the triangle into two equal parts.

Finally, we look at case three where

$$\begin{aligned} |1 - z_0| &= |-\lambda - z_0| \\ \sqrt{(1-\frac{1}{3})^2 + 0^2} &= \sqrt{(a+\frac{1}{3})^2 + b^2} \\ (\frac{2}{3})^2 &= (a+\frac{1}{3})^2 + b^2 \end{aligned}$$

This case is analogous to case 2. There are corresponding distinct values of γ along the imaginary axis such that N_λ is conjugate to N_γ where N_γ is symmetric with respect to the real axis. Therefore, N_λ is symmetric with respect to the median of T_λ into two congruent triangles. □

What other types of triangles are there? Perhaps a more interesting question would be "do other types of triangles imply something about the Newton function to which they correspond?" As a matter of fact, there are other implications to be made.

Proposition 9 *If T_λ is an equilateral triangle then N_λ is symmetric with respect to each median of T_λ.*

Proof. Solving the equation $z = 1$, we get the cube roots of unity. Define $\mu : z \rightarrow z^3 - 1$ and calculate

$$N_\mu(z) = z - \frac{z^3 - 1}{3z^2} = \frac{2z^3 + 1}{3z^2}.$$

It is easily verified that for $A(z) = ze^{\frac{2}{3}\pi i}$

$$N_\mu(z) = A \circ N_\mu \circ A^{-1}(z).$$

We then can apply proposition 6, the proposition that states if two cubics have similar triangles, then their Newton functions are conjugate to each other. Hence, N_λ is symmetric about each median of T_λ. □

We also have triangles that are scalene.

Proposition 10 *If T_λ is scalene, then N_λ is not symmetric with respect to any line in the plane.*

Proof. Let us start with a line l in the plane. Also, let us assume that the roots of p_λ are distinct and form a triangle T_λ. We want to see that the line l will separate the plane into two distinct hemispheres which do not contain the same number of roots, in other words, are not symmetric. Consider on the hemisphere with the greater number of roots. We can find a point $p \in l$ that is not a root of p_λ such that p and one of the roots r are equidistant to any particular point on l. However, $N_\lambda(r)$ is a fixed point of N_λ and $N_\lambda(p)$ is a not fixed point of N_λ. Thus, they cannot be equidistant to any point in l. In this case, the only way that symmetry will occur is if we have the situation where the line l splits the plane into two hemispheres that contain exactly one root. Let k be the line determined by the roots of p_λ which are not in l. If l does not intersect k orthogonally then for each root $r \in k$ there exists a point $r^{'}$ not in k such that $r^{'}$ and r are equidistant to any point in l, but $N_\lambda(r)$ and $N_\lambda(r^{'})$ are not equidistant to any point in l. If l intersects k orthogonally then l does not intersect the midpoint of the line segment determined by the roots in k since T_λ is scalene. So the two roots in k are not equidistant to any point in l. This means for each $r \in k$ there exists a point $r^{'} \in k$ which is not a root of p_λ such that r and $r^{'}$ are equidistant to any point in l, but $N_\lambda(r)$ and $N_\lambda(r^{'})$ are not equidistant to any point in l. □

Now, let us dive into a closer examination of N_λ.

4.2 Newton's Method for Cubic Polynomial N_λ

In Chapter 3, we proved that any complex cubic polynomial with at least two distinct roots is topologically conjugate to

$$N_\lambda = \frac{2z^3 - z^2 - \lambda^2}{3z^2 - 2z - \lambda^2}$$

where $\lambda = \frac{a-b}{2c-(a+b)}$ is the only parameter with a, b, c as the coefficients of our cubic polynomial. We have also discovered the graphical representation of the behavior of quadratic polynomials in Chapter 2. This Newton function, N_λ, is that in which we will be referring to for the remainder of the discussion. We want to utilize the ability of computer graphics created with the program MatLab which will enable us to describe and visualize the dynamics of cubic polynomials.

Recall from Chapter 3 the proof of Proposition 4. We found in our calculations that there is an extra critical point for $z = \frac{1}{3}$. It is this extra critical point that allows us to view the parameter space where we plot $N_\lambda(\frac{1}{3})$ for each value of λ. Since we have already chosen our function, the question that naturally follows is "what is the behavior of N_λ for different values of λ?"

Unlike the spaces created with a rainbow coloring, which simply indicate the rate of convergence for points, we will color the parameter space with only three colors. Each color will correspond to the root to which that point is attracted. A picture of the space created with these criteria is shown in figure 4.1.

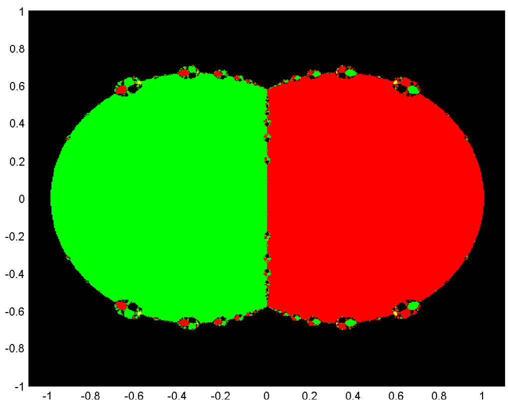

Figure 4.2: The parameter space created with $z = \frac{1}{3}$.

In these cases, black represents points attracted to 1, red represents those points attracted to λ, and green corresponds to points that are attracted to $-\lambda$. We can see that there are two "circles" in the plane. One centered at $\frac{1}{3}$ with radius $\frac{2}{3}$, and the other centered at $-\frac{1}{3}$ with radius $\frac{2}{3}$. We can zoom in and investigate the behavior of the clusters that exist on the perimeter of these "circles".

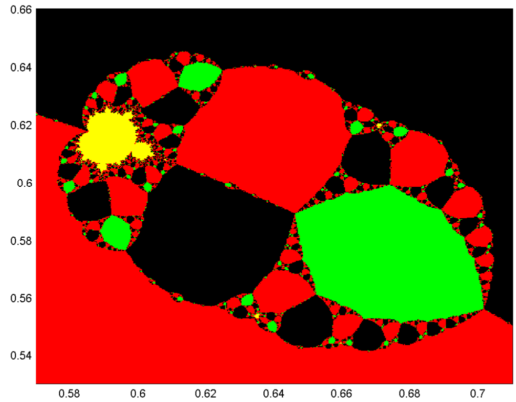

Figure 4.3: Excerpt from parameter space figure.

In this figure, we have a new color that we did not see before we zoomed in on the figure. Let us again zoom in to examine closely the shape of this area.

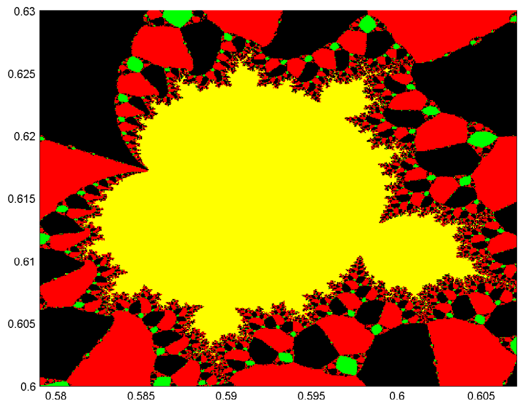

Figure 4.4: The set of periodic points from our parameter space.

From previous experience, we know that the shape of this figure is that of the Mandlebrot set. This raises the question: “How do the points that are colored yellow behave?”

Now, to investigate this question about N_λ, let us focus on the behavior of a particular point in the plane. The Julia set of a complex function is the set of all points on the boundary between the set of points that escape to infinity and the set of points that do not escape to infinity. We see this with the following example.

Consider the value $\lambda = .589 + .605i$. This is a value located in the Mandelbrot set of our parameter space. Let's look at the convergence of all the points under Newton's method. We see this illustration is Figure 4.5.

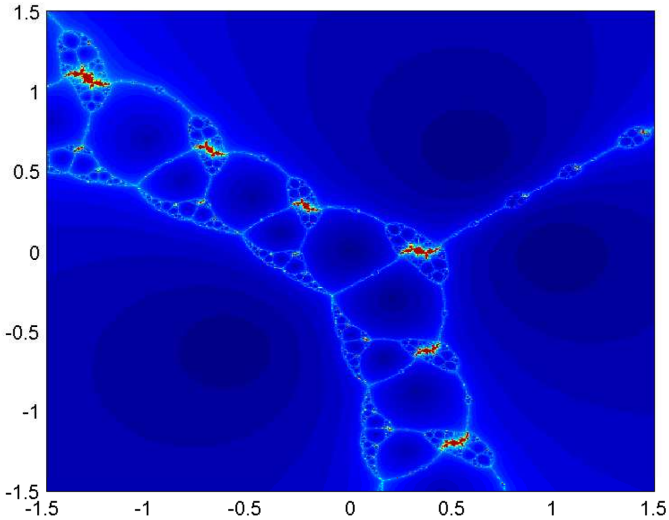

Figure 4.5: N_λ for $\lambda = .589 + 605i$ from -1.5 to 1.5.

Now, we zoom in on the portion in the center that looks like it is glowing. As we zoom in, we see this part taking shape. It looks like a rabbit. This is our Julia set for the particular polynomial created with $\lambda = .589 + .605i$.

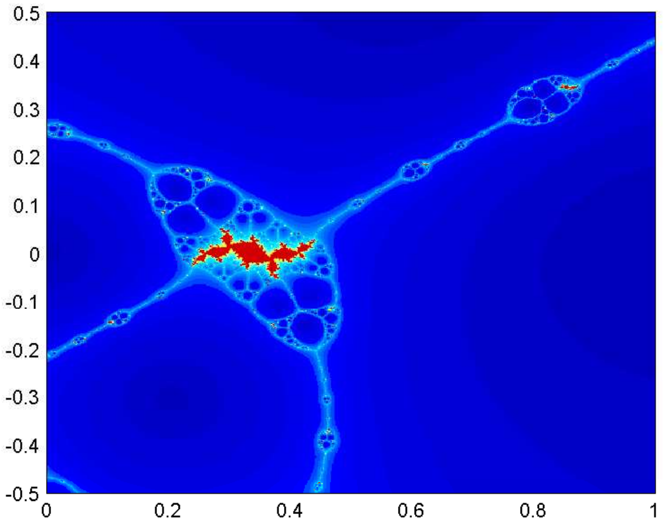

Figure 4.6: N_λ for $\lambda = .589 + .605i$ from 0 to 1.5.

The rabbit is not the only figure that we can see with a particular value of λ. We also gets pictures that illustrate parts of nature like elephants, spiders, and many more.

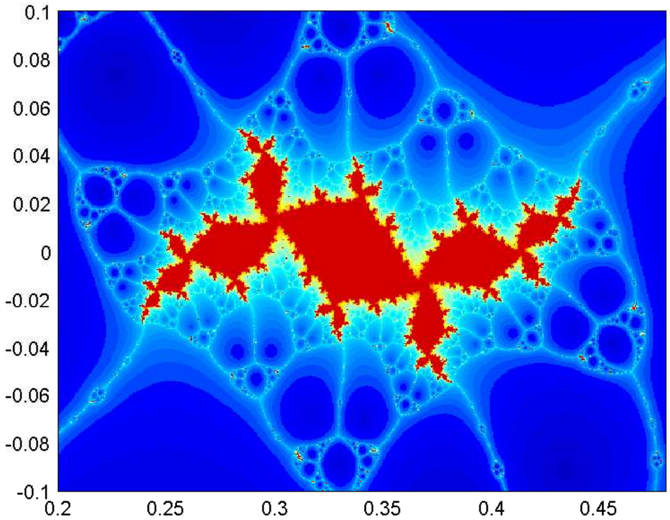

Figure 4.7: "The Rabbit": N_λ for $\lambda = .589 + .605i$ from 0.2 to .48.

We have accomplished our goal of viewing the dynamics of cubic polynomials after the iterations of Newton's method. The graphical nature of the iterations gave us very nice properties that allow us to describe the behavior of the points in the plane. This leads us to ponder a question about repeating the same investigation with higher degree polynomials. Following these investigations, would we be able to make assumptions and develop theory for polynomials of nth degree? In the future, this is a question that deserves some attention.

Appendix A

MatLab Code

A.1 Newton's Method on Quadratic Polynomials

```
%Shannon Miller
%Marshall University
%2006

%This program will plot convergence of the values for the Newton
%function of a quadratic polynomial with a particular coefficients.
%Simply change the values for the min and max of the x axis and y
%axis to zoom in and/or out.  Figures 2.1 and 2.2 with
%lambda = .589 +.605i.

function NMQ(a,b,c)

% default setting
if (nargin < 7)
    min_re = -5;
    max_re = 5;
    min_im = -5;
    max_im = 5;
```

```
    n_re = 500;
    n_im = 500;
    tol = 0.01;
end
coeff = [a b c];
polyRoots = roots(coeff)
format compact;
max_steps = 10;

% stepsize
delta_re = (max_re-min_re)/n_re; delta_im = (max_im-min_im)/n_im;
x = min_re:delta_re:max_re; y = min_im:delta_im:max_im;
[X,Y] =
meshgrid(x,y); Z = X + i*Y;
for j = 1:n_im + 1
    for k = 1:n_re + 1    % one pixel z = (j,k)
        z = Z(j,k);
        if z == 0
            z = tol;
        end
        m = 0;
        flag = 0;
        while (flag == 0)
            % iteration
            z = z - (a*z.^2 + b*z + c)./(2*a*z + b);
            if norm(a*z.^2 +b*z +c) <= tol
                % we are next to a zero
                flag = 1;
            end
```

```
            if m > max_steps
                flag = 1;
            end
        m = m + 1;
        end
      % assign color according to number of steps
        Z(j,k) = m;
    end
end
% plot the result
%colormap(hot);
colormap(prism(10)); brighten(0.5);
%image(Z)
pcolor(X,Y,Z)
%axis off;
shading flat;
```

A.2 Newton's Method on Cubic Polynomials

```
%Shannon Miller
%Marshall University
%2006

%This program will plot convergence of the values for the Newton
%function of a cubic polynomial for any choice of coefficients.
%Simply change the values for the min and max of the x axis and y
%axis to zoom in and/or out.  Figure 4.1 illustrates.

function NMC(a,b,c,d)
```

```
% default settings
    min_re = -1.5;
    max_re = 1.5;
    min_im = -1.5;
    max_im = 1.5;
    n_re = 300;
    n_im = 300;
    tol = 0.01;

coeff = [a b c d]; polyRoots = roots(coeff)
format compact;
max_steps = 20;
% stepsize
delta_re = (max_re-min_re)/n_re; delta_im = (max_im-min_im)/n_im;
x = min_re:delta_re:max_re; y = min_im:delta_im:max_im;
[X,Y]=meshgrid(x,y); Z = X + i*Y;
for j = 1:n_im + 1
    for k = 1:n_re + 1    % one pixel z = (k,j)
        z = Z(j,k);
        if z == 0
            z = tol;
        end
        m = 0;
        flag = 0;
        while (flag == 0)
            % iteration
            z = z - (a*z.^3 + b*z.^2 + c*z +d)./(3*a*z.^2 + 2*b*z + c);
            if norm(a*z.^3 + b*z.^2 + c*z +d) <= tol
                % we are next to a zero
```

```
                flag = 1;
            end
            if m > max_steps
                flag = 1;
            end
        m = m + 1;
        end
      % assign colour according to number of steps
        Z(j,k) = m;
    end
end
% plot the result
colormap(hot);
%colormap(prism(20));
brighten(0.5);
%image(Z)
pcolor(X,Y,Z)
%axis off;
shading flat;
```

A.3 Newton's Method on Lambda Cubic

```
%Shannon Miller
%Marshall University
%2006

%This program will plot the rate convergence for values of the Newton
%function of a cubic polynomial with a particular value of lambda.
%Simply change the values for the min and max of the x axis and y
%axis to zoom in and/or out.  Figures 4.5, 4.6 and 4.7 with
```

```
%lambda = .589 +.605i.

function NMNLambda(lambda)

% default setting
if (nargin < 7)
    min_re = 1.5;
    max_re = 1.5;
    min_im = -1.5;
    max_im = 1.5;
    n_re = 200;
    n_im = 200;
    tol = 0.01;
end
%forms x and y vectors of n points between min and max default values
x = linspace(min_re, max_re, n_re); y = linspace(min_im, max_im,
n_im);
format compact;
max_steps = 50;
% stepsize
[X,Y] = meshgrid(x,y); Z = X + i*Y;
for j = 1:n_im
    for k = 1:n_re     % one pixel z = (k,j)
        z = Z(j,k);
        if z == 0
            z = tol;
        end
        m = 0;
        flag = 0;
```

```
        while (flag == 0)
            % iteration
            z = z - (z.^3 - z.^2 - (lambda.^2)*z + lambda.^2)./
                                        (3*z.^2 - 2*z - lambda.^2);
            if norm(z.^3 - z.^2 - (lambda^2)*z + lambda^2) <= tol
                % we are next to a zero
                flag = 1;
            end
            if m > max_steps
                flag = 1;
            end
        m = m + 1;
        end
      % assign colour according to number of steps
        Z(j,k) = m;
    end
end
% plot the result
%colormap(hot);
colormap(jet(50)); brighten(0.5);
%image(Z)
pcolor(X,Y,Z)
%axis off;
shading flat;
```

A.4 Parameterspace

```
%Shannon Miller
%Marshall University
%2006
```

```
%This program will plot the convergence of z = 1/3 for a particular
%value of lambda. Simply change the values for the min and max of the
%x axis and y axis to zoom in and/or out.  Figures 4.1, 4.2 and 4.3
%show this.


function parameterspaceTest(iter)

% default setting
    min_re = 1.3;
    max_re = -1.3;
    min_im = 1;
    max_im = -1;
    n_re = 500;
    n_im = 500;
    tol = 0.0001;
format compact;
%forms x and y vectors of 300 points between min and max default values
x = linspace(min_re, max_re, n_re); y = linspace(min_im, max_im,
n_im);
%forms n_re x n_im matrix of x + iy values
[X,Y] = meshgrid(x,y); Lambda = X + i*Y; z = ones(n_re);
z=z*(1/3);
for m = 1:iter
    z = z - (z.^3 - z.^2 - Lambda.^2.*z +
Lambda.^2)./(3*z.^2 - 2*z - Lambda.^2); end

green = abs(z - Lambda) < tol;
```

```
green = green*6;
red=abs(z+Lambda)<tol;
red = red*9;
black = abs(z - 1) < tol;
black = black*13; colors = zeros(n_re) + green + red + black;
yellow = colors == 0; colors = colors + yellow; colors(1, :) = 16;

%set the color map
colormap(hsv); brighten(0.5);
%Map the Lambda matrix to the color map
pcolor(X,Y,colors) shading flat;
```

Bibliography

[1] Blanchard, Paul.,"Complex Analytic Dynamics on the Riemann Sphere," *Bulletin of the American Mathematical Society*, **Vol. 11** (1984), 85–144.

[2] Carleson, Lennart and Gamelin, Theodore W., *Complex Dynamics*, Springer-Verlag , 2005.

[3] Devaney, Robert L., *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, 2003.

[4] Devaney, Robert L., *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics*, Addison-Wesley Publishing Co., 1990.

[5] Gilbert, William J., "Generalizations of Newton's Method," *Fractals*, **Vol. 9, No. 3**, (2001), 251–262.

[6] Hubbard, J., Schleicher, D., and Sutherland, S., "How to find all roots of complex polynomials by Newton's Method," *Inventiones mathematicae*, **Vol. 146** (2003), 1–33.

[7] Lynch, Stephen, *Dynamical Systems with Applications Using MatLab*, Birkhäuser, 2004.

[8] Milnor, John, *Dynamics in One Complex Variable*, F. Vieweg and Sohn, 1999.

[9] Peitgen, H., Jürgens, H., and Saupe, D., *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, 1992.

[10] Roberts, Gareth E. and Horgan-Kobelski, Jeremy, "Newton's versus Hayley's Method: A Dynamical Systems Approach," *International Journal of Bifurcation and Chaos*, **Vol. 14 No. 10**, (2004), 3459-3475.

[11] Saff, E. B., and Snider, A. D., *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, Prentice-Hall, Inc., 1976.