2008

# A study of present value maximization for the monopolist problem in time scales

Keshav Prasad Pokhrel

# A STUDY OF PRESENT VALUE MAXIMIZATION FOR THE MONOPOLIST PROBLEM IN TIME SCALES

Keshav Prasad Pokhrel

Thesis submitted to the Graduate College of Marshall University in partial fulfillment of the requirements for the degree of

Master of Arts

in

Mathematics

Dr. Bonita A. Lawrence, Ph.D., Committee Chairperson

Dr. Ralph Oberste-Vorth, Ph.D., Committee member

Dr. Basant Karna, Ph.D., Committee member

Department of Mathematics

Huntington, West Virginia

July 2008

**PRESENT VALUE MAXIMIZATION OF MONOPOLIST IN TIME SCALES**

**Keshav Prasad Pokhrel**

## ABSTRACT

The present value of the investment of the monopolist for the continuous case is given by $PV = \int_0^\infty e^{-rt} q(t)p(t)dt$ with demand condition $p(t) = f(t) - q(t) - a_1 q'(t) - a_2 q''(t)$,(C), and the present value for the discrete case is $PV = \sum_0^\infty \beta^t q_t p_t$ with demand condition $p_t = f_t - q_t - \alpha q_{t-1}$, (D), provided $0 < \beta \leq 1$.

We will discuss various conditions and possibilities of maximization of present value of a monopolist. Basically we are focused on the Continuous (C) and Discrete (D) cases. In the (C), there is an exponential approach of growth if and only if $a_2 \neq 0$. The boundary conditions in (C) generate some mathematical issues. The first derivative of the quantity $q'(t)$ has finite jump at $t = 0$. If $a_2 = 0$ then the jump is similar to the jump of $q(t)$ at $t = 0$. If the sufficient condition for the C problems are satisfied, then the demand equation is unstable. Finally, in (C) the maximum positive discount rate depends on $a_1$ and $a_2$ that yields finite maximum present value.

In (D) we do not need to have any adjustment as long as $\alpha \neq 0$. The sufficient conditions for maximum present value are satisfied for all $t \in (0, 1)$. The optimal path is uniquely determined by the boundary condition and the choice of discount factor $\beta$. The stability of discount factor is a major player in (D) problem. The stable demand condition implies the existence of bounded finite maximum present value for all $\beta \leq 1$.

General results for time scales with right dense points are established. A study of issues that arise when unifying (C) and (D) is included in the analysis.

## Acknowledgement

I would like to gratefully acknowledge the support of my advisor and committee chair Dr. Lawrence. I appreciate her passion and dedication for work. My sincere gratitude goes to Dr. Bohner. Thank you for piquing the problem of my interest.

I would also like to thank Dr. Oberste- Vorth and Dr. Karna for agreeing to serve on my committee. Apart from from subject matter Dr. Karna's help in technical and social aspects is unforgettable. In every likelihood I am proud to be a part of this family.

I am greatly indebted to the various writers in economics, differential equations, calculus of variation and time scales. I have been influenced by various researchers. My genuine appreciation to all the researchers whose work has helped me directly or indirectly to prepare this paper. I take this opportunity to thank Marshall University Graduate School for summer thesis research grant.

# Contents

## 1. Introduction

The development of science and technology is an indispensable tool for a prosperous society. It is widely believed that the development of technology has a high degree of correlation with the development of mathematics. The development of technology is expanding the horizon of mathematics to a wider range. The rigorous involvement of mathematicians in the development of medical science, computer science, business, economics, and finance has given an ample amount of opportunity and challenges for those who want to be involved in the field of mathematics and related areas. Nowadays an imbalance between demand and supply of gasoline is a political and financial hot topic in the respective fields. Besides that it is an issue of every kitchen because of the energy crisis.

Monopoly, the exclusive control of a commodity or service in a given market, or control that makes possible the fixing of prices and the virtual elimination of free competition, is a bi-product of the corporate world. It is natural that every business person wants to maximize profit. Moreover, they also want to maximize the current worth of their investment so that they can feel always secure in their business. The monopolist is not out of this box.

Our study will focus on the maximization of the present value of the investment of a monopolist under certain demand conditions. It is not a survey, rather is a mathematical analysis of the respective theory. In this paper we have tried to touch a small corner of demand and supply theory. The concept of the problem is taken from a paper "Continuous and Discrete Time Approaches to a Maximization Problem" by L.G. Telser and R.L. Graves [1]. An effort has been made to study the problem in time scale analysis as well.

Fundamental concepts and definitions of some technical terms from economics are mentioned in Chapter 2. The concept of present value and the derivation of the formula for present value using integration is developed [27, 28, 29]. This gives a sense of strong bonding of calculus and financial mathematics. The Euler - Lagrange equation for the calculus of variation is mentioned with proof. This is the main theory that we have used for the purpose of maximization. In Chapter 3 we have tried to summarize the definitions and some fundamentals of time scale calculus [5].

In Chapter 4 we have studied two types of monopolist demand conditions, continuous and discrete. In the continuous case, (C), some mathematical conditions must be satisfied to insure bounded maximum present value. An extensive discussion of the variation of the parameter method is used to solve nonhomogeneous differential equations [19]. The second order differential equation (derived from the Euler-Lagrange equation) must have real roots ($\lambda_1$ and $\lambda_2$). Moreover, for the given admissible path the jump in $q(t)$ is prohibited while the jump in $q'(t)$ is generally required. For a bounded maximum Present Value (PV) it is verified that $\lim_{t\to\infty} e^{-rt}q(t)q'(t) = 0$ for ordinary differential equation and $\lim_{t\to\infty} e^{-rt}q(t)q^{\triangle}(t) = 0$ for time scales with right dense points. The maximum value of the characteristic root of the Euler equation occurs for that value of $r$ which makes the discriminant of the roots of the Euler equation zero. For the bounded maximum value in the discrete case, (D), the discount factor must satisfy $\beta \in (0, 1)$ and $1 - \alpha\beta^2 > 0$.

The interpretation of the results by comparing the continuous and discrete cases is the most interesting part of the problem. This interpretation can also be found in chapter 4. The necessity of perfect forecasting in the perfectly competitive market is a major problem for stability of the differential equation. However, forecasting always helps the producers to judge the market and plan accordingly. The major difference in the continuous and discrete problems depends on the derivative of the continuous curve. This implies the slope is same whether it approaches from left or right. Thus the right and left derivatives are equal. This implies for any point $t$ there exist a neighborhood $N_\delta(t)$ such that the future value is the same as the past value. Hence it is completely predictable [1]. Thus, the complete symmetry of the demand relation over time is not practical.

Finally, an effort has been made to study the same problem on time scales. Due to some technical reasons the problem is narrowed down to the right dense case only. Without right dense points we have to face the partial derivative of $q^{\sigma\triangle(t)}$ with respect to $q(\sigma(t))$. The derivation of Euler-Lagrange equation for time scales was developed by R . Hilcher, V. Zeidan [12] and Ahlbrandt, Bohner, Ridenhour [11].

## 2. Preliminaries

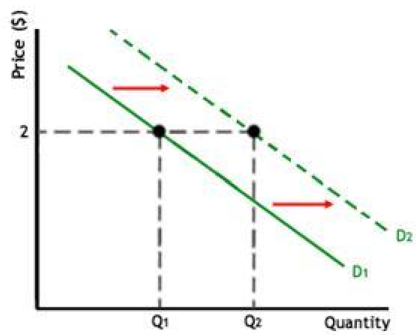
Economics is the branch of social science that studies the production, distribution, and consumption of goods and services. The term economics comes from the Greek for oikos (house) and nomos (custom or law), meaning "rules of the house".
The primary purpose of this chapter is to discuss some fundamental definitions of the terms in economics, a derivation of present value and the derivation of the Euler-Lagrange equation. The Demand function, revenue function, cost function, present value, face value and the derivation of present value will be the nuclear part of the discussion.

### 2.1. **Fundamental Economics Terms.**

In every market, there are both buyers and sellers. The buyers' willingness to buy a particular good (at various prices) is known as the "demand" for that good. The sellers' willingness to supply a particular good (at various prices) is referred to as the "supply" of that good. If the price of the good increases, then the demand on the market decreases. Thus there exists an inverse relation between demand and price. The graphical representation of the demand and price relation is called the demand curve. Because of the inverse relation the demand curve has a negative slope. In economics it is also called downward sloping.
The general form of the demand equation can be written $p = a + bq$, where

$$p = \text{price},$$
$$q = \text{quantity demanded},$$
$$a = \text{price when there is no demand, and}$$
$$b = \text{slope of the demand curve}$$

*Changes in preferences:* As people's preferences for goods and services change over time, the demand curve for these goods and services will also shift. For example, as the price of gasoline has risen, automobile buyers have demanded more fuel efficient, "economy" cars and fewer gas-guzzling, "luxury" cars. This change in preferences could be illustrated by a shift to the right in the demand curve for economy cars and a shift to the left in the demand curve for luxury cars.




In the above figure, we can see the change in demand without change in price. Therefore price is not the only factor that causes a change in demand. In our analysis we will discuss the major factors that affect demand.

**Definition 1.** *Cost Function:*

The amount or equivalent sum paid or charged for some commodity can be described by the cost function. Mathematically, the cost function is defined as

$$C(x) = F(x) + V(x)$$

where $x$ = total quantity produced; $C(x)$ = cost function; $F(x)$ = fixed costs; $V(x)$ = variable costs.

Fixed costs represents the cost when total quantity production equals zero. These cost do not vary depending on production or sales levels, such as rent, property tax, insurance, or interest expense.

Variable costs are costs of labor, material or overhead that change according to the change in the volume of production units. The total variable cost changes with increased production, the total fixed costs stays the same.

**Definition 2.** *Revenue Function:*

The amount of money that a company actually receives during a specific period, including discounts and deductions for returned merchandize is can be described by the Revenue Function. It is the "gross income" figure from which costs are subtracted to determine the net income.

Revenue is calculated by multiplying the price at which goods or services are sold by the number of units or amount sold.

$R(x) = \mathrm{P(x)} \times \mathrm{(x)}$

where $x$ = total quantity sold; R(x) = revenue function and P(x)= price per unit.

**Definition 3.** *Face Value*

The nominal value or dollar value of a security stated by the issuer is called the Face Value. For stocks, it is the original cost of the stock shown on the certificate. For bonds, it is the amount paid to the holder at maturity. In terms of mathematics, it is the sum of money invested plus interest over the course of time.

**Definition 4.** *Present Value*

The current worth of a future sum of money or stream of cash flow given a specified rate of return is called the Present Value. Future cash flows are discounted at the discount rate, and the higher the discount rate, the lower the Present Value of the future cash flows. This is also known as " the discounted value".

2.2. **Mathematics of present value.** Consider the following example to motivate our discussion. Let \$1000 be invested at the rate of 10% per annum. By the end of year the sum of money will be equal to \$1100 (\$ 1000+ 0.1* \$1000). The difference between the sum after one year and current sum of money is \$100.

Thus, the rate of change in the sum of money per year = rate of interest * invested sum.

Numerically this is equivalent to $\left\{\frac{\$1100-\$1000}{year} = 0.1 * \$1000\right\}$.
Let $r$ be the interest rate at which sum of the money accumulates. Suppose $B(t)$ is the balance in any bank account at time $t$. The value of the account increases over the course of time at the given rate of interest. This can be described by the equation:

$$\frac{d}{dt}B(t) = rB(t).$$

This implies,

$$\frac{dB(t)}{B(t)} = rdt,$$
$$lnB(t) = rt + c,$$

$$B(t) = e^{rt+c}. \tag{1}$$

When t=0, $B(0) = e^{r*0+c} = e^c$. Hence

$$B(t) = e^{rt}e^c = B(0)e^{rt}$$

and

$$B(0) = B(t)e^{-rt}. \tag{2}$$

In financial mathematics, $B(0)$ is also termed as the Present Value (PV) and $B(t)$ as the Face Value (FV).
If the interest is calculated more frequently the investment is worth more. The idea is that money available at the present time is worth more than the same amount in the future, due to its potential earning capacity. Over the course of time the account become less valuable. The bank ( or any investment business) increases the value in such a way that the sum retains the same intrinsic value over the course of time.
From equation (2) we have

$$e^{rt} = \frac{B(t)}{B(0)}.$$

Thus $e^{rt}$ represents the ratio of the current balance to the balance when $t = 0$.

Let $B(t_1)$ and $B(t_2)$ be the balances at time $t_1$ and $t_2$, respectively. The intrinsic value of the balance is equal if

$$B(t_1) = B(t_2)$$

$$\text{which implies} \quad B(0)e^{rt_1} = B(0)e^{rt_2}.$$

Thus

$$\$1 \quad \text{at time} \quad t_1 \text{is equivalent to,} \quad e^{r(t_1-t_2)} \text{at time} \quad t_2.$$

This is also called an unit conversion formula.

**Example 1:** A sum of \$1000 is invested at the rate of 10% per annum. What is the rate of exchange between the value after 5 years and 9 years from now? Let $t_1 = 5$; $t_2 = 9$ and $r = 0.1$. Suppose $\$_1$and $\$_2$ are values of the account at time $t_1$ and $t_2$, respectively.

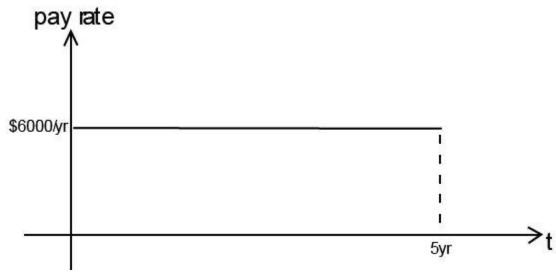
Thus, by conversion formula $\$_1 1 = \$_2 e^{0.1(9-5)}$. This implies $\quad \$_1 1 = \$_2 e^{0.4}$

where $\$_1 1$ represents the amount of money at time $t_1$. Thus \$1 after 5 years is equal to \$1.49 after 9 years.

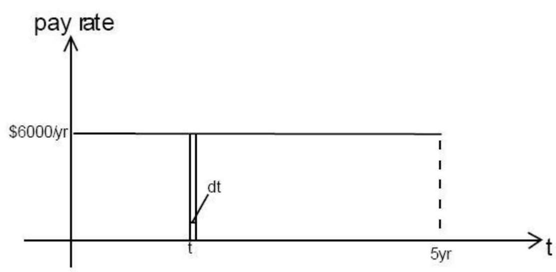
2.3. **Continuous Money Stream.** If the sum of money is accumulated continuously instead of for fixed time intervals, such a payment stream is called continuous payment stream.

The continuous payment stream is not precisely the same as that of the discrete payment stream but it can be used to approximate it. Moreover, in the continuous payment stream we can use a powerful mathematical tool "integration".

**Example 2 :** A constant payment of \$500 per month (\$ 6000 per year) for years can be represented by the graph:




From the diagram the value of money is the sum of the slices of rectangles within the boundary of time. The money at time $t$ is the money between time $t$ and $t + \triangle t$. Hence, we can introduce the Riemann sum and eventually an integral.




In the above diagram the area representing the total sum of money paid for 5 years can be divided into a number rectangular slices and hence can be used to analyze the continuous money stream with future value, present value and other financial parameters.

Suppose, $R(t)$= (payment/unit time )*(total time )

Thus,

$$\begin{aligned} R(t) &= \int (\text{payment rate})dt. \\ &= \int dR(t) \end{aligned}$$

**Theorem 1.** *Let the annual rate of income of some money stream be given by $R(t)$ at time $t$ where $0 \leq t \leq T$ and the interest assumed to be compounded continuously at the rate $r$ per annum. Then, the present value $PV$ ($B(0)$ i.e., balance when $t = 0$),* *of the money stream is given by:*

$$PV = \int_0^T R(t)e^{-rt}dt.$$

*Proof.* : Let $R(t)$ = balance at time $t$,

$T$ = end of the time interval,

$n$ = number of subintervals for [0.T]

(in terms of compound interest: total number of times the money compounded. )

This implies the interval $[0, T]$ is divided into $n$ intervals.

Thus,

$$\triangle t = \frac{T}{n}; \quad \text{where} \quad 0 = t_1 < t_2 < \ldots < t_n = T$$

and $r$ = rate of interest compounded continuously.

The present value can be approximated by the following sum:

$$PV \approx R(t_1)e^{-rt_1}\triangle t + R(t_2)e^{-rt_2}\triangle t + \ldots, +R(t_n)e^{-rt_n}\triangle t.$$

$$\therefore PV = \sum_{i=1}^{n} R(t_i)e^{-rt_i}\triangle t.$$

Now $\triangle t \to 0$implies $\quad n \to \infty$.

Letting $n \to \infty$, P.V. can be written as

$$PV = \lim_{n\to\infty} \sum_{i=1}^{n} R(t_i)e^{-rt_i}\triangle t.$$

$$\therefore \qquad PV = \int_0^T R(t)e^{-rt}dt$$

□

## 2.4. **Euler-Lagrange Equation.**

**Lemma 2.** *Fundamental Theorem of Calculus of Variation*

*If $f$ is a smooth function over the interval [a,b] and*

$$\int_a^b f(x)h(x)dx = 0$$

*for every function $h \in C^\infty[a,b]$ with $h(a) = 0$, then $f(x)$ is identically zero on the interval $(a,b)$.*

*Proof:* Let $f$ be a function satisfying the hypothesis.
Let $r(x)$ be a function such that $r(a) = 0, r(b) = 0 \quad \text{and} \quad r > 0, \quad x \in (a,b)$satisfying.$\because$ $\int_a^b f(x)h(x)dx = 0$.
For example $r(x) = -(x-a)(x-b)$.
Suppose, $\quad h = rf$. Then

$$\int_a^b f(x)h(x)dx = \int_a^b f(x)r(x)f(x)dx = \int_a^b f^2(x)r(x)dx = 0.$$

Since $r(x)f^2 \geq 0$ for $r \in (a,b)$, $f = 0$ for all $x \in [a,b]$.

**Theorem 3.** *Let $f(t)$ be be a real variable.*
*Suppose, $J = \int_a^b F(t, f(t), f'(t))dt$ where $F$ is given by*
*$F : R \times X \times Y \to R$ with continuous first partial derivatives*
*where, $\quad f : R \to X$ and $f' : R \to Y$.*
*The Euler-Lagrange equation is the ordinary differential equation*

$$F_x(t, f(t), f'(t)) - \frac{d}{dt}F_y(t, f(t), f'(t)) = 0$$

*where $F_x$ and $F_y$ are the partial derivatives of $F$ with respect to $x$ and $y$ respectively.*

*Proof:* Let us consider $f(a) = c$, $f(b) = d$.
The function $f(t)$ is supposed to extremize the functional

$$\int_a^b F(t, f(t), f'(t))dt.$$

We assume that $F$ has continuous first order partial derivatives,

and let $g_\epsilon(t) = f(t) + \epsilon\eta(t)$.

$\therefore g'_\epsilon(t) = (f'(t) + \epsilon\eta'(t))$.

Note that, $\frac{\partial g_\epsilon(t)}{\partial \epsilon} = \eta(t)$ and $\frac{\partial g'_\epsilon(t)}{\partial \epsilon} = \eta'(t)$

where $\eta(t)$ is differentiable function satisfying $\eta(a) = \eta(b) = 0$.

Define

$$J(\epsilon) = \int_a^b F(t, g_\epsilon(t), g'_\epsilon(t))dt.$$

The total derivative of $J$ with respect to $\epsilon$ is given by

$$\frac{dJ}{d\epsilon} = \int_a^b \frac{dF}{d\epsilon}(t, g_\epsilon(t), g'_\epsilon(t))dt.$$

By the definition of total derivative

$$\begin{aligned} \frac{dF}{d\epsilon} &= \frac{\partial F}{\partial t}\frac{\partial t}{\partial \epsilon} + \frac{\partial F}{\partial g_\epsilon}\frac{\partial g_\epsilon}{\partial \epsilon} + \frac{\partial F}{\partial g'_\epsilon}\frac{\partial g'_\epsilon}{\partial \epsilon} \\ &= 0 + \frac{\partial F}{\partial g_\epsilon}\eta(t) + \frac{\partial F}{\partial g'_\epsilon}\frac{\partial}{\partial \epsilon}\eta'(t). \end{aligned}$$

$$\therefore \frac{dJ}{d\epsilon} = \int_a^b \left\{ \eta(t)\frac{\partial F}{\partial g_\epsilon} + \eta'(t)\frac{\partial F}{\partial g'_\epsilon} \right\} dt.$$

Thus $\epsilon = 0$ implies $g_\epsilon = f$ and since $f$ is an extreme value, it follows that $J'(0) = 0$.

Then the following are equivalent,

$$J'(0) = \int_a^b \left\{ \eta(t)\frac{\partial F}{\partial f} + \eta'(t)\frac{\partial F}{\partial f'} \right\} dt = 0$$

$$
\begin{aligned}
&= \int_a^b \eta(t)\frac{\partial F}{\partial f}dt + \int_a^b \eta'(t)\frac{\partial F}{\partial f'}dt \\
&= \int_a^b \eta(t)\frac{\partial F}{\partial f}dt + \left\{\eta(t)\frac{\partial F}{\partial f'}\right\}_a^b - \int_a^b \eta(t)\left\{\frac{\partial F}{\partial f'}\right\}dt \\
&= \int_a^b \left\{\frac{\partial F}{\partial f} - \frac{d}{dt}(\frac{\partial F}{\partial f'})\right\}\eta(t)dt + \left\{\frac{\partial F}{\partial f'}\eta(t)\right\}_a^b \\
&= \int_a^b \left\{\frac{\partial F}{\partial f} - \frac{d}{dt}(\frac{\partial F}{\partial f'})\right\}\eta(t)dt
\end{aligned}
$$

provided $\eta(a) = \eta(b) = 0$.

By the fundamental theorem of calculus of variation

$$
\frac{\partial F}{\partial f} - \frac{d}{dx}(\frac{\partial F}{\partial f'}) = 0
$$

which is the Euler -Lagrange equation.

## 3. Time Scales Calculus

The calculus of time scales was introduced by Stefan Hilger in his Ph.D. thesis (Universität Würzburz, 1988) in order to unify the discrete and continuous analysis. A time scale is an arbitrary non-empty closed subset of the real numbers. It is usually denoted by $\mathbb{T}$.

Thus $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ are some examples of time scales.

But $\mathbb{Q}, \mathbb{R} - \mathbb{Q}\{irrationals\}, \mathbb{C}$ and $(0,1)$ i.e. the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales. We move through the time scale using forward and backward jump operators. The gaps in the time scale are measured by a function $\mu$, defined in terms of forward jump operator, $\sigma$.

**Definition 5.** *Forward and Backward jump operator:*

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

The backward jump operator is the operator $\rho : \mathbb{T} \to \mathbb{T}$ given by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

Note 1: If $\sigma(t) > t$, we say that $t$ is $right - scattered$, while if $\rho(t) < t$ we say that $t$ is $left - scattered$. The points which are right-scattered and left-scattered at the same time are called isolated.

Note 2 : If $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense.

Note 3 : If $t > \inf\mathbb{T}$ and $\rho(t) = t$, then $t$ is called left-dense.

The forward jump operator $\sigma(t)$ is not always same as $t$. The difference between $\sigma(t)$ and $t$ is called Graininess.

**Definition 6.** *Graininess:*

The *Graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

Note 4: If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T} - \{m\}$ . Otherwise $\mathbb{T}^k = \mathbb{T}$.

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup\mathbb{T}), sup\mathbb{T}] & \text{if } \sup T < \infty \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

Note 5: Let $f : \mathbb{T} \to \mathbb{R}$ be a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$ *i.e.*, $f^\sigma = f \circ \sigma$.

Using $\sigma$ we define the delta derivative of a function $f$ in a natural way.

**Definition 7.** *Differentiation:*

Assume $f : \mathbb{T}^k \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\triangle(t)$ to be the number with the property that given any $\epsilon > 0$ there exists a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of $t$ such that

$$|[f(\sigma(t)) - f(s)] - f^\triangle(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U$$

where $f^\triangle(t)$ is called delta derivative of $f$ at $t$.

We will introduce the delta derivative $f^\triangle$ for a function $f$ defined on $\mathbb{T}$. It is expressed as

(i) $f^\triangle = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and (ii) $f^\triangle = \triangle f$ is the forward difference operator if $\mathbb{T} = \mathbb{Z}$.

**Theorem 4.** *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following.*

*(i) If $f$ is differentiable at $t$, then $f$ is continuous at $t$.*

*(ii) If $f$ is continuous at $t$, then $f$ is differentiable at $t$ with*

$$f^\triangle(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

*(iii) If* $t$ *is right- dense, then* $f$ *is differentiable at* $t$ *iff the limit*

$$\lim_{s\to\ t} \frac{f(t)-f(s)}{t-s}$$

*exists as a finite number. In this case*

$$f^{\triangle}(t) = \lim_{s\to\ t} \frac{f(t)-f(s)}{t-s}.$$

*(iv) If* $f$ *is differentiable at* $t$*,then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\triangle}(t).$$

Now we introduce the most powerful fundamentals delta of derivatives: sum rule, product rule, quotient rule and the transformation of the sigma function in terms of the original function and its derivative.

**Theorem 5.** *Assume* $f$*,* $g$*:* $\mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$*. Then:*
*(i) The sum* $f+g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t$ *with*

$$(f+g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t).$$

*(ii) For any constant* $\alpha$*,* $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t$ *with*

$$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t).$$

*(iii) The product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t$ *with*

$$\begin{aligned}(fg)^{\triangle}(t) &= f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t)\\ &= f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).\end{aligned}$$

*(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at $t$ with*

$$\left\{\frac{1}{f}\right\}^{\triangle}(t) = \frac{-f^{\triangle}(t)}{f(t)f(\sigma(t))}$$

*(v) If $g(t)g(\sigma(t)) \neq$ then $\frac{f}{g}$ is differentiable at $t$ and*

$$\left\{\frac{f}{g}\right\}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}$$

Apart from the differentiability we need a few more conditions for integrability of the function.

**Definition 8.** *Regulated, rd-continuous and pre-differentiation*

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exists at all right-dense points in $\mathbb{T}$ and its left-sided limits exists at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd - continuous$ provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. It is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable in the region of differentiation $D$, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k - D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and $f$ is differentiable at each each $t \in D$. Assume $f : \mathbb{T} \to \mathbb{T}$ is regulated function. Any function $F$ in theorem is called a $pre - antiderivative$ of $f$ if $F^{\triangle}(t) = f(t)$.

**Theorem 6.** *Existence of Pre-Antiderivative*

*Let $f$ be regulated. Then there exists a function $F$ which is pre-differentiable with region of differentiation $D$ such that $F^{\triangle}(t) = f(t)$ holds for all $t \in D$.*

The indefinite integral of a regulated function $f$ is given by

$$\int f(t)\triangle t = F(t) + C$$

where $C$ is an arbitrary constant and $F$ is a pre-antiderivative of $f$. We define the Cauchy integral by:

$$\int_r^s f(t)\triangle t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $\quad t \in \mathbb{T}^k$.

*TABLE*: Time scale derivative and Antiderivative for $\quad \mathbb{T} = \mathbb{R} \quad or \quad \mathbb{T} = \mathbb{Z}$

| Time $\mathbb{T}$ | $\mathbb{R}$ | $\mathbb{Z}$ |
|---|---|---|
| Backward jump operator$\rho(t)$ | $t$ | $t-1$ |
| Forward jump operator $\sigma(t)$ | $t$ | $t+1$ |
| Graniness$\mu(t)$ | 0 | 1 |
| Derivative $f^{\triangle}(t)$ | $f'(t)$ | $\triangle f(t)$ |
| Integral $\int_a^b f(t)\triangle t$ | $\int_a^b f(t)dt$ | $\sum_{t=a}^{b-1} f(t)(\text{if} \quad a < b)$ |
| Rd-continuous $f$ | continuous $f$ | any $f$ |

**Theorem 7.** *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\triangle\tau = \mu(t)f(t).$$

Some fundamental laws of integration are summarized in the following theorem including two laws of integration by parts.

**Theorem 8.** *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_r d$, then*

*(i)* $\int_a^b [f(t) + g(t)]\triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t$;

*(ii)* $\int_a^b (\alpha f)(t)\triangle t = \alpha \int_a^b f(t)\triangle t;$

*(iii)* $\int_a^b f(t)\triangle t = -\int_a^b f(t)\triangle t;$

*(iv)* $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t$

*(v)* $\int_a^b f(\sigma(t))g^{\triangle t}(t) = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(t)\triangle t;$

*(vi)* $\int_a^b f(t)g^{\triangle}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t;$

*(vii)* $\int_a^a f(t)\triangle t = 0$

*(viii) If* $\quad |f(t)| \leq g(t) on [a,b),$ *then* $|\int_a^b f(t)\triangle t| \leq \int_a^b g(t)\triangle t;$

*(ix) If* $f(t) \geq 0$ *for all* $a \leq t \leq b$, *then* $\int_a^b f(t)\triangle t \geq 0.$

The interesting aspects of time scales calculus is that the integration can also be performed even if the domain of the function is a set of integers. Thus integration of any function depends upon the domain of the function.

**Theorem 9.** *Let* $a, b, c \in \mathbb{T}$ *and* $f \in C_{rd}$

*(i) If* $\mathbb{T} = \mathbb{R}$,

$$\int_a^b f(t)\triangle t = \int_a^b f(t)dt$$

*where the integral on the right is the usual Riemann integral from calculus.*

*(ii) If [a,b] consists of only isolated points, then*

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a{=}b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b \end{cases}$$

*(iii) if* $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b \\ 0 & \text{if } a{=}b \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b \end{cases}$$

**First order Linear Equation**

Now we move to the differential equations that contain the delta derivative.

**Definition 9.**

Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation

$$y^{\triangle} = f(t, y, y^{\sigma}) \tag{3}$$

is called a first order dynamic equation, or sometimes called a differential equation. If

$f(t, y, y^{\sigma}) = f_1(t)y + f_2(t)$ or $f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t)$

for the functions $f_1$ and $f_2$, then (3) is called a linear equation. A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of (3) if $y^{\triangle}(t) = f(t, y(t), y(\sigma(t)))$ is satisfied for for all $t \in \mathbb{T}^k$. The general solution of (3) is defined to be the set of all solutions of (3). We will require a condition known as regressivity on functions as well as dynamic equations. They are defined as follows:

**Definition 10.**

We say that the function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^k$.

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\Re$ or $\Re(\mathbb{T})$ or $\Re(\mathbb{T}, \mathbb{R})$.

The generalized exponential function is denoted by $e_p(t, s)$. It is used to solve differential equations.The basic properties of exponential function can be summarized as follows:

If $p$, $q \in \Re$, then $e_0(t, s) = 1$ and $e_p(t, t) = 1$, $\frac{1}{e_p(t,s)} = e_{\ominus p}(s, t)$,

$e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$ and $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$ where " circle plus" $p \oplus q$ is defined by $p \oplus p \equiv= p(t) + q(t) + \mu(t)p(t)q(t)$ for all $t \in \mathbb{T}^k$ , $p, q \in \Re$. and " circle minus" $\ominus p$ is defined as

$(\ominus p)(t) = \frac{p(t)}{1+\mu(t)p(t)}$ for all $t \in \mathbb{T}^k$.

Note: If $p \in \Re$, then the first order linear dynamic equation

$$y^{\triangle} = p(t)y$$

is called regressive.

**Theorem 10.** *Suppose first order dynamic equation $y^{\triangle} = p(t)y$ is regressive and fix $t_0 \in \mathbb{T}$. Then $e_p(., t_0)$ is a solution of the initial value problem*

$$y^{\triangle} = p(t)y, \qquad y(t_0) = 1.$$

**Second Order Linear Equations**

The general form of a the second order linear dynamic equation is

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t)$$

where $p, q, f \in C_{rd}$.

Let us consider an operator $L_2 : C^2_{rd} \to C_{rd}$ defined by

$$L_2y(t) = y^{\triangle\triangle}(t) + p(t)y^{\triangle}(t) + q(t)y(t) \quad \text{for} \quad t \in \mathbb{T}^k.$$

Thus the general form of the second order equation can be written as

$$L_2y = f,$$

where $L_2y = 0$ is called the homogenous dynamic equation.

**Theorem 11.** *The operator $L_2 : C^2_{rd} \to C_{rd}$ is a linear operator, i.e; $L_2(\alpha y_1 + \beta y_2) = \alpha L_2(y_1) + \beta L_2(y_2)$ for all $\alpha,\ \beta \in \mathbb{R}$ and $y_1, y_2 \in C^2_{rd}$.*

**Definition 11.**

The second order linear dynamic equation $y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t)$ is called regressive provided $p, q, f \in C_{rd}$ are such that

$$1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^k.$$

with this in mind, we establish conditions for existence and uniqueness of a solution.

**Theorem 12.** *Existence and Uniqueness:*
*Assume that the second order linear dynamic equation* $y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t)$ *is regressive. If* $t_0 \in \mathbb{T}^k$*, then the initial value problem*
$L_2 y = f(t)$, $y(t_0) = y_0$, $y^{\triangle}(t_0) = y_0^{\triangle}$,
*where* $y_0$ *and* $y_0^{\triangle}$ *are given constants, has a unique solution, and this solution is defined on the whole time scale* $\mathbb{T}$.

# 4. Continuous and Discrete Time Approach to a Maximization problem

One of the the fundamental goals of the project is to seek a maximum present value with a given rate of interest. The problem is narrowed down by taking the present value maximization case of a monopolist. The generalization of the discrete and continuous case of the the problem in time scale calculus is the primary concern of the work. We will see differences between discrete and continuous cases.

4.1. **Continuous(C) Monopoly Problem.** The Present Value (PV ) of the monopoly problem is given by

$$PV = \int_0^\infty e^{-rt} q(t)p(t)dt \tag{4}$$

where,

$$p(t) = \text{price per unit at time } t,$$

$$q(t) = \text{total quantity demanded at time } t,$$

$$r = \text{rate of interest.}$$

The monopolist's demand condition can be written as:

$p(t) = f(t) - q(t) - a_1 q'(t) - a_2 q''(t)$ provided, $f(t) = 0$ and $q(t) = 0$ for $t < 0$

where $f(t) =$ size of the market.

Note: $f(t)$ is taken in such away that $p(t)$ is non-negative.

**Definition 12.** *Admissible quantity path*

Our solution will come from a class of functions known as admissible paths. Admissible quantity $q(t)$ is a continuous curve with a continuous first derivative except at a finite number of points such that $|e^{-\frac{rt}{2}} q(t)|$ and $|e^{-\frac{rt}{2}} q'(t)|$ are bounded.

**Definition 13.** *Interior Curve and Boundary Curve*

If the integrals

$$\int_0^\infty e^{-rt} q^2(t) dt$$

and

$$\int_0^\infty e^{-rt} q'^2(t) dt$$

are finite then the admissible quantity curve is an interior curve otherwise the curve is a boundary curve.

The PV(4) is said to have interior solution if the the integral $\int_0^\infty e^{-rt} q(t)p(t) dt$ is finite for all admissible interior curves. Similarly, the integral (4) is said to have edge solution for a given $r$ if the value of the integral is finite for all admissible interior curves and for some boundary curves.

4.2. **Maximization of Present Value in continuous case.** The present value for the monopolist in the continuous case is given by: $PV = \int_0^\infty e^{-rt} q(t)p(t) dt$

where $\quad p(t) = f(t) - q(t) - a_1 q'(t) - a_2 q''(t)$

Therefore, PV can be written as,

$$PV = \int_0^\infty e^{-rt} [f - q - a_1 q' - a_2 q''] q \quad dt. \tag{5}$$

Equation (5) can also be written as

$$PV = \left[-a_2 e^{-rt} q q'\right]_0^\infty + \int_0^\infty e^{-rt} [fq - q^2 - (a_1 + a_2 r) q q' + a_2 q'^2] dt \tag{6}$$

or, equivalently,

(7)

$$PV = \left[-a_2 e^{-rt} q q'\right]_0^\infty + \int_0^\infty e^{-rt} [fq - q^2 - a_1 q q'] dt - a_2 r \int_0^\infty e^{-rt} q q' dt + \int_0^\infty a_2 e^{-rt} q'^2 dt$$

To verify this note that, from equation (7), considering the integral $a_2 r \int_0^\infty e^{-rt} qq' dt$ and integrating by parts we get:

$$-a_2 \int_0^\infty e^{-rt} qq' dt = a_2 [qq' e^{-rt}]|_0^\infty + a_2 r \int_0^\infty \left[qq'' + q'^2\right] \frac{e^{-rt}}{-r} dt$$

This implies,

$$(8) \qquad -a_2 r \int_0^\infty e^{-rt} qq' dt = a_2 \left[qq' e^{-rt}\right]_0^\infty - a_2 \int_0^\infty qq'' e^{-rt} dt - a_2 \int_0^\infty e^{-rt} q'^2 dt$$

Finally, substituting the equivalent expression of the integral from (8) into (7) yields,

$$PV = \left[-a_2 e^{-rt} qq'\right]_0^\infty + \int_0^\infty e^{-rt}[fq - q^2 - a_1 qq'] dt + \left[a_2 e^{-rt} qq'\right]_0^\infty$$

$$-a_2 \int_0^\infty qq'' e^{-rt} dt - a_2 \int_0^\infty e^{-rt} q'^2 dt + a_2 \int_0^\infty e^{-rt} q'^2 dt$$

Thus we return to,

$$P.V. = \int_0^\infty e^{-rt}[fq - q^2 - a_1 qq' - a_2 qq''] dt$$

provided, $\lim_{t\to\infty} e^{-rt} qq' = 0$. This verifies the validity of equation (6).
Now equation (6) is equivalent to

$$(9) \qquad PV = q(0)q'(0) + \int_0^\infty e^{-rt}[fq - q^2 - (a_1 + a_2 r) qq' + a_2 q'^2] dt$$

Since, $q(0)$ and $q'(0)$ are constants. It is sufficient to maximize the integral on the right side of (9).

Suppose, $F(t, q, q') = e^{-rt}[fq - q^2 - (a_1 + a_2 r) qq' + a_2 q'^2]$.

For the Present Value maximization, we need to maximize $\int_0^\infty F(t, q, q') dt$.

For maximization, the function $q(t)$ must satisfy the Euler-Lagrange equation:

$$(10) \qquad F_q - \frac{d}{dt} F_{q'} = 0.$$

Moreover, the sufficient condition for the maximum PV is given by the Legendre's condition :

$$F_{qq'} < 0.$$

Differentiating the function $F(t,q,q')$ partially with respect to $q$ and $q'$ respectively gives us

$$F_q = e^{-rt}[f - 2q - (a_1 + a_2 r)q']$$

and

$$F_{q'} = e^{-rt}[-(a_1 + a_2 r)q + 2a_2 q'].$$

This implies, $\frac{d}{dt}F_{q'} = e^{-rt}[-(a_1 + a_2 r)q' + 2a_2 q''] - re^{-rt}[-(a_1 + a_2 r)q + 2a_2 q']$. Substituting the values of $F_q$ and $\frac{d}{dt}F'_q$ in equation (10) we obtain

$$e^{-rt}[f - 2q - (a_1 + a_2 r)q'] - e^{-rt}[-(a_1 + a_2 r)q' + 2a_2 q''] + re^{-rt}[-(a_1 + a_2 r)q + 2a_2 q'] = 0,$$

$$f - 2q - (a_1 + a_2 r)q' + (a_1 + a_2 r)q' - 2a_2 q'' - r(a_1 + a_2 r)q + 2a_2 q' = 0,$$

$$-2a_2 q'' + 2a_2 r q' - (2 + a_1 r + a_2 r^2)q + f = 0.$$

$$\therefore 2a_2 q'' - 2a_2 r q' + (2 + a_1 r + a_2 r^2)q - f = 0. \tag{11}$$

The corresponding characteristic equation is given by:

$$2a_2 m^2 - 2a_2 m + (2 + a_1 r + a_2 r^2) = 0$$

This implies,

$$m = \frac{2a_2 r \pm \sqrt{4a_2^2 r^2 - 8a_2(2 + a_1 r + a_2 r^2)}}{4a_2}, \text{ or}$$

$$m = \frac{r}{2} \pm \frac{\sqrt{-a_2^2 - 4a_2 - 2a_1 a_2 r}}{2a_2}. \tag{12}$$

Suppose, $m = \lambda_1$ and $\lambda_2$.

There are three possibilities :

- $\lambda_1$ and $\lambda_2$ are real and unequal.
- $\lambda_1$ and $\lambda_2$ are real and equal.
- $\lambda_1$ and $\lambda_2$ are complex.

**Claim:** The Characteristic roots of the Euler equation (11) must be real for bounded maximum present value.

*Proof*: Let us consider an admissible quantity path: $q = Me^{\delta t}$, $\delta < \frac{r}{2}$, $M = q(0)$. Now equation (9) can be written as

$$PV = a_2 q(0)q'(0) + \int_0^\infty e^{-rt} qf dt - \int_0^\infty e^{-rt}[e^{2\delta t} + (a_1 + a_2 r)Me^{\delta t} M\delta e^{\delta t} - a_2\delta^2 M^2 e^{2\delta t}]dt.$$

This implies

$$P.V. = a_2 q(0)q'(0) + \int_0^\infty e^{-rt} fq dt + M^2 \int_0^\infty e^{(2\delta - r)t}[1 + (a_1 + a_2 r)\delta - a_2\delta^2]dt.$$

Thus

$$PV = a_2 q(0)q'(0) + \int_0^\infty e^{-rt} fq dt + [1 + (a_1 + a_2 r)\delta - a_2\delta^2]\frac{e^{(2\delta - r)t}}{2\delta - r}|_0^\infty.$$

$$\therefore PV = a_2 q(0)q'(0) + \int_0^\infty e^{-rt} fq dt + \frac{1}{2\delta - r}[1 + (a_1 + a_2 r)\delta - a_2\delta^2].$$

$\frac{1}{2\delta - r} \to -\infty$ as $\delta \to (\frac{r}{2})^-$

Thus PV can be arbitrarily large if $\lim_{\delta \to (\frac{r}{2})^-} [1 + (a_1 + a_2 r)\delta - a_2\delta^2] < 0.$

which is equivalent to:

$$1 + a_1\frac{r}{2} + a_2(\frac{r}{2})^2 < 0.$$

To avoid the possibility of arbitrarily large PV, it is necessary that

$$1 + a_1\frac{r}{2} + a_2(\frac{r}{2})^2 \geq 0 \tag{13}$$

The discriminant of the equation (12) can be written as

$$-a_2^2 r^2 - 4a_2 - 2a_1 a_2 r i.e., \quad -4a_2[a_2(\frac{r}{2})^2 + a_1\frac{r}{2} + 1],$$

where, $a_2 < 0$. This implies the discriminant is non negative. Hence the characteristic roots ($\lambda_1$ and $\lambda_2$) are real. Ultimately, the Present Value is bounded above if the characteristic roots of the demand equation are real. Considering the following example

**Example 3:** Let $f(t) = \frac{Ke^{\frac{rt}{2}}}{1+t}$. Then,

$$\begin{aligned}
\int_0^\infty e^{-rt}\{f(t)\}^2\,dt &= \int_0^\infty e^{-rt}\left\{\frac{Ke^{\frac{rt}{2}}}{1+t}\right\}^2 dt \\
&= \int_0^\infty \frac{e^{-rt}}{(1+t)^2}K^2e^{rt}dt \\
&= K^2\int_0^\infty \frac{1}{(1+t)^2}dt \\
&= K^2[-1/(1+t)]_0^\infty \\
&= K^2
\end{aligned}$$

which is finite. Substituting the value of $f(t)$ in the price equation

$$p(t) = \frac{Ke^{\frac{r}{2}}}{1+t} - Me^{\delta t} - a_1M\delta e^{\delta t} - a_2M\delta^2e^{\delta t}$$

or,

$$p(t) = e^{\delta t}\left\{\frac{Ke^{(\frac{r}{2})t}}{1+t} - M(1 + a_1\delta + a_2\delta^2)\right\}.$$

If $K > 0$ is large enough so that price is positive then the market grows. For the positive bounded value, the function must satisfy the inequality (13). Thus there exists bounded positive maximum value provided:

- $K > 0$ and large enough so that $p(t)$ is positive.
- $1 + a_1\frac{r}{2} + a_2(\frac{r}{2})^2 \geq 0$ with $a_2 < 0$. Let's now consider the relevant cases.

**Case 1** The characteristic roots ($\lambda_1$ and $\lambda_2$ ) are not equal.

The complimentary function of the equation (11) can be written as:

$$q_c(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{14}$$

To find the particular solution of the equation (11), we therefore let

$$q_p(t) = v_1(t) y_1 + v_2(t) y_2 \tag{15}$$

where $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Then,

$$q_p'(t) = v_1'(t) y_1 + v_2'(t) y_2 + v_1(t) y_1' + v_2(t) y_2'$$

We impose the condition

$$v_1'(t) y_1 + v_2'(t) y_2 = 0 \tag{16}$$

leaving

$$q_p'(t) = v_1(t) y_1' + v_2(t) y_2'. \tag{17}$$

From this, we find

$$q_p''(t) = v_1'(t) y_1' + v_2'(t) y_2' + v_1(t) y_1'' + v_2(t) y_2''. \tag{18}$$

Substituting (15), (17), and (18) into (11) we obtain

$$2a_2[v_1'(t)y_1' + v_2'(t)y_2' + v_1(t)y_1'' + v_2(t)y_2''] - 2a_2[v_1(t)y_1' + v_2(t)y_2'] + A[v_1(t)y_1 + v_2(t)y_2] - f = 0$$

where $A = (2 + a_1 r + a_2 r^2)$, or, equivalently,

$$2a_2[v_1'(t)y_1' + v_2'(t)y_2'] + [y_2'' - 2a_2 r y_1' + A y_1] v(t) + [y_2'' - 2a_2 r y_2' + A y_2] v - 2(t) = f.$$

Since $y_1$ and $y_2$ are solutions of corresponding homogenous differential equation for (11), the expressions in the last two brackets in the above equation are identically zero. This leaves

$$v_1'(t)y_1' + v_2'(t)y_2' = \frac{f}{2a_2}$$

(19) $$\therefore \lambda_1 v_1'(t)y_1 + \lambda_2 v_2'(t)y_2 = \frac{f}{2a_2}.$$

Solving equations (16) and (19) by using Cramer's rule we get:
$v_1'(t) = -f(2a_2)^{-1}(\lambda_1 - \lambda_2)^{-1}e^{-\lambda_1 t}$ and
$v_2'(t) = f(2a_2)^{-1}(\lambda_1 - \lambda_2)^{-1}e^{-\lambda_2 t}$.
Thus, we obtain the functions $v_1(t)$ and $v_2(t)$ defined by

$$v_1(t) = (2a_2)^{-1}(\lambda_1 - \lambda_2)^{-1}\int_0^t f(s)e^{-\lambda_1 s}ds$$

$$v_2(t) = -(2a_2)^{-1}(\lambda_1 - \lambda_2)^{-1}\int_0^t f(s)e^{-\lambda_2 s}ds$$

Thus the general solution of equation (11) is
$q(t) = q_c(t) + q_p(t)$
$\therefore q(t) = [c_1 + (\lambda_1 - \lambda_2)^{-1}(2a_2)^{-1}\int_0^t f(s)e^{-\lambda_1 s}ds]e^{\lambda_1 t} + [c_2 - (\lambda_1 - \lambda_2)^{-1}(2a_2)^{-1}\int_0^t f(s)e^{-\lambda_2 s}ds]e^{\lambda_2 t}$
Thus, $q(t) = (\lambda_1 - \lambda_2)^{-1}\left[\frac{c_1}{(\lambda_1 - \lambda_2)^{-1}} + (2a_2)^{-1}\int_0^t f(s)e^{-\lambda_1 s}ds]e^{\lambda_1 t}\right]$
$+ (\lambda_1 - \lambda_2)^{-1}\left[\frac{c_2}{(\lambda_1 - \lambda_2)^{-1}} + (2a_2)^{-1}\int_0^t f(s)e^{-\lambda_2 s}ds\right]e^{\lambda_2 t}$
This implies
(20)
$$q(t) = (\lambda_1 - \lambda_2)^{-1}\left\{\left[C_1 + (2a_2)^{-1}\int_0^t f(s)e^{-\lambda_1 s}ds\right]e^{\lambda_1 t} + \left[C_2 - (2a_2)^{-1}\int_0^t f(s)e^{-\lambda_2 s}ds\right]e^{\lambda_2 t}\right\}$$

where$C_1 = \frac{c_1}{(\lambda_1 - \lambda_2)^{-1}}$ and $C_2 = \frac{c_2}{(\lambda_1 - \lambda_2)^{-1}}$
Now by using boundary condition
$q(0) = (\lambda_1 - \lambda_2)^{-1}(C_1 + C_2)$ and $C_2 = (2a_2)^{-1}\int_0^\infty e^{-\lambda_2 s}f(s)ds$
we obtain, $C_1 = (\lambda_1 - \lambda_2)q(0) - C_2$ i.e., $C_1 = (\lambda_1 - \lambda_2)q(0) - (2a_2)^{-1}\int_0^\infty e^{-\lambda_2 s}f(s)ds$

Thus, the general solution can be written as

$$q(t) = (\lambda_1-\lambda_2)^{-1}\left[(\lambda_1-\lambda_2)q(0) - (2a_2)^{-1}\int_0^\infty e^{-\lambda_2 s} f(s)ds + (2a_2)^{-1}\int_0^t f(s)e^{-\lambda_1 s}ds\right] e^{\lambda_1 t}$$

$$+(\lambda_1-\lambda_2)^{-1}\left[(2a_2)^{-1}\int_0^\infty e^{-\lambda_2 s} f(s)ds - (2a_2)^{-1}\int_0^t f(s)e^{-\lambda_2 s}ds\right]. \tag{21}$$

$$\therefore \quad q(t) = q(0)e^{\lambda_1 t} + A_1 e^{\lambda_1 t}\int_0^t e^{-\lambda_1 s} f(s)ds - A_1 e^{\lambda_1 t}\int_0^\infty e^{-\lambda_2 s} f(s)ds$$

$$+A_1 e^{\lambda_2 t}\int_t^\infty e^{-\lambda_2 s} f(s)ds$$

where $\quad A_1 = (\lambda_1-\lambda_2)^{-1}(2a_2)^{-1}$.

This yields, (By using fundamental theorem of calculus and Leibnitz notation for the derivative: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$)

$$q'(t) = q(0)\lambda_1 e^{\lambda_1 t} + A_1\lambda_1 e^{\lambda_1 t}\int_0^t e^{-\lambda_1 s} f(s)ds + A_1 e^{\lambda_1 t}e^{-\lambda_1 t} f(s) - A_1\lambda_1 e^{\lambda_1 t}\int_0^\infty e^{-\lambda_2 s} f(s)ds$$

$$-A_1\lambda_1 e^{\lambda_1 t}\lim_{h\to\infty} e^{-\lambda_2 h} f(h)$$

$$+(\lambda_1-\lambda_2)^{-1}\left[(2a_2)^{-1}\lambda_2 e^{\lambda_2 t}\int_t^\infty e^{-\lambda_2 s} f(s)ds + (2a_2)^{-1}\lambda_2 e^{\lambda_2 t}\lim_{h\to\infty}\left\{e^{-\lambda_2 h} - e^{\lambda_2 t}\right\} f(t)\right]$$

$$\therefore q'(t) = q(0)\lambda_1 e^{\lambda_1 t} + A_1\lambda_1 e^{\lambda_1 t}\int_0^t e^{-\lambda_1 s} f(s)ds - A_1\lambda_1 e^{\lambda_1 t}\int_0^\infty e^{-\lambda_2 s} f(s)ds$$

$$+A_1\lambda_2 e^{\lambda_2 t}\int_t^\infty e^{-\lambda_2 s} f(s)ds$$

This implies,

$$q(t)q'(t) = q^2(0)\lambda_1 e^{2\lambda_1 t} + q(0)A_1\lambda_1 e^{2\lambda_1 t}\int_0^t e^{-\lambda_1 s} f(s)ds - q(0)A_1\lambda_1 e^{2\lambda_1 t}\int_0^\infty e^{-\lambda_2 s} f(s)ds$$

$$+q(0)A_1\lambda_2 e^{(\lambda_1+\lambda_2)t}\int_0^\infty e^{-\lambda_1 s} f(s)ds + q(0)A_1\lambda_1 e^{(2\lambda_1)t}\int_0^t e^{-\lambda_1 s} f(s)ds + A_1^2\lambda_1 e^{2\lambda_1 t}\left[\int_0^t e^{-\lambda_1 s} f(s)ds\right]^2$$

$$-A_1^2\lambda_1 e^{2\lambda_1 t}\left[\int_0^t e^{-\lambda_1 s}f(s)ds\int_0^\infty e^{\lambda_2 s}ds\right]+A_1^2\lambda_1 e^{(\lambda_1+\lambda_2)t}\left[\int_0^\infty e^{-\lambda_2 s}f(s)ds\right]^2$$
$$-q(0)A_1\lambda_1 e^{2\lambda_1 t}\int_0^\infty e^{-\lambda_2 s}f(s)ds-A_1^2\lambda_1 e^{2\lambda_1 t}\left[\int_0^\infty e^{-\lambda_2 s}f(s)ds\int_0^t e^{\lambda_1 s}f(s)ds\right]$$
$$+A_1^2\lambda_1 e^{2\lambda_1 t}\left[\int_0^\infty e^{-\lambda_2 s}f(s)\right]^2-A_1^2\lambda_2 e^{(\lambda_1+\lambda_2)t}\left[\int_0^\infty e^{-\lambda_2 s}f(s)\int_t^\infty e^{-\lambda_2 s}f(s)ds\right]$$
$$+q(0)A_1\lambda_2 e^{(\lambda_1+\lambda_2)t}\int_t^\infty +A_1^2\lambda_1 e^{(\lambda_1+\lambda_2)t}\left[\int_t^\infty e^{-\lambda_2 s}f(s)ds\int_0^t e^{-\lambda_2 s}f(s)ds\right]$$
$$-A_1^2\lambda_1 e^{(\lambda_1+\lambda_2)t}\left[\int_t^\infty e^{-\lambda_2 s}f(s)ds\int_0^\infty e^{-\lambda_2 s}f(s)ds\right]+A_1^2\lambda_2 e^{2\lambda_2 t}\left[\int_t^\infty e^{-\lambda_2 s}f(s)ds\right]^2.$$

For $\lambda_1 < r/2$ and $\lambda_2 > r/2$ , $2\lambda_1 < r$ , $2\lambda_2 > r$ and $\lambda_1 + \lambda_2 = r$.

Finally, multiplying $q(t)q'(t)$ by $e^{-rt}$ and allowing $t \to \infty$, we obtain the desired condition.That is,

$$\lim_{t\to\infty} e^{-rt}q(t)q'(t) = 0.$$

**Case 2** The characteristic roots ($\lambda_1$ and $\lambda_2$ ) are equal.

In this case the discriminant in equation (12) is equal to zero. This implies $a_2r^2 + 2a_2r + 4 = 0$. Moreover, for the equal roots $\lambda_1 = \lambda_2 = \frac{r}{2}$.

The characteristic equation of demand equation is $a_1n^2 + a_1n + 1 = 0$. When $n = r/2$, we have $a_1n^2 + a_1n + 1 = 0$ which implies $a_2(r/2)^2 + a_1(r/2) + 1 = 0$ i.e., $a_2r^2 + a_1r + 4 = 0$.

Therefore $r/2$ is also a characteristic root of demand equation. Suppose that another root of demand equation is $\sigma$.

**Example 4:** Let $f(t) = \frac{e^{rt/2}}{1+t}$. This implies $\left|e^{-rt/2}f(t)\right| = \left|\frac{1}{1+t}\right|$ and $\left|e^{-rt/2}f'(t)\right| = \left|\frac{r/2}{1+t} - \frac{1}{(1+t)^2}\right|$ both are bounded for all value of $t$. Thus $f$ is an admissible function for $q = e^{rt/2}$ and the integral

$$\begin{aligned}\int_0^\infty e^{-rt}q(t)f(t)dt &= \int_0^\infty \frac{1}{1+t}dt \\ &= [\ln(1+t)]_0^\infty \text{ is undefined.}\end{aligned}$$

This implies the possibility of an edge solution.

Let $f(t) \leq Ce^{\delta t}$ with $\delta < r/2$. For $C = 1$ and $f(t) = e^{\delta t}$.

The complimentary function of equation (11) can be written as

$$q_c = c_1 e^{rt/2} + c_2 t e^{rt/2}.$$

Suppose, $y_1 = e^{rt/2}$ and $y_2 = te^{rt/2}$.

$\therefore q_c = c_1 y_1 + c_2 y_2$

To find the particular solution of the equation (11), we therefore let

$q_p = v_1(t)y_1 + v_2(t)y_2$ i.e., $q_p = v_1(t)e^{rt/2} + v_2(t)te^{rt/2}$.

Thus $q_p'(t) = v_1'(t)y_1 + v_2'(t)y_2 + v_1(t)y_1' + v_2(t)y_2'$. We impose the condition

$$v_1'(t)y_1 + v_2'(t)y_2 = 0 \quad i.e., \quad v_1' e^{rt/2} + v_2' t e^{rt/2} = 0 \tag{22}$$

leaving $q_p'(t) = v_1(t)y_1' + v_2(t)y_2'$. From this, we find

$q_p''(t) = v_1'(t)y_1' + v_2'(t)y_2' + v_1(t)y_1'' + v_2(t)y_2''$.

By variation of parameters, we can write $v_1'(t)y_1' + v_2'y_2' = \frac{f}{2a_2}$, this implies

$$v_1'(t)(r/2)e^{(rt/2)} + v_2'(t)[e^{(rt/2)} + r/2te^{(rt/2)}] = f(2a_2)^{-1}$$

$$i.e., v_1'(t)(r/2) + v_2'(t)[1 + (r/2)] = e^{\delta t}(2a_2)^{-1}e^{-rt/2}$$

$$\therefore v_1'(t)r/2 + v_2'(t)[1 + r/2] = (2a_2)^{-1}e^{(\delta - r/2)t}. \tag{23}$$

Solving equations (22) and (23) by using Cramer's rule we get

$$v_1'(t) = \frac{-(2a_2)^{-1}e^{(\delta - r/2)t}t}{1 + rt/2 - rt/2}.$$

$$\begin{aligned}
\therefore v_1'(t) &= -(2a_2)^{-1}e^{(\delta-r/2)t}t. \\
\text{This implies} \quad v_1(t) &= -(2a_2)^{-1}\int_0^t e^{(\delta-r/2)s}sds \\
&= (2a_2)^{-1}\left[-\frac{se^{(\delta-r/2)s}}{\delta-r/2}+\frac{e^{(\delta-r/2)s}}{(\delta-r/2)^2}\right]_0^t \\
&= (2a_2)^{-1}\left[-\frac{te^{(\delta-r/2)t}}{\delta-r/2}+\frac{e^{(\delta-r/2)t}}{(\delta-r/2)^2}+\frac{1}{(\delta-r/2)^2}.\right]
\end{aligned}$$

Similarly, $\quad v_2'(t) = (2a_2)^{-1}e^{(\delta-r/2)t}$

This implies $\quad v_2(t) = (2a_2)^{-1}\left[\frac{e^{(\delta-r/2)t}}{\delta-r/2}\right]_0^t$

$$\therefore v_2(t) = (2a_2)^{-1}\left[\frac{e^{(\delta-r/2)t}}{\delta-r/2}-\frac{1}{\delta-r/2}\right].$$

Thus the particular solution is given by

$$q_p = (2a_2)^{-1}\left[-\frac{te^{(\delta-r/2)t}}{\delta-r/2}+\frac{e^{(\delta-r/2)t}}{(\delta-r/2)^2}+\frac{1}{(\delta-r/2)^2}\right]e^{rt/2}$$

$$+(2a_2)^{-1}\left[\frac{e^{(\delta-r/2)t}}{\delta-r/2}-\frac{1}{\delta-r/2}\right]te^{rt/2}$$

Or,

$$q_p = (2a_2)^{-1}\left\{\frac{-te^{(\delta-r/2)t}}{\delta-r/2}+\frac{e^{(\delta-r/2)t}}{(\delta-r/2)^2}-\frac{1}{(\delta-r/2)^2}+\frac{te^{(\delta-r/2)t}}{(\delta-r/2)}-\frac{t}{\delta-r/2}\right\}e^{rt/2}.$$

Finally,

$$\begin{aligned}
q_p(t) &= \frac{e^{\delta t}(2a_2)^{-1}}{(\delta-r/2)^2}-(2a_2)^{-1}\left\{\frac{1}{(\delta-r/2)^2}e^{rt/2}+\frac{1}{(\delta-r/2)}te^{rt/2}\right\} \\
&= \frac{e^{\delta t}(2a_2)^{-1}}{(\delta-r/2)^2}-(2a_2)^{-1}\left\{A_1y_1(t)+A_2y_2(t)\right\} \\
&= \left[(2a_2)(\delta-r/2)^2\right]^{-1}e^{\delta t}.
\end{aligned}$$

Since $y_1$ and $y_2$ are linearly independent solutions of the corresponding homogenous equation, $A_1y_1(t) + A_2y_2(t) = 0$ where $A_1 = \frac{1}{(\delta - r/2)^2}$ and $A_2 = \frac{1}{(\delta - r/2)}$.
Thus for $f = e^{\delta t}$ the general solution of Euler equation (11) for equal roots is given by

$$q(t) = c_1e^{rt/2} + c_2te^{rt/2} + \left[(2a_2)(\delta - r/2)^2\right]^{-1} e^{\delta t}. \tag{24}$$

Now $q(0) = c_1 + [(2a_2)^{-1}(\delta - r/2)^{-2}]$. This implies, $c_1 = q(0) - [(2a_2)^{-1}(\delta - r/2)^{-2}]$.
Setting $c_2 = 0$ equation (24) can be written as

$$q(t) = \left\{q(0) - \left[(2a_2)(\delta - r/2)^2\right]^{-1}\right\} e^{rt/2} + \left[(2a_2)^{-1}(\delta - r/2)^{-2}\right]^{-1} e^{\delta t}.$$

We have $c_1 = \left\{q(0) - [(2a_2)(\delta - r/2)^2]^{-1}\right\}$ and
suppose, $d_1 = [(2a_2)^{-1}(\delta - r/2)^{-2}]^{-1}$. This implies, $q'(t) = r/2c_1e^{rt/2} + \delta d_1e^{\delta t}$

$$\begin{aligned} \therefore q(t)q'(t)e^{-rt} &= e^{-rt}\left\{r/2c_1^2e^{rt} + \delta c_1d_1e^{rt/2}e^{\delta t} + r/2d_1e^{rt/2}e^{\delta t} + \delta d_1^2e^{2\delta t}\right\} \\ &= r/2c_1^2 + \delta c_1d_1e^{(\delta - r/2)t} + r/2d_1e^{(\delta - r/2)t} + \delta d_1^2e^{(2\delta - r)t} \end{aligned}$$

For $\delta < r/2$, $(\delta - r/2) < 0$ and $(2\delta - r/2) < 0$.

$$\therefore \lim_{t\to\infty} q(t)q'(t)e^{-rt} = r/2c_1^2 \neq 0.$$

From inequality (13) and $a_2 < 0$ we can conclude that the maximum value of the characteristic root of Euler equation occurs for that $r$ which makes the discriminant of the Euler equation zero. If $a_2 < 0$ then $a_2r^2 + 2a_1r + 4 = 0$ has real roots of opposite sign then largest admissible $r$ is the positive root of $a_2r^2 + 2a_1r + 4 = 0$. Thus the largest admissible $r$ is the positive root of $a_2r^2 + 2a_1r + 4 = 0$. The positive root is the value of $r$ which makes the discriminant equal to zero and $q(t)$ grows most rapidly. Thus we can conclude that for the given pair of numbers $(a_1, a_2)$, there exists a positive $r$ such that the present value is bounded and maximum . In this case the demand equation and the Euler equation have common characteristic root (r/2).

4.3. **Dicrete(D) Monopoly Problem.** The Present Value of discrete monopoly problem is given by:

$$PV = \sum_{t=0}^{\infty} \beta^t q_t p_t \tag{25}$$

where, $p_t$ = price per unit at time $t$

$q_t$ = quantity demanded at time $t$

$\beta$ = discount factor and $0 < \beta \leq 1$.

The monopolist demand condition for the discrete case is given by

$$p_t = f_t - q_t - \alpha q_{t-1}. \tag{26}$$

The relation between interest rate $r$ and the discount factor $\beta$ is given by :

$$\beta = e^{-r}.$$

Similar to the continuous case we define admissible paths and interior and boundary sequences.

**Definition 14.** *Admissible quantity path*

The quantity q(t) is said to be admissible if $\beta^{-\frac{t}{2}} q_t$ is bounded for all sequences $\{q_t\}$ . An admissible sequence is an *interior sequence* if $\sum_{t=0}^{\infty} \beta^t q_t^2$ is finite, if not the sequence is said to be a *boundary sequence.*

Note: In the continuous case(C) if the demand equation implies the bounded PV then the the quantity purchased at time $t$ must depend on the prices before and after time $t$. But in the discrete case for bounded maximum present value, the quantity purchased at time $t$ depends on past prices and the past size of market f(t). That is the trend of consumers' behavior can be used to predict the future trend.

The PV of the discrete monopoly problem can be written as

$$PV = \sum_{t=0}^{\infty} \beta^t q_t[f_t - q_t - \alpha q_{t-1}].$$

This implies that

$$PV = q_0(f_0 - q_0) + \beta q_1(f_1 - q_1 - \alpha q_0) + \beta^2 q_2(f_2 - q_2 - \alpha q_1) + \beta^3 q_3(f_3 - q_3 - \alpha q_2) + .....$$

$$+\beta^t q_t(f_t - q_t - \alpha q_{t-1}) \ldots$$

Taking partial derivatives with respect to $q_0$, $q_1$,..., and $q_t$ respectively, we get

$$\frac{\partial PV}{\partial q_0} = f_0 - 2q_0 - \alpha\beta q_1,$$

$$\begin{aligned} \frac{\partial PV}{\partial q_1} &= \beta f_1 - 2\beta q_1 - \alpha\beta q_0 - \alpha\beta^2 q_2, \\ &= \beta[f_1 - \alpha q_0 - 2q_1 - \alpha\beta q_2] \end{aligned}$$

$$\begin{aligned} \frac{\partial PV}{\partial q_2} &= \beta^2 f_2 - 2\beta^2 q_2 - \alpha\beta^2 q_1 - \alpha\beta^2 q_3 \\ &= \beta^2[f_2 - \alpha q_1 - 2q_1 - 2q_2 - \alpha\beta q_3], \end{aligned}$$

and

$$\frac{\partial PV}{\partial q_t} = \beta^t[f_t - \alpha q_{t-1} - 2q_t - \alpha\beta q_{t+1}]. \tag{27}$$

Thus for maximum PV it is necessary that

$\frac{\partial PV}{\partial q_0} = 0$, $\frac{\partial PV}{\partial q_1} = 0$,..., and $\frac{\partial PV}{\partial q_t} = 0$.

From equation (27) we can write: $\beta^t[f_t - \alpha q_{t-1} - 2q_t - \alpha\beta q_{t+1}] = 0$.

Thus, $[f_t - \alpha q_{t-1} - 2q_t - \alpha\beta q_{t+1}] = 0$ i.e.,

$$\alpha q_{t-1} + 2q_t + \alpha\beta q_{t+1} = f_t. \tag{28}$$

For the finite maximum PV it is necessary to show that the difference equation (28) must not have characteristic root of modulus $\beta^{1/2}$ [6].

This is similar to the continuous case where the characteristic roots must be real and must not be equal to $r/2$.

The corresponding system is equivalent to a system with Toeplitz matrix [25].

$$\begin{bmatrix} 2 & \beta^{1/2} & 0 & 0 & 0 & \dots \\ \beta^{1/2} & 2 & \beta^{1/2} & 0 & 0 & \dots \\ 0 & \beta^{1/2} & 2 & \beta^{1/2} & 0 & \dots \\ \dots\dots\dots\dots\dots\dots\dots & & & & & \end{bmatrix}$$

Maximizing PV is equivalent to finding conditions to insure that the Toeplitz matrix is positive definite [6]. This is equivalent to the conditions such that the zeros of

$$\beta^{1/2}\alpha w + 2 + \beta^{1/2}\alpha w^{-1} = 0 \tag{29}$$

are not of modulus 1, [6]. This implies

$$w = \frac{1 \pm \sqrt{1 - \beta\alpha^2}}{\beta^{1/2}\alpha}$$

To insure that $w$ is not of modulus of 1, we need $(1 - \beta\alpha^2) > 0$.

A substitution of $w = \beta^{1/2}z$ returns us to (28) and therefore $z = \beta^{-1/2}w$.

This implies that the modulus of $z$ can not be $\beta^{-1/2}$ and the zero for equation (28) can not have modulus $\beta^{1/2}$. Let $\mu_1$ and $\mu_2$ be the roots of the equation (29). Suppose $\mu_1 = \frac{-1+\sqrt{1-\alpha^2\beta}}{\alpha\beta}$ and $\mu_2 = \frac{-1-\sqrt{1-\alpha^2\beta}}{\alpha\beta}$.

For real roots the discriminant must be non negative. Moreover, $|\mu_1| < |\mu_2|$. For $|\mu_1| \neq |\mu_2|$ it is necessary and sufficient that the discriminant satisfy

$$1 - \alpha^2\beta > 0. \tag{30}$$

This implies $1 > \alpha^2\beta$ or, $\alpha < \pm\frac{1}{\sqrt{\beta}}$. Hence $-1 < \alpha < 1$ for $0 < \beta \leq 1$.

This yields $|\alpha| < 1$.

Thus for all $\alpha$ and all $\beta s$ in the range, there exists a bounded maximum present value.

Generally, the discount factor satisfies $\beta \in (0, 1)$.

4.4. **Differences Between the (C) and (D) problems.** Lets look at some important differences that exist between the continuous problem, (C), and the discrete problem, (D). **1**. In C problem when $a_2 = 0$ then equation (11) can be written as: $(2 + a_1 r)q = f$. This implies,

$$q(t) = \frac{f(t)}{2 + a_1 r}. \tag{31}$$

Similarly PV can be written as

$$PV = \int_0^\infty e^{-rt}[f(t) - q(t) - a_1 q'(t)]q(t)dt.$$

This implies the quantity path is discontinuous at certain point. Thus there exists the necessity for an adjustment. Suppose $q(t)$ is discontinuous when $t = 0$ and $a_2 < 0$. Let us approximate the quantity path on two intervals $[k, h]$ and $(h, \infty)$ with different slopes at $t = 0$.

Suppose, $q(0^-) = q(0) = q(0^+)$ but $q'(0^-) \neq q(0^+)$.

Consider, $q'(0^-) = \frac{q(t)-q(k)}{t-k}$ if $t, k < 0, t > k$ and $q'(0^+) = \frac{q(t)-q(h)}{t-h}$ if $t, h > 0, t < h$.

Considering two terms containing the first derivatives from equation (6)

$$\left[-e^{-rt}qq'\right]_{0^-}^{\infty} + \int_{0^-}^{\infty} e^{-rt}q'^2 dt$$

and integrating in the interval [k,h] implies,

$$-\left\{q(0^+)q'(0^+) - q(0^-)q'(0^-) + \lim_{t\to\infty} e^{-rt}q(t)q'(t) - q(0^+)q'(0^+)\right\} + \int_k^h e^{-rt}q'^2 dt$$

$$
\begin{aligned}
&= -\left\{q(0^-)q'(0^-)\right\} + \int_k^0 e^{-rt}q'^2(0^-)dt + \int_0^h e^{-rt}q'^2(0^+)dt \\
&= -\left\{q(0^-)q'(0^-)\right\} + \frac{1}{-r}q'^2(0^-)[1-e^rk] + \frac{a_2}{-r}q'^2(0^+)[e^{hr}-1] \\
&= -\left\{q(0^-)q'(0^-)\right\} + \frac{1}{-r}[q'^2(0^-)(1-e^{-rk}) - q'^2(0^+)(1-e^{-rt})].
\end{aligned}
$$

note that $\lim\limits_{k\to 0^-}(1-e^{-rk}) = 0$ and $\lim\limits_{h\to 0^+}(1-e^{-rh}) = 0$. Again integrating in the interval $[h, \infty]$ we get

$$
-\left[\lim_{t\to\infty} e^{-rt}qq' - q(0^+)q'(0^+)\right] + \int_h^\infty e^{-rt}q'^2(0^+)dt
$$

$$
\begin{aligned}
&= -[-q(0^+)q'(0^+)] + 1\int_h^\infty e^{-rt}q'^2(0^+)dt \\
&= -[-q(0^+)q(0^+)] + \frac{1}{-r}q'^2(0^+)\left[e^{-rt}\right]_h^\infty \\
&= q(0^+)q'(0^+) + \frac{1}{-r}q'^2(0^+)e^{-rh}
\end{aligned}
$$

where $\lim\limits_{h\to 0^+}(e^{-rh}) = 1$. Thus the Euler path is continuously differentiable for all $t > 0$ and the only discontinuity is the jump at $q'(0)$.

Suppose for the Euler path $q'(0)$ exists and that $q(t)$ has a jump at $t = 0$. This implies $a_2 q'^2(t) < 0$ because $a_2 < 0$. Hence the value of the integral is arbitrarily small. This jump would make integral approach to $-\infty$. Moreover the slope of the path at $t = 0$ does not necessarily equal the given slope $q'(0)$. This ultimately leads to the conclusion that the jump in $q'(t)$ at $t = 0$ is required but the jump in $q(t)$ is rather prohibited. In (D) the demand equation is first order difference equation. There is a geometric approach throughout the problem.

**2**. If the monopolist's demand equation is stable then the characteristic equation of the demand relation

$$p(t) = f(t) - q(t) - a_1 q'(t) - a_2 q''(t) \tag{32}$$

is given by $a_2 m^2 + a_1 m + 1 = 0$ and has roots with negative real parts. Let the roots be $\sigma_1$ and $\sigma_2$.

The complimentary function for the demand relation is given by

$$q_c(t) = c_1 e^{\sigma_1 t} + c_2 e^{\sigma_2 t}.$$

To find the particular solution of the the equation (32) we let $q_p(t) = w_1(t)z_1 + w_2(t)z_2$ where $z_1 = e^{\sigma_1 t}$ and $z_2 = e^{\sigma_2 t}$

This implies $q_p'(t) = w_1'(t)z_1 + w_1(t)z_1 + w_2'(t)z_2 + w_2(t)z_2$.

Imposing

$$w_1'(t)e^{\sigma_1 t} + w_2'(t)e^{\sigma_2 t} \tag{33}$$

we can write $q_p'(t) = w_1(t)z_1 + +w_2(t)z_2$.

By variation of parameter we can write $v_1'(t)z_1' + v_2'(t)z_2' = (f - p)(2a_2)^{-1}$

. That is,

$$\sigma_1 w_1'(t)e^{\sigma_1 t} + \sigma_2 w_2'(t)e^{\sigma_2 t} = (f - p)(a_2)^{-1}. \tag{34}$$

Using Cramer's rule to solve (33) and (34) we get

$$w_1'(t) = (\sigma_1 - \sigma_2)^{-}(a_2)^{-1}(f - p)e^{-\sigma_1 t}.$$

Integrating we get

$$w_1(t) = (\sigma_1 - \sigma_2)^{-1}(a_2)^{-1} \int_0^t (f - p)e^{-\sigma_1 s} ds$$

Similarly,

$$w_2(t) = -(\sigma_1 - \sigma_2)^{-1}(a_2)^{-1}\int_0^t (f-p)e^{-\sigma_2 s}ds$$

Thus

$$q_p(t) = e^{\sigma_1 t}(\sigma_1 - \sigma_2)^{-1}(a_2)^{-1}\int_0^t (f-p)e^{-\sigma_1 s}ds + e^{\sigma_2 t}(\sigma_1 - \sigma_2)^{-1}(a_2)^{-1}\int_0^t (f-p)e^{-\sigma_2 s}ds.$$

Therefore the general solution of the demand equation is given by

(35)

$$q(t) = (\sigma_1 - \sigma_2)^{-1}\left\{e^{\sigma_1 t}[c_1 + a_2^{-1}\int_0^t e^{-\sigma_1 s}(f-s)ds] + e^{\sigma_2 t}[c_2 - a_2^{-1}\int_0^t e^{-\sigma_2 s}(f-s)ds].\right\}$$

From equation (35) we see that there is no provision for future prices after time $t$. This implies the profit maximizing monopolist can find an quantity path with infinite present value. This is a kind of short-sided consumer behavior. Since the demand equation is stable, for perfect competition it is assumed that the consumer confidence level is high and no consumer deferment of purchases occurs with the hope of a change in price. The competition in the market always helps the consumer. In fact it prevents the exploitation of the consumer. In such a situation forecasting always helps the producer to improve inventory, output and employment policies without exploiting the consumer.

**3**. Let us consider an unstable demand condition. Because of the unstable demand the consumer forecasting is necessary but forecasting only may not be sufficient.

In (D) problem an unbounded maximum present value is impossible if the demand relation is stable and $f$ grows at a geometric rate less than $\beta^{-1/2}$ for $\beta \leq 1$. Moreover, the quantity demanded at time $t$ is a function of a weighted moving average of the past prices. Therefore it can be interpreted as a forecast of future prices. This result strongly contrasts with the result in (C) where perfect foresight is necessary for a finite maximum value.

Basically, the difference between (C) and (D) depends on the nature of the continuous curve with a continuous first derivative. In such a case the slope is same whether it is taken from the right or the left. This implies the demand is completely predictable. That is, there is no uncertainty. The same phenomenon could be used by modifying the discrete case. This helps to express the discrete problem with more clarity.

**4**. Discrete approximation of continuous problem.

Let $q'(t) = q_t - q_{t-1}$. The symmetric first difference is given by $q'(t) = q_{t+1} - q_{t-1}$. The perfect $\beta$-symmetric difference

$$q'(t) = \beta^{1/2} q_{t+1} - \beta^{-1/2} q_{t-1}.$$

The future term $q_{t+1}$ is discounted by the factor $\beta^{1/2}$ and past term is increased by $\beta^{-1/2}$. This implies a connection between past and future. Let us assume the demand relation is the perfect $\beta$-symmetric difference equation

$$2q_t + \alpha(\beta^{1/2} q_{t+1} - \beta^{1/2} q_{t-1}) = f_t - p_t. \tag{36}$$

Substituting the value of $p_t$ in equation (25) we get

$$PV = \sum_{0}^{\infty} \beta^{\triangle}[f_t - 2q_t - \alpha(\beta^{1/2} q_{t+1} - \beta^{1/2} q_{t-1})]q_t.$$

This yields,

$$PV = [q_0 f_0 - 2q_0^2 - \alpha\beta^{1/2} q_0 q_1] + \beta[q_1 f_1 - 2q_1^2 - \alpha\beta^{1/2} q_1 q_2 - \alpha\beta^{-1/2} q_1 q_0]$$

$$+\beta^2[q_2 f_2 - 2q_2^2 - \alpha\beta^{1/2} q_2 q_3 + \alpha\beta^{-1/2} q_2 q_1]$$

Now differentiating PV with respect to $q_0$, $q_1$, $q_2$,…, and $q_t$ respectively,

$$\frac{\partial PV}{\partial q_0} = \left\{ f_0 - 4q_0 - \alpha\beta^{1/2} q_1 \right\} + \beta[-\alpha\beta^{-1/2} q_1].$$

$$\frac{\partial PV}{\partial q_1} = f_1 - 4q_1 - \alpha(\beta^{1/2} q_2 - \beta^{-1/2} q_0) - \alpha(\beta^{-1/2} q_0 - \beta^{1/2} q_2).$$

$$\frac{\partial PV}{\partial q_2} = f_2 - 4q_2 - \alpha(\beta^{1/2}q_3 - \beta^{-1/2}q_1) - \alpha(\beta^{-1/2}q_1 - \beta^{1/2}q_3).$$

Similarly,

$$\frac{\partial PV}{\partial q_t} = f_t - 4q_t - \alpha(\beta^{1/2}q_{t+1} - \beta^{-1/2}q_{t-1}) - \alpha(\beta^{-1/2}q_{t-1} - \beta^{1/2}q_{t+1}).$$

For the maximum PV it is necessary that $\frac{\partial P.V.}{\partial q_0} = 0$, $\frac{\partial PV}{\partial q_1} = 0 \ldots$, $\frac{\partial P.V.}{\partial q_t} = 0$. This yields

$$f_t - 4q_t - \alpha(\beta^{1/2}q_{t+1} - \beta^{-1/2}q_{t-1}) - \alpha(\beta^{-1/2}q_{t-1} - \beta^{1/2}q_{t+1}) = 0.$$

$$\therefore f_t = 4q_t.$$

The above equation exhibits similar behavior as the continuous case when $a_2 = 0$, i.e.; an instantaneous adjustment to the optimal quantity path. But for this problem a finite maximum value always exists without any restriction on $\beta$. The complimentary function $[a_2r^2 + a_1r + 1 = 0]$ of the demand equation must satisfy
$(\sigma_1 + \sigma_2) = -\frac{a_1}{a_2}$ and $(\sigma_1\sigma_2)^{-1} = a_2$. This implies $r = -\frac{a_1}{a_2}$. Therefore, $a_1 + a_2r = 0$. Thus $a_2 = 0$ implies $a_1 = 0$. If the demand equation is missing $q''$ then it also misses $q'$. Thus there is no perfectly symmetric first order symmetric differential equation. In this case the role of symmetry is controversial in the real world. This is because such symmetry totally eradicates the difference between past and future. First, it is impossible to change the decision of past. Also if there is perfect foresight then there is not necessary to change the past decision. Practically, perfect forecasting is impossible!! Thus the decision maker has to consider current circumstances as much as possible.

## 5. Time Scale Analysis

After the separate analysis of (C) and (D) cases, we intend to combine them by using time scale analysis. In this section we focus our study on describing and generalizing the problem using time scale calculus. Because of the similarity in calculation most of the steps involving have been omitted. The present value under the monopoly condition is given by

$$PV = \int_0^\infty e_{-r}(t,t_0)p(t)q(\sigma(t))\triangle t$$

where $\qquad p(t) = f(t) - q(\sigma(t)) - a_1 q^\triangle - a_2 q^{\triangle\triangle}$. This implies

$$PV = \int_0^\infty e_{-r}(t,t_0)\left[f(t) - q(\sigma(t)) - a_1 q^\triangle - a_2 q^{\triangle\triangle}\right] q(\sigma(t))\triangle t. \tag{37}$$

Equation (37) can also be written as

$$PV = \left[-a_2 e_{-r}(t,t_0)q(\sigma(t))q^\triangle\right]_0^\infty + \int_0^\infty e_{-r}(t,t_0)\left[f(t)q(\sigma(t)) - q(\sigma(t))^2 - (a_1 + a_2 r)q(\sigma(t))q^\triangle\right]\triangle t$$
$$+ \int_0^\infty e_{-r}(t,t_0)a_2(q^{\sigma\triangle})^2\triangle t. \tag{38}$$

Now if $\sigma(t) = t$, that is, for right dense points in the time scales, equation (38) can be written as

$$PV = -a_2 q(0)q^\triangle(0) + \int_0^\infty L(t,q(t),q^\triangle(t))\triangle t$$

provided

$$\lim_{t\to\infty} e_{-r}(t,t_0)q(t)q^\triangle(t) = 0$$

where

$$L(t,q(t),q^\triangle(t)) = e_{-r}(t,t_0)\left[f(t)q(t) - q(t)^2 - (a_1 + a_2 r)q(t)q^\triangle + a_2(q^\triangle)^2\right].$$

For maximum present value it is sufficient to maximize $L(t,q(t),q^\triangle(t))$. From the paper by Dr. Bohner and Dr. Ridenhour [11] the Euler-lagrange equation time scale

$\mathbb{T}$ is

$$L_y(t, y(t), y^{\triangle}(t)) - [L_r(t, y(t), y^{\triangle}(t)]^{\triangle} = 0$$

where Lagrangian $L_r(t, y, r)$ for $t \in \mathbb{T}$, $y \in R^n$ and $r \in R^n$. Thus, the Euler-Lagrange equation can also be written as

$$L_{q^{\sigma}} - [L_{q^{\triangle}}]^{\triangle} = 0. \tag{39}$$

For $q(\sigma(t)) = q(t)$ equation (39) can be written as $L_{q(t)} - \left[L_{q^{\triangle}(t)}\right]^{\triangle} = 0$. This returns us to (C) where $\mathbb{T} = [0, \infty)$. For this time scale every point is right dense. This implies $q(\sigma(t)) = q(t)$.

Thus,

$$L(t, q(t), q^{\triangle}) = e_{-r}(t, t_0)\left[f(t)q(t) - (q(t))^2 - (a_1 + a_2 r)q(t)q^{\triangle} + a_2(q^{\triangle})^2\right].$$

Applying Euler-Lagrange equation in $L(t, q(t), q^{\triangle})$ we can write

$$2a_2 q^{\triangle\triangle} - 2a_2 r q^{\triangle} + (2 + a_1 r + a_2 r^2)q - f = 0. \tag{40}$$

Suppose, $m = \lambda_1$ and $\lambda_2$. For the bounded maximum present value, the roots of the equation (40) must be real.

**Assertion:** The Characteristic roots of the Euler-Lagrange equation (40) must be real for bounded maximum present value.

**Case 1:** The characteristic roots ($\lambda_1$ and $\lambda_2$ ) are not equal.

Let us further suppose $\lambda_1 < r/2$ and $\lambda_2 > r/2$. The general solution of the Euler-Lagrange equation is given by

$$q(t) = (\lambda_1 - \lambda_2)^{-1}\left\{\left[C_1 + (2a_2)^{-1}\int_0^t f(s)e_{-\lambda_1}(s, s_0)\triangle s\right]e_{\lambda_1}(t, t_0)\right\}$$

$$+(\lambda_1 - \lambda_2)^{-1}\left\{\left[C_2 - (2a_2)^{-1}\int_0^t f(s)e_{-\lambda_2}(s, s_0)\triangle s\right]e_{\lambda_2}(t, t_0)\right\}. \tag{41}$$

where $C_1 = \frac{c_1'}{(\lambda_1-\lambda_2)^{-1}}$ and $C_2 = \frac{c_2'}{(\lambda_1-\lambda_2)^{-1}}$.
Now by using boundary condition $q(0) = (\lambda_1 - \lambda_2)^{-1}(c_1 + c_2)$ and
$C_2 = (2a_2)^{-1} \int_0^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s$ we obtain
$C_1 = (\lambda_1 - \lambda_2) q(0) - C_2$ i.e. $C_1 = (\lambda_1 - \lambda_2) q(0) - (2a_2)^{-1} \int_0^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s$.
Thus the general solution can be written as

$$q(t) = (\lambda_1 - \lambda_2)^{-1} \left[ (\lambda_1 - \lambda_2) q(0) - (2a_2)^{-1} \int_0^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s \right]$$

$$+ (\lambda_1 - \lambda_2)^{-1} \left[ (2a_2)^{-1} \int_0^t f(s) e_{-\lambda_1}(s, s_0) \triangle s \right] e_{\lambda_1}(t, t_0)$$

$$+ (\lambda_1 - \lambda_2)^{-1} \left[ (2a_2)^{-1} \int_0^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s - (2a_2)^{-1} \int_0^t f(s) e_{-\lambda_2}(s, s_0) \triangle s \right] \tag{42}$$

$$q(t) = q(0) e_{\lambda_1}(t, t_0) + A_1 e^{\lambda_1}(t, t_0) \int_0^t e^{-\lambda_1}(s, s_0) f(s) \triangle s$$
$$- A_1 e^{\lambda_1}(t, t_0) \int_0^\infty e^{-\lambda_2}(s, s_0) f(s) \triangle s$$
$$+ A_1 e^{\lambda_2}(t, t_0) \int_t^\infty e^{-\lambda_2}(s, s_0) f(s) \triangle s$$

provided $A_1 = (\lambda_1 - \lambda_2)^{-1}(2a_2)^{-1}$. This yields

$$q^\triangle(t) = q(0)\lambda_1 e_{\lambda_1}(t, t_0) + A_1 \lambda_1 e_{\lambda_1}(t, t_0) \int_0^t e_{-\lambda_1}(t, t_0) f(s) \triangle s + A_1 e_{\lambda_1}(t, t_0) e^{-\lambda_1}(t, t_0) f(t)$$

$$- A_1 \lambda_1 e_{\lambda_1}(t, t_0) \int_0^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s - A_1 \lambda_1 e_{\lambda_1}(t, t_0) \lim_{h \to \infty} e_{-\lambda_2}(h, t_0) f(h)$$

$$+ A_1 \left[ \lambda_2 e_{\lambda_2}(t, t_0) \int_t^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s + \lambda_2 e_{\lambda_2}(t, t_0) \lim_{h \to \infty} \left\{ e_{-\lambda_2}(h, t_0) - e_{\lambda_2}(t, t_0) \right\} f(t) \right]$$

$$q^\triangle(t) = q(0)\lambda_1 e_{\lambda_1}(t, t_0) + A_1 \lambda_1 e_{\lambda_1}(t, t_0) \int_0^t e_{-\lambda_1}(t, t_0) f(s) \triangle s$$
$$- A_1 \lambda_1 e_{\lambda_1}(t, t_0) \int_0^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s$$
$$+ A_1 \lambda_2 e_{\lambda_2}(t, t_0) \int_t^\infty e_{-\lambda_2}(s, s_0) f(s) \triangle s.$$

For $\lambda_1 < r/2$ and $\lambda_2 > r/2$ : $2\lambda_1 < r$ ; $2\lambda_2 > r$ and $\lambda_1 + \lambda_2 = r$

Applying $t \to \infty$ in the product of $q(t)$ , $q^{\triangle t}$ and $e_{-r}(t, t_0)$ we get

$$\therefore \lim_{t\to\infty} e_{-r}(t, t_0) q(t) q^{\triangle}(t) = 0.$$

**Case 2** The characteristic roots ($\lambda_1$ and $\lambda_2$ ) are equal.

In this case the discriminant of the characteristic equation of (40) is equal to zero. This implies $a_2r^2 + 2a_2r + 4 = 0$. Moreover, for the equal roots $\lambda_1 = \lambda_2 = \frac{r}{2}$.

The characteristic equation of demand equation is $a_1n^2 + a_1n + 1 = 0$. When $n = r/2$,

$a_1n^2 + a_1n = 1 = 0$ implies $a_2(r/2)^2 + a_1(r/2) + 1 = 0$ i.e., $a_2r^2 + a_1r + 4 = 0$.

Therefore $r/2$ is also a characteristic root of demand equation. As the results in chapter 4 show us, (C) and (D) have very distinct characteristics. Unification of these two problems requires an adjustment to (D). This is certainly a valuable undertaking for further study by the author but it may lead away from the practical applications that has motivated this work. Study of the generalized problem has shown that successful unification requires the determination of a function

$$L \equiv L(t, q(\sigma(t)), q^{\triangle}(t)$$

resulting from the original integral to be maximized. Then, the Euler-Lagrange equation for timr scale can be applied and the solution will follow from the theory of second order dynamic equation on time scales, [5].

## 6. Conclusion

In (C) there is an exponential approach to the equilibrium quantity path when $a_2 \neq 0$. If $a_2 = 0$, the demand equation reduces to a first order differential equation.In this case we need to have instantaneous adjustment to the quantity path. For $a_2 = 0$ there exists a jump in $q'(t)$ at $t = 0$ while a jump in $q(t)$ is prohibited. In (D) there is no instantaneous adjustment as long as $\alpha \neq 0$. For $\alpha \neq 0$, $q_t$ approaches the equilibrium path at a geometric rate. For the stable demand equation, it is assumed that the consumer confidence level is high and no consumers defer their purchases in the hopes of a change price. Competition always helps to prevent the exploitation of the consumer. In such a case, forecasting always helps producers improve improve output and employment. For the unstable demand condition, perfect forecasting is necessary. In (D) there exists an unbounded maximum PV for the characteristic roots of modulus $\beta^{-1/2}$ or $\beta^{1/2}$. The Discrete approximation of the continuous problem leads towards the perfect symmetry in the demand condition. Perfect symmetry totally eradicates the difference between the past and future demand. First it is impossible to change the past and if there is perfect foresight then it is not necessary to change the past decisions. The role of symmetry is controversial in real world applications. For time scales when $(q(\sigma(t)) = q(t))$ the maximization of PV has close similarity with (C). We have verified the existence of a finite maximum PV if $\sigma(t) = t$. To extend the results to scattered points, i.e. a general time scale, the general problem must be adjusted to meet the parameters of the Euler-Lagrange equation for time scales. Research required to obtain similar results for a general time scale continuous.

# 7. Appendix

**Stable and Unstable Demand:** We know that business is usually good when prices are rising and usually not so good when prices are falling; the number of shoes that will be bought at three dollars a pair will be greater if it is known that the price is increasing at the rate of fifty cents per week than if the price may decrea at the rate of fifty cents per week. A good approximation for such a situation is to assume a law of demand as follows

$$q = a + bp + h\frac{dp}{dt},$$

where $a < 0$, $b > 0$ and $h > 0$, and the quantity $\frac{dp}{dt}$ is the rate of increasing price. The coefficient $h$ is also called the sensitivity of demand. If the sensitivity of demand is greater then the offer then that situation is **unstable**. If the sensitivity of offer is greater then the demand is **stable**. In stable case the price tends toward definite limits. The simple monopoly problem depends upon three relations:(a) between the amount produced and cost of it; (b) between the demand and price; (c) relation between the amount produced and the amount demanded. For any of the particular functions we can choose a more general form of the function. In particular, the relation between the price and the demand can be generalized as

$$q = a + bp + h\frac{dp}{dt} + k\frac{d^2p}{dt^2}.$$

The more general form of the above second order equation can also be taken according to the theoretical and practical interest and of the problem depending on the parameters involved.

# References


[1] Tesler, Lester G. and Graves, Robert L. *Continuous and Discrete Time Approaches to a Maximization Problem*. The Reiview of Economic Studies, Vol. 35, No.3.(Jul., 1968), pp.307-325.

[2] Evans, griffith C. *Mathematical Introduction to Economics*. McGraw-Hill, New York, 1930.

[3] Gelfand, I.M. and Fomin S.V. (1964) *Calculus of Variation*. Prentice-Hall, Inc. Englewood Cliffs, New Jersy.

[4] Tesler, Lester G.and Graves, Robert L. (1972) *Functional Analysis in Mathematical Economics*. The University of Chicago Press, Chicago.

[5] Bohner, Martin and Peterson, Allen. (2001) *Dynamic Equations on Time Scales*. Birkhauser, Boston.

[6] Tesler, Lester G. and Graves, Robert L. *An Infinite-Horizon Time Quadratic Problem as Applied to a Monopoly Problem*. Econormertica Vol. 35, No. 2. (Apr., 1967 ), pp. 234-272.

[7] Kailath, T and sayed, A.H.(1999) *Fast Reliable Algoriths for Matrices with Structure*. Society for Industrial and Applied Mathematics, Philadelphia.

[8] Lancaster, Kelvin (1969) *Mathematical Economics*. The Macmillan Company, New York.

[9] Hestenes, Magnus R. (1980)*Calculus of Variation and Optimal Control Theory* . Robert E. Krieger Publishing Company, New York.

[10] Allen, R.G.D (1959) *Mathematical Economics*. Macmillan and Co. Ltd., New York.

[11] Ahlbrandt, Calvin D. and Bohner, Martin *Hamiltonion Systems on Time Scales* . Journal of mathematical Analysis and Applications 250,(2000), pp. 561-578.

[12] Hilcher, Roman and Zeidan, Vera *Calculus of Variations on Time Scales : Weak Local Piecewise* $C^1_{rd}$ *Solutions with Variable Endpoints*. Journal of Mathematical Analysis and Applications. 289 (2004), pp. 143-166.

[13] Bohner, Martin and Guseinov, Gusein Sh. *Partial Differentiation on Time Scales* . Dynamic System and Applications, 13 (2004),pp. 351-379.

[14] Bohner, Martin and Guseinov Gusein Sh. *Double Integral Calculus of Variations On Time Scales* . Computers and mathematics with Applications, 54 (2007), pp. 45-57.

[15] Naidu, D. Subbaram and Imaura, Yoshiko *Unified Approach for Euler-Lagrange Equation Arising in Calculus of variations*. Preceedings of the American Control Conference, Denver, Colorado June 4-6, 2003.

[16] Bohner, Martin *Calculus of Variations on Time Scales* . Dynamic systems and Applications 13 (2004), pp. 339-349.

[17] Agarwal, Ravi P. and Bohner, Martin *Quadratic Functional for Second Order Matrix Equations on Time scales*. Nonlinear Analysis 33(1998), pp. 675-692.

[18] Ahlbrandt, Calvin; Bohner, Martin and Voepel, Tammy *Variable Change for Sturm - Liouville Differential Expressions on Time Scales*. Journal of Differential Equations and Applications, (2003) Vol. 9(1), pp. 93- 107.

[19] Ross, Shepley L. *Differential Equations*. John Wiley and Sons Pte. Ltd.(2004).

[20] Bartle, Robert G. *Introduction to Real Analysis*. Jon Wiley and Sons, Inc. New York(2002).

[21] Thomas Jr., George B., Finney, Ross L. and Weir Maurice D. *Calculus and Analytic Geometry*. Addison-Wesley (1996).

[22] Royden, H.L. *Real Analysis 3rd Edition*. Prentice Hall, Upper Saddle River, New Jersey (1988).

[23] Stewart, James *Calculus*. Brooks/ Cole Publishing Company(1999) .

[24] Kelly, Walter, Peterson, Allan *The Theory of Differential Equations Classical and Quantitative*. Pearson Education inc. Upper saddle River, New Jersey(2004).

[25] Gray, Robert A. *Toeplitz and Cicculant Matrices* . Stanford University.

[26] Bliss, Gilbert A. *Lectures on the Calculus of Variations*. University of Chicago Press, Chocago, (1946).

[27] http://www.math.duke.edu/ cbray/0405Spring/32L/presentvalue.pdf.

[28] http://financial-dictionary.thefreedictionary.com/Economic.

[29] Finan, Marcel B. *Present and Future Value of a Continuous Income Stream*. Arkansas Tech University.

**Do not worry about your difficulties in mathematics, I assure you that mine are greater.**

**-Albert Einstein**